A Produce-to-Stock System with Advance Demand Information and Secondary Customers

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Abstract: We consider a manufacturer (i.e., a capacitated supplier) that produces to stock and has two classes of customers. The primary customer places orders at regular intervals of time for a random quantity, while the secondary customers request a single item at random times. At a predetermined time the manufacturer receives advance demand information regarding the order size of the primary customer. If the manufacturer is not able to fill the primary customer's demand, there is a penalty. On the other hand, serving the secondary customers results in additional profit; however, the manufacturer can refuse to serve the secondary customers in order to reserve inventory for the primary customer. We characterize the manufacturer's optimal production and stock reservation policies that maximize the manufacturer's discounted profit and the average profit per unit time. We show that these policies are threshold-type policies, and these thresholds are monotone with respect to the primary customer's order size. Using a numerical study we provide insights into how the value of information is affected by the relative demand size of the primary and secondary customers. © 2007 Wiley Periodicals, Inc. Naval Research Logistics 54: 331–345, 2007

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1. INTRODUCTION

Parts suppliers (or manufacturers) often have two sets of customers: Original Equipment Manufacturers (OEMs), who usually order in large quantities; and Service Parts (SP) customers, who usually order in small quantities. In serving these two markets, suppliers give a higher priority to the OEMs and make sure that they are able to satisfy the OEM’s orders on time by building inventories. However, this practice can lead to high inventory costs, particularly if the OEM’s demands are highly variable. One way to deal with high variability in demand (while keeping a low inventory) is through the use of advance demand information. With advance demand information, the OEM informs the manufacturer about the size of his order in advance, so the manufacturer can better manage her operations and provide better service to the OEM. Regarding the timing of the advance demand information, the manufacturer and the OEM have opposing interests. The manufacturer would like to delay releasing demand information to get a better estimate of his needs and to retain flexibility in serving his own customers. Therefore, when the manufacturer negotiates her contract with the OEM, it is critical for the manufacturer to understand the value of the OEM’s information as well as the effects of the timing of the information on her total profit. Furthermore, the manufacturer also needs to know how she can effectively use the OEM’s information to make better production and inventory decisions.

In order to gain insights into the above questions, in this paper we consider the case where a manufacturer produces to stock and serves an OEM customer as well as SP customers. The OEM customer orders periodically, but his order quantity is random, while the SP customer orders arrive randomly, each requesting one item. The OEM informs the manufacturer about the quantity of his order at a specified time before the point that he needs his order to be delivered. When the information about the order size is received, the manufacturer can use that information to schedule her production and to build enough additional inventory so that she can satisfy the OEM’s complete order on time. Since the OEM is the primary customer, the manufacturer has a contractual obligation to fill

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the OEM’s order. If the manufacturer will not have enough on-hand inventory to satisfy the OEM’s order through normal operations, she will use different mechanisms (e.g., overtime, subcontracts, substitution, rush delivery) to satisfy the OEM’s order on time. Although these mechanisms result in a cost that reduces the manufacturer’s overall profit, they can save the manufacturer from paying a large contractual penalty cost. SP customers are secondary customers; therefore, the manufacturer can reject their orders without penalty (other than the lost sale) if she needs to reserve her inventory for her primary customer.

We focus on the following questions: (i) How do the manufacturer’s optimal production and inventory policies change after receiving the demand information from the OEM? (ii) When should the manufacturer reject a random SP order and how is this decision affected by having advance information? (iii) How do factors such as the manufacturer’s capacity, variability in the OEM’s order size, and the relative profit of the SP customer affect the value of advance information?

Our study is motivated by a situation faced by the automobile industry. For a given part, there often exists a single supplier (or manufacturer) who supplies this part to the OEM as well as SP customers. The OEM is often an assembly plant and its demand is characterized by high volumes and regular orders with contracts specifying advance order times and order due dates. Due to the high cost of stockouts for the OEM, these contracts often include a very high penalty cost per unit time of delay in order delivery to the OEM. Therefore, in order to avoid the large penalty cost and maintain a good relationship with the OEM, the manufacturer often uses overtime, rush delivery, or subcontracting to fill the OEM’s order on time.

The SP market, on the other hand, often consists of small repair shops that order the part depending on the immediate needs of their customers. They are characterized by random order arrivals, mostly single-item orders, and often no order lead time. In contrast to the case with the OEM, if the manufacturer were not able to supply a part to a SP customer, the SP customer would find the part for repair at an alternative supplier. Therefore, there is no contractual penalty cost and the manufacturer only loses the opportunity to profit from selling one item when she does not serve a SP customer. Carr and Duenyas [2] present an example of such a system where a manufacturer of car side mirrors receives orders from repair shops (i.e., SP customers) that need mirrors to replace the mirrors of cars in their shops.

2. LITERATURE REVIEW

Since two major characteristics of our model are “advance demand information” and “stock reservation/rationing,” we present our literature review under these two topics.

Advance demand information has recently been studied in different production/inventory settings. Karaesmen et al. [12] look at the effects of having advance order information in a produce-to-stock system with a single customer. Order inter-arrival and job processing times are geometrically distributed, and all orders arrive a pre-specified number of periods in advance of their due dates. They find that the optimal production policy does not have a simple structure, and therefore they propose an alternative heuristic policy and show that their proposed policy performs well.

Lu, Song, and Yao [14] study the impacts of advance demand information on performance measures of an assemble-to-order system with batch Poisson arrivals. They find that advance demand information is more effective in improving order fill rates than an equivalent reduction in component lead times. Dellaert and Melo [3] examine a single-product system in a make-to-stock environment with a single class of customers ordering in advance of their actual needs. Their problem is to determine the optimal size of a production lot so that they meet delivery requirements with minimal average costs. They show that the optimal policy found using a Markov decision model is too complex; therefore, they present a cost-effective heuristic policy.

Özer [16] considers a system with a single warehouse and multiple retailers. Each retailer has a demand distribution that determines a quantity that is ordered for various demand lead times. In this system with advance demand information, Özer develops a close-to-optimal heuristic to investigate the benefit of advance demand information and its impact on the allocation of inventories. He finds that advance demand information can be used to substitute for inventory. He also finds that an inventory manager is only interested in advance demand information up to the retailer lead time, plus one decision period.

Gallego and Özer [7] consider a periodic-review, multi-echelon inventory system with advance demand information. In their model, the advance information takes the form of a change in demand distributions. Customers are given a choice of delivery times, each with a different price (i.e., earlier delivery times are available at a higher price). Gallego and Özer [7] determine optimal echelon-stock policies for this situation.

Papers that study systems that include the option of rejecting an order so inventory can be saved for other more valuable customer classes (i.e., stock rationing) generally do not examine the effect of advance demand information. Topkis [20] considers the problem of stock rationing in a system with $n$ demands of varying importance. Inventory is built up through orders from an outside source. When a customer arrives, the firm must decide whether to serve that customer demand or to reject the demand for potential use to satisfy a more important class of demands later. He finds that the optimal policy is determined by a set of rationing levels such that demands are
only filled if the inventory is above the rationing level of that class (and there are no unsatisfied demands of more important classes). Kaplan [11] derives similar results for the case where items that are refused due to rationing are backordered instead of lost.

Ha [8] considers the stock rationing problem in a single-item, make-to-stock production system with several demand classes and lost sales. Demands arrive according to a Poisson process and production times are exponentially distributed. He shows that the optimal policy is made up of a series of monotonic stock rationing levels. Ha also examines the effect of changing the ratio between the value of the different classes on the effectiveness of the inventory rationing. He finds that as the difference between the values of the different classes increases, the manufacturer saves more when rationing inventory. Ha [9] extends these results in systems where production times follow an Erlang distribution.

Carr and Duenyas [2] look at the problem of a single manufacturer who produces two products. Product 1 is produced to stock with no backorders, but the manufacturer is contractually obligated to meet demand for this class of orders (OEM customer), and there is a penalty cost when an order is not filled. Product 2 is produced to order for aftermarket customers. The manufacturer has the option of accepting or rejecting the aftermarket order. The demand process for both is Poisson, and the production times are exponentially distributed. They study the optimal production and also the optimal order acceptance policy and show that these policies are threshold-type policies.

Deshpande et al. [4] study the case where inventory is pooled across two demand classes with different arrival rates and shortage costs. When on-hand inventory falls below a threshold, low-priority demands are backordered while high-priority demands continue to be filled from available inventory. They find that this type of policy performs better than maintaining separate stocks for each demand class and that the threshold policy is close to optimal, especially when there is a significant mix of high- and low-priority customers.

To the best of our knowledge, there is no paper that studies stock rationing decisions when advance demand information is available. Therefore, in this paper we consider a system that has both features. We study the structure of the manufacturer’s optimal production as well as the optimal rationing decision in a make-to-stock system with two classes of customers, namely the OEM and the SP customers. This paper also provides insights into how the OEM and the SP markets affect the value of advance demand information.

The rest of the paper is structured as follows. In Section 3, we introduce our problem and we show how it can be formulated as a Markov Decision Process. We also characterize the optimal production and the optimal stock rationing policies in that section. In Section 4, we provide some insights into how the value of information changes with various system parameters such as the variability in OEM’s order size, the manufacturer’s capacity, and the relative size of the OEM and the SP market.

3. PROBLEM DESCRIPTION

Consider a single manufacturer with a production capacity of μ items per unit time who produces a single product according to a produce-to-stock routine. There is a primary customer, the OEM, who needs his orders to be shipped at the end of every cycle of length T. We define an order cycle as the time between two consecutive shipments to the OEM. The manufacturer does not know the OEM’s order size until Tm units of time (Tm < T) after the beginning of an order cycle when she receives advance information about the OEM’s exact order size. However, using the OEM’s order history the manufacturer knows the probability distribution Pr(q), which is the probability that the OEM’s order size is q in a given order cycle. If at the shipment time the manufacturer does not have enough inventory to satisfy the full order of the OEM, she incurs a shortfall penalty cost C for every unsatisfied item. This shortfall penalty cost can be considered to represent overtime cost, subcontracting cost, or rush delivery costs incurred to fill the OEM order beyond normal operations.

Demands from SP customers for a single item arrive according to a Poisson process with rate λ per unit time. If the demand of the SP customer is not immediately satisfied, the manufacturer receives a profit of R. We assume that the gross profit gained by serving the SP customer is less than the shortfall penalty for one unit of the OEM’s order (i.e., R < C). If R ≥ C, the service policy will be trivial: one would always serve the SP customer to take the guaranteed profit now and accept the risk of taking a penalty later. However, in practice, the penalty for not filling the OEM order is very high so R < C is reasonable.¹ Note that the standard profit (profit × quantity) for the OEM customer is not included in the model, since it is independent of the manufacturer’s production and service policy. This base profit level is due to the fact that the manufacturer will eventually satisfy all the OEM’s demand either through normal operations or through using special measures (which results in a loss of profit of C per unit of shortfall from the base profit level).

Let α be the time discount factor, xα(τ) be the manufacturer’s inventory level at time τ, Nα(τ) be the accumulated

¹ See Demir Barlas, 2004, Part-Tracking Smarts in The E-Business Executive Daily. It is noted there that penalties for late delivery charges are up to $5,000 a minute in situations involving OEM production line shutdowns.
OEM shortfalls up to time $\tau$, and $M_\pi(\tau)$ be the accumulated SP sales up to time $\tau$ under policy $\pi$. The manufacturer’s objective is to find an optimal control policy $\pi^*$ to maximize the discounted total profit over an infinite horizon:

$$\max_\pi V^\pi = E\left[\sum_{\tau=0}^{\infty} e^{-\alpha \tau} [h_x(\tau) - C \delta_t] d\tau + \int_0^\infty e^{-\alpha \tau} C dN_\pi(\tau) + \int_0^\infty e^{-\alpha \tau} R dM_\pi(\tau)\right]$$  \hspace{1cm} (1)$$

or the average profit over an infinite horizon:

$$\max_\pi V^\pi_a = \lim_{\Gamma \to \infty} \frac{1}{\Gamma} E\left[\sum_{\tau=0}^{\Gamma} -h_x(\tau) d\tau - CN_\pi(\Gamma) + RM_\pi(\Gamma)\right]$$ \hspace{1cm} (2)$$

In (1), $V^\pi$ is the expected discounted profit function under policy $\pi$ starting from time 0. In the rest of the paper we will focus on the discount-profit problem. However, we will show that the theoretical results in the discounted-profit model also apply to the average-profit problem.

### 3.1. Markov Decision Process Formulation

To analyze the manufacturer’s optimal policy, we assume that the manufacturer’s production time follows an exponential distribution with average time of $1/\mu$. This allows us to utilize the Markov Decision Process (MDP) to characterize the structure of the optimal production and stock rationing policies. When the structure of the optimal policies is revealed, it becomes intuitively clear that our assumption regarding exponentially distributed production times is not a restrictive assumption, and the characteristics of the optimal policies also hold for systems with non-exponential production times.

Before we present our MDP model, we need to discretize the order cycle of length $T$ into $T$ equal and non-overlapping infinitesimal periods of length $\delta t$, where $\delta t \to 0$. The time discretization approach has been used in the literature to model a situation where a decision can be made at any point of time (see Gallego and van Ryzin [5], Bitran and Mondschein [1], Maglaras and Meissner [15] as examples of such time discretization in the field of dynamic pricing).

Since SP interarrival times and production times are exponentially distributed, then the probability of one production completion in an interval of length $\delta t$ is $\mu \delta t + o(\delta t)$, and the probability of one SP customer arrival in an interval of length $\delta t$ is $\lambda \delta t + o(\delta t)$. On the other hand, the probability of two production completions, two SP arrivals, or a production completion and a SP arrival in an interval of length $\delta t$ are, respectively, $\mu^2 (\delta t)^2 + o((\delta t)^2)$, $\lambda^2 (\delta t)^2 + o((\delta t)^2)$, and $\mu \lambda (\delta t)^2 + o((\delta t)^2)$, where $o(x)$ is such that $[o(x)/x] \to 0$ as $x \to 0$ (see Ross [18]). We choose $\delta t$ to be small enough so that the probability of more than one events during an interval of length $\delta t$ is almost zero.

Figure 1 shows how the order cycle is discretized. As Figure 1 shows, the entire order cycle is uniformly discretized into periods of length $\delta t$. There are four distinct time intervals during one order cycle. First is the time interval from period 1 to period $T_m - 1$, which is from the beginning of the order cycle until just before time $T_m$. During this interval, the order quantity of the OEM customer is not known. Second is $T_m$ in which the information regarding the OEM order quantity is received. Third is the time interval from period $T_m + 1$ until period $T - 1$, which is the time from when order information arrives until just before the OEM’s order is shipped. Finally, there is period $T$, which corresponds to the last period of length $\delta t$ before time $T$. At the end of period $T$, the OEM order is shipped.

We now formulate our problem as an MDP model in which

- **decision epochs** are the beginning of each of the time interval of length $\delta t$;
- **state of the system** is the triplet $(x, t, z)$, where $x$ represents the manufacturer’s on-hand inventory, $x \in Z^+$ and $Z^+$ is the set of non-negative integer numbers; $t$ is the number of periods of length $\delta t$ since the last delivery was made to the OEM customer ($t = 1, 2, \ldots, T$); and $z$ is the quantity of the OEM’s order for that cycle. For periods $1 \leq t \leq T_m$

![Figure 1. Information arrival and order delivery times in one order cycle.](image-url)
where the OEM’s order size is not known, we use a dot “·” in place of $z$. For $t > T_m$, we use the actual OEM’s order size $q$ in place of $z$. Finally, we let $\Omega$ be the state space.

- **action** with respect to production process is to **produce** one more item or keep the machine **idle**. On the other hand, with respect to the arriving order of the SP customers, the actions are to **serve** (satisfy) the order (from available inventory) or to **reject** the order.

We follow the same approach as in Lippman [13] and use the concept of “potential” production completion. Under the action “produce,” when a potential production completes in an interval, the inventory is increased by one. However, under action “idle,” when a potential production completes, the inventory remains the same.

We consider the following sequence of events within a given period (i.e., interval of length $\delta t$). The manufacturer chooses her policy at the beginning of each period. An arrival of a SP customer, a completion of a production, and the arrival of the advance order information all occur at the end of a period. Deliveries of product to the OEM customer are made at the end of period $T$. If a product is completed in a period and a SP demand arrives in the same period, under action “serve,” the newly completed product can be used to serve the SP demand. Similarly, a product that is completed during period $T$ can be used to fill the order of the OEM customer.

In the following, we discuss the optimality equation of the MDP model under the discounted-profit criterion for four phases of an order cycle by focusing on what happens over a small interval of time $\delta t$. With slight abuse of notation, in the following we will use $h$, $\mu$, $\lambda$, and $\alpha$ in place of $h\delta t$, $\mu\delta t$, $\lambda\delta t$, and $\alpha\delta t$.

\[
V(x,t,\cdot) = -hx + e^{-\alpha} \left[ \mu H_\mu V(x,t,\cdot) + \lambda H_\lambda V(x,t,\cdot) \right]
\]

\[
+ (1 - \mu - \lambda) H_0 V(x,t,\cdot), \quad \text{for } t = 1, 2, \ldots, T_m - 1
\]

\[
V(x,T_m,\cdot) = -hx + e^{-\alpha} \sum q P_r(q) \left[ \mu H_\mu V(x,T_m,q) \right]
\]

\[
+ \lambda H_\lambda V(x,T_m,q) + (1 - \mu - \lambda) H_0 V(x,T_m,q)
\]

\[
V(x,t,q) = -hx + e^{-\alpha} \left[ \mu H_\mu V(x,t,q) + \lambda H_\lambda V(x,t,q) \right]
\]

\[
+ (1 - \mu - \lambda) H_0 V(x,t,q), \quad \text{for } t = T_m + 1, \ldots, T - 1
\]

\[
V(x,T,q) = -hx + e^{-\alpha} \left[ \mu H_\mu^T V(x,T,q) + \lambda H_\lambda^T V(x,T,q) \right]
\]

\[
+ (1 - \mu - \lambda) H_0^T V(x,T,q),
\]

where $H_\mu$, $H_\lambda$, and $H_0$ are operators for $t = 1, \ldots, T - 1$, and they are defined as

\[
H_\mu V(x,t,z) = \max \left\{ V(x+1,t+1,z), V(x,t+1,z) \right\}
\]

\[
H_\lambda V(x,t,z) = \left\{ \begin{array}{ll}
\max \{ R + V(x-1,t+1,z), V(x,t+1,z) \} & \text{if } x > 0 \\
V(x,t+1,z) & \text{if } x = 0
\end{array} \right.
\]

\[
H_0 V(x,t,z) = V(x,t+1,z).
\]

where $H_\mu$ corresponds to the production decision, and $H_\lambda$ is associated with the admission decision for an arriving SP order. When a SP order arrives in a period, if the product is available, the manufacturer chooses either to satisfy or to reject the order. $H_0$ is associated with the case where there is no production completion or SP order arrival in the period.

In the last period of each order cycle, operators $H_\mu^T$, $H_\lambda^T$, and $H_0^T$ are defined as

\[
H_\mu^T V(x,T,q) = \max \left\{ -C[q-x-1] + V([x+1-q]^+,1,\cdot), -C[q-x] + V([x-q]^+,1,\cdot) \right\}
\]

\[
H_\lambda^T V(x,T,q) = \left\{ \begin{array}{ll}
\max \{ R - C[q-x-1] + V([x+1-q]^+,1,\cdot), V([x-q]^+,1,\cdot) \} & \text{if } x > 0 \\
- C[q-x] + V([x-q]^+,1,\cdot) & \text{if } x = 0
\end{array} \right.
\]

\[
H_0^T V(x,T,q) = -C[q-x] + V([x-q]^+,1,\cdot).
\]

These operators are similar to those in (3)-(5), except that they also include the shipment of the OEM order at the end of period $T$. If the inventory level is higher than $q$, the OEM order will be fully satisfied; otherwise, the manufacturer will fulfill the orders with available inventory and pay the shortage penalty for the rest.

The optimality equations under the average-profit criterion in the four phases of an order cycle are

\[
g + V(x,t,\cdot) = -hx + \mu H_\mu V(x,t,\cdot) + \lambda H_\lambda V(x,t,\cdot)
\]

\[
+ (1 - \mu - \lambda) H_0 V(x,t,\cdot), \quad \text{for } t = 1, 2, \ldots, T_m - 1
\]

\[
g + V(x,T_m,\cdot) = -hx + \sum q P_r(q) \left[ \mu H_\mu V(x,T_m,q) \right]
\]

\[
+ \lambda H_\lambda V(x,T_m,q) + (1 - \mu - \lambda) H_0 V(x,T_m,q)
\]

\[
g + V(x,t,q) = -hx + \mu H_\mu V(x,t,q) + \lambda H_\lambda V(x,t,q)
\]

\[
+ (1 - \mu - \lambda) H_0 V(x,t,q), \quad \text{for } t = T_m + 1, \ldots, T - 1
\]

\[
g + V(x,T,q) = -hx + \mu H_\mu^T V(x,T,q) + \lambda H_\lambda^T V(x,T,q)
\]

\[
+ (1 - \mu - \lambda) H_0^T V(x,T,q),
\]

\[
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\]

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\]
where \( g \) represents \( gst \), the optimal average cost per period. Note that both the average- and the discounted-profit model use the same operator functions.

### 3.2. Characteristics of the Optimal Policy

We investigate the characteristics of the optimal policy following the approach in [8]. We first define a set of optimality conditions and decision rules and then show that profit function \( V(x,t,z) \) satisfies the conditions in both average-profit and discounted-profit cases.

For any function \( f \), let \( \Delta_x f(x,t,z) = f(x+1,t,z) - f(x,t,z) \). We define the set of functions as \( \mathcal{F} \) such that if \( f(x,t,z) \in \mathcal{F} \) then,

- **upper bound**: \( \text{F1: } \Delta_x f(x,t,z) \leq C \), for \( x \geq 0 \).
- **concavity**: \( \text{F2: } \Delta_x f(x,t,z) \) is non-increasing in \( x \).
- **threshold levels**: \( \text{F3: } \Delta_x f(x,t+1,z) \geq 0 \) for \( x < P(t,z) \), where \( P(t,z) = \min \{ x | \Delta_x f(x,t+1,z) < 0 \} \).

\[
\text{F4: } \Delta_x f(x-1,t+1,z) \leq R \text{ for } x > Q(t,z), \text{ where } Q(t,z) = \max \{ x | \Delta_x f(x-1,t+1,z) > R \}.
\]

- **supermodularity**: \( \text{F5: } \) For any time \( T_m < t \leq T \), \( \Delta_x f(x,t,q) \) is non-decreasing in \( q \).
- **recurrence**: \( \text{F6: } \) For any time \( T_m < t \leq T \) and integer \( u \), \( \Delta_x f(x,t,q) = \Delta_x f(x+u,t,q+u) \).

To have some intuition on the above conditions, we apply the condition set \( \mathcal{F} \) to our profit function \( V(x,t,z) \). Condition \( \text{F1} \) implies that the marginal benefit of producing one more item is less than or equal to the shortfall penalty for an OEM demand. Condition \( \text{F2} \) implies that the marginal benefit of increasing inventory level, \( x \), is non-increasing in \( x \). Condition \( \text{F3} \) implies that producing is better than idling when \( x < P(t,z) \). Condition \( \text{F4} \) implies that accepting an SP order is better than rejecting if the inventory \( x > Q(t,z) \). Condition \( \text{F5} \) implies that the marginal benefit of inventory increases in \( q \), when the OEM’s order quantity is known. Condition \( \text{F6} \) implies that the marginal benefit of inventory does not change if the inventory level and the OEM’s order quantity increase with the same amount.

In the following, we show that the structure of functions in \( \mathcal{F} \) is preserved under the operators defined in (3)–(8). The proof of Lemma 1 is presented in the Appendix to save space.

**LEMMA 1:** If \( f(x,t,z) \in \mathcal{F} \), then \( H_\mu f, H_\lambda f, H_0 f, H_\lambda^T f, H_\mu^T f, H_0^T f \in \mathcal{F} \).

By (6) and (7), we can conclude the following proposition. The proof is presented in the Appendix.

**PROPOSITION 1:** At Period \( T \), if the current inventory level \( x \) is less than \( q \), then the optimal production policy is to produce; if the current inventory level \( x \) is less than or equal to \( q \), then the optimal service policy is to reject SP demand.

Before we analyze the structure of the optimal policy, in Figure 2 we present an example of the typical structure of the optimal policy. In this example, the OEM demand in a cycle (i.e., \( q \)) is uniformly distributed between 1 and 8. Since the OEM’s order quantity can be of any value between 1 and 8, the complete graph of the optimal policy consists of 8 different sub-graphs, one for each potential value of \( q \). Figure 2 shows the optimal policy only for \( q = 2, 5 \), and 8. As these figures depict, the optimal policy generally has the following properties:

- The optimal policy can be divided into two parts: the optimal policy for the time periods that the order size \( q \) is not known (i.e., \( 1 \leq t \leq T_m \)) and the optimal
policy for the time periods that the order size \( q \) is known (i.e., \( T_m + 1 \leq t \leq T \)).

- In the former phase, where \( 1 \leq t \leq T_m \), the optimal policy depends on the distribution of \( q \), not its ultimate value. Therefore, in each order cycle, the optimal production and service policies before the information arrival will be the same, regardless of the actual value of \( q \) in that cycle. So in Figure 2, the optimal policies are the same in the three subgraphs for \( 1 \leq t \leq T_m \).

- There are two thresholds, namely the production threshold and the service (i.e., stock reservation) threshold. In Figure 2 the production threshold is indicated by a dashed line while the service threshold is indicated by a solid line. At any time \( t \), the production threshold is in fact a base-stock level that specifies when the manufacturer should stop producing, while the service threshold specifies an optimal stock-reservation level where an order of a SP customer should be rejected if the inventory is below that level.

For profit function \( V(x,t,z) \) in both the average-profit and the discounted-profit models, consider functions \( P(t,z) \) and \( Q(t,z) \) as follows:

\[
P(t,z) = \min\{x | x \in \mathbb{Z}^+, \Delta_x V(x,t+1,z) < 0\} \quad (9)
\]

\[
Q(t,z) = \max\{x | x \in \mathbb{Z}^+, \Delta_x V(x-1,t+1,z) > R\}. \quad (10)
\]

Theorem 1 characterizes the structure of the optimal production policy and the optimal service policy. We will show that \( P(t,z) \) constitutes a time-dependent base-stock level, and \( Q(t,z) \) is the time-dependent stock reservation level for both average-profit and discounted-profit models. The proof is presented in the Appendix.

**THEOREM 1**: For both discounted- and average-profit models, the optimal policy is characterized by switching curves \( P(t,z) \) and \( Q(t,z) \) defined in (9) and (10) as follows:

1. For any given \( t \) and \( z \), there exists a base-stock level \( P(t,z) \) such that for \( x \geq P(t,z) \), the optimal production policy is to idle; otherwise, when \( x < P(t,z) \), the optimal production policy is to produce.
2. For any given \( t \) and \( z \), there exists a stock-reservation level \( Q(t,z) \) such that for \( x \leq Q(t,z) \), the optimal service policy is to reject arriving SP orders; otherwise, when \( x > Q(t,z) \), the optimal service policy is to serve arriving SP orders.
3. Both base-stock level \( P(t,q) \) and stock reservation level \( Q(t,q) \) are non-decreasing in \( q \) for \( T_m < t \leq T \).
4. At any period \( T_m < t \leq T \) and any integer \( u \), the optimal policies at states \( (x,t,q) \) and \( (x+u,t,q+u) \) are the same.

Note that Part 4 of Theorem 1 implies

\[
P(t,q + u) = u + P(t,q) : t = T_m, T_{m+1}, \ldots, T.
\]

\[
Q(t,q + u) = u + Q(t,q) : t = T_m, T_{m+1}, \ldots, T.
\]

In other words, the structure of the optimal base-stock and stock-reservation levels (i.e., the trend in the thresholds \( P \) and \( Q \)) remains the same in periods \( T_m, T_{m+1}, \ldots, T \). The only change is that those thresholds move upward as the OEM’s order \( q \) increases.

**3.3. Relationship between Thresholds \( P(t,z) \) and \( Q(t,z) \)**

We show that under the optimal conditions, the base-stock level \( P(t,z) \) cannot be less than the reservation level \( Q(t,z) \).

Please refer to the Appendix for the proof.

**THEOREM 2**: In both the discounted-profit and average-profit cases, \( P(t,z) \geq Q(t,z) \) for any \( t \) and \( z \).

Generally, the production threshold is higher than the reservation threshold, \( P(t,z) > Q(t,z) \). However, there are cases where the two thresholds are equal, i.e., \( P(t,z) = Q(t,z) \). These cases often occur when \( R \) is relatively small. Please refer to the Appendix for a detailed discussion.

Figure 2 is an example of the general cases where \( P(t,z) \neq Q(t,z) \). As Figure 2 shows, in general, at any time \( t \) as the inventory \( x \) increases, the optimal policy changes from (produce/reject) to (produce/serve) to (idle/serve). In this case, action (idle/reject) is never optimal. When \( P(t,z) = Q(t,z) \), however, the optimal policy changes from (produce/reject) to (idle/reject) to (idle/serve) as \( x \) increases. Figure 3 presents one of those cases.

According to Theorem 1, the optimal production policy is to idle for \( x \geq P(t,z) \), and the optimal service policy...
is to reject when \( x \leq Q(t, z) \). Therefore, if it happens that \( x = P(t, z) = Q(t, z) \), then the optimal policy is (idle/reject) at state \((x, t, z)\).

COROLLARY 1: If we have \( x = P(t, z) = Q(t, z) \), then the optimal policy is (idle/reject) at state \((x, t, z)\). For any state \((x', t, z)\) where \( x' > x \) the optimal policy is (idle/serve), and for any state \((x'', t, z)\) where \( x'' < x \), the optimal policy is (produce/reject).

Corollary 1 implies that, at any given time period, (produce/reject) and (idle/serve) will never be found optimal for different inventory levels, which is also shown in Figure 2.

COROLLARY 2: At any time \( t \), if there is a state \((x, t, z)\) where the optimal policy is (idle/reject), then there is no state \((y, t, z)\) where the optimal policy is (produce/serve). Conversely, if there is a state \((x, t, z)\) where the optimal policy is (produce/serve), then there is no state \((y, t, z)\) where the optimal policy is (idle/reject).

4. NUMERICAL ANALYSIS

Since there is no closed-form analytical solution for the optimal threshold levels and the optimal profit, it is difficult to study the potential value of the advance order information analytically. Therefore, in this section we use an extensive numerical study to evaluate the value of information and to analyze the effects of system parameters on the value of information. Specifically, we investigate the following questions:

1. What is the potential value of the advance order information? Under what circumstances does the order information have (or does not have) a significant value?
2. How does the manufacturer’s capacity (relative to the market size) affect the value of advance order information?
3. Does information have more value in systems with secondary customers or without them?
4. How does the value of information change when the manufacturer increases her capacity in response to an increase in the secondary market?
5. How do the relative value and the relative size of the OEM and SP markets affect the value of advance order information?

In our numerical study we use our average-profit model in order to isolate the effects of main parameters such as manufacturing capacity, demand, or customer profits and costs from the effects of time discount factor \( \alpha \). We will measure the value of information in terms of the following metric:

\[
\Delta(T_m) = \frac{\text{Profit}(T_m) - \text{Profit}(T)}{\text{Profit}(T)} \times 100\%,
\]

where \( \text{Profit}(T_m) \) is the manufacturer’s expected profit per unit time when she receives advance information at time \( T_m \). The relative saving \( \Delta(T_m) \) shows how much receiving information at time \( T_m \) can increase the manufacturer’s profit compared to when an OEM does not provide advance order information (i.e., \( T_m = T \)). This metric gives an indication of how beneficial advance demand information can be for the manufacturer.

Our numerical study includes a total of 2880 scenarios generated over a wide range of parameters. For details of how these scenarios were generated, see Appendix B. We considered an order cycle of length \( T = 4 \), which is discretized into \( T \geq 4000 \) periods of length \( \delta t \leq 0.001 \). For each of the 2880 scenarios, we considered five different times for order information to arrival, namely \( T_m = 0, 1, 2, 3 \) and 4. This corresponds to \( T_m/T = 0, 0.25, 0.5, 0.75, \) and 1.

4.1. Value of Information

After evaluating 14400 (i.e., \( 2880 \times 5 \)) cases in our numerical study, we observed that \( \Delta(T_m) \), the manufacturer’s saving due to advance order information, has an average of 5.4% and can be as high as 29%. We observed that the value of information increases as the ratio of \( \Delta(T_m)/T \) decreases. In general, the maximum benefit corresponds to the cases where the manufacturer’s capacity is low, the secondary customer is less valuable (i.e., \( R \) is much smaller than \( C \)), the variability of the order information is high.

In the following sections, we further describe how system parameters such as the manufacturer’s capacity, SP and OEM market size and profit affect the value of information. In each section we use one or two representative examples (among 2880 scenarios of our numerical study) to describe the impact of a particular system parameters on the value of information. The representative example better highlights the general trend that we observed in our study of 2880 cases.

4.2. Effects of Capacity and Demand

It has been shown in the literature on information sharing and advance demand information (e.g., [6, 10, 21]) that the value of information increases as the manufacturer’s capacity increases. In our problem, however, we observed a contrasting behavior depending on the relative size of the OEM and SP markets. We will use the ratio \( \lambda_O = E[q]/T \), which is
Figure 4. The effect of the manufacturer’s capacity on the value of information (i.e., \( \Pi(T_m) \)). Left: OEM market is twice as large as the SP market. Right: The SP market is twice as large as the OEM market.

the OEM’s average demand per unit time, as a measure of the OEM’s market size. Similarly, we will use \( \lambda \) as the measure of the SP market size. In the following two cases, \( R = 9 \), \( C = 18 \), \( h = 1 \), and the OEM’s order size is uniformly distributed in the range \((1,15)\) (therefore \( \lambda_O = 2 \)). We change the manufacturer’s capacity so that the ratio of capacity over total demand \( (\mu/([\lambda_O + \lambda])) \) ranges from 0.33 to 2.

CASE I: We examine how the value of information changes as manufacturer’s capacity changes when the OEM market is twice as large as the SP market. The example in Figure 4 (left) shows that if the OEM market is significantly larger than the SP market; regardless of the timing of the information, the value of information is higher where the capacity is higher (i.e., for higher values of \( \mu/([\lambda_O + \lambda]) \)). The reason is that, when the OEM market is large, the potential for a large shortfall penalty cost (due to the larger demand of OEM) is high. Therefore, manufacturers with small capacity need to produce continuously and reject most of the SP customers to keep up with the large relative demand of the OEM, especially in systems with large shortfall cost. This means that information regarding the OEM’s order size does not have a significant effect on the production and service policies, which implies that the information does not have much value. However, as the capacity increases the manufacturer does not need to produce all the time and can use information to decide whether to serve a SP customer or not. Consequently, she can use the information about the OEM’s order size to reduce her inventory cost and reduce the potential for a shortage by making better production and service decisions.

CASE II: We examine how the value of information changes as manufacturer’s capacity changes when the OEM market is half as large as the SP market. When the SP market is significantly larger than the OEM market, as Figure 4 (right) shows, the value of information decreases as manufacturer’s capacity increases.\(^2\) When the SP market is large, a large portion of the manufacturer’s profit comes from the SP customers. Therefore, the decision to accept or to reject a SP customer becomes a significant factor in the manufacturer’s profit, especially for manufacturers with smaller production capacity. Manufacturers with small production capacity can use advance information to strike a balance between the number of served SP customers and the inventory reserved for the OEM. High-capacity manufacturers, on the other hand, do not need information as much, since they have the capacity to serve all the SP customers and can respond to the OEM demand with a lower risk of suffering a penalty cost for not filling the OEM order.

We note that when \( \mu/\lambda_O \rightarrow 0 \) (a very unrealistic case), regardless of the size of the SP and the OEM markets, the value of information is close to zero. This is because the manufacturer’s optimal policy is to always produce and reject the SP customers regardless of the OEM’s order information.

4.3. Effects of SP and OEM Markets

In this section we study the following questions: Does information have more value in systems with secondary customers or without them? How does the value of information change when the manufacturer increases her capacity in response to an increase in the secondary market? In order to address these questions we compare the following two cases.

CASE I: We first examine the value of advance order information as we increase the secondary (SP) market demand

\(^2\) Note that this does not mean that the manufacturer’s total profit decreases.
while keeping the manufacturer’s capacity constant and the OEM demand constant. This case represents situations where the secondary market demand increases; however, the manufacturer does not have the ability to increase her capacity. Figure 5 (left) shows the results for one of our experiments in which we increase the SP demand from \( \lambda = 0 \) to \( \lambda = 8 \). In Figure 5 \( \mu = 4, R = 9, C = 18, h = 1 \), and the OEM’s order size is uniformly distributed in the range (3, 13).

As Figure 5 (left) shows, the value of information decreases as the size of the SP market increases. Note that as the SP market increases, the ratio \( \mu / (\lambda_O + \lambda) \) decreases. When \( \lambda \) is between 0 and 2, the SP market is smaller than the OEM market. Thus, an increase in the SP market results in a decrease in ratio \( \mu / (\lambda_O + \lambda) \), which in turn leads to a decrease in the value of information (consistent with the behavior of the system shown in Figure 4 (left). However, as \( \lambda \) increases beyond 2, the SP market becomes larger than the OEM market. Although the ratio \( \mu / (\lambda_O + \lambda) \) decreases, in this case the value of information remains the same. The reason is that in Figure 5 (left) while the ratio \( \mu / (\lambda_O + \lambda) \) decreases, the capacity \( \mu \) and ratio \( \mu / \lambda_O \) remain constant. Thus, the manufacturer uses information to minimize her shortfall penalty cost of the OEM customers, as well as to increase her profit by accepting more SP demand. However, the ability of the manufacturer to satisfy SP demands is limited by her capacity. Therefore, as \( \lambda \) increases, the manufacturer’s potential profit (even with perfect information) reaches a limit, as the manufacturer finds himself following a policy of producing constantly. This results in the value of information \( \Delta(T_m) \) approaching a limit (see Figure 5, left).

CASE II: Case II is designed to study how the simultaneous increase in the SP market and manufacturer’s capacity affects the value of information of the OEM. Consider a case where a manufacturer has a capacity of \( \mu = 2 \) and only serves her primary customer with market of size \( \lambda_O = 2 \). Also, assume that (due to marketing activities such as advertising) there is a new (secondary) market with size \( \lambda = 1 \). Consequently, the manufacturer increases her capacity to respond to the increase in demand (i.e., the new capacity is \( \mu = 3 \)). We represent this case by \( (\lambda = 1, \mu = 3) \).

Figure 5 (right) shows how the value of information changes as the SP market grows and the manufacturer’s capacity is increased accordingly. Figure 5 includes the following cases: \( (\lambda = 1, \mu = 3), (\lambda = 2, \mu = 4), (\lambda = 4, \mu = 6), (\lambda = 6, \mu = 8), \) and \( (\lambda = 8, \mu = 10) \). In Figure 5 \( R = 9, C = 18, h = 1 \), and the OEM’s order size is uniformly distributed in the range (1, 15). As Figure 5 depicts, the value of information does not increase monotonically. In fact, it first increases and then decreases. Note that in all cases the ratio of capacity over total demand is constant (i.e., \( \mu / (\lambda + \lambda_O) \) = 1); therefore, the effect of capacity is neutralized.

The reason that the value of information is higher at \( \lambda = 1 \) compared to \( \lambda = 0 \) is that in both cases the size of the SP market is less than the size of the OEM market. Therefore, in the case where \( \lambda = 1 \) the information is critical for the decision to accept or reject an arriving SP demand, since it affects the inventory that should be reserved for the OEM customer. On the other hand, when SP demands are added, information can be used at every SP customer arrival epoch to determine whether to accept or to reject the SP customer. In other words, there are more points in time that the information will be useful. This is the main reason why information is more valuable at \( \lambda = 1 \) compared to \( \lambda = 0 \).

In contrast, when the SP market becomes larger than the OEM market (i.e., the case with \( \lambda = 4, \mu = 6 \)), then
the SP market becomes a larger source of profit than the OEM, and the optimal policy focuses on serving a larger portion of SP customers, especially when the OEM’s penalty cost is not very large. As the production policy is geared more toward serving the larger SP market, advance order information has less effect on the decision of serving or rejecting a SP customer. Thus, the relative value of information declines as the SP market grows. For example, in an extreme case where the SP market is 1000 times larger than the OEM market, and the OEM’s shortfall penalty cost is not very large, then the optimal policy becomes close to a base-stock policy, where the optimal base-stock level is not significantly affected by the OEM’s advance demand information. In other words, the information loses its value since it is less important for making production and service decisions.

4.4. Effects of Customer Profit Ratio

In our numerical study we use the ratio $R/C$ to represent the relative value of SP and OEM customers. In this section, we examine the effects of this ratio on the value of information. As our representative example, we consider a case in which $\lambda = 4$, $\mu = 10$, $C = 18$, $h = 3$, and $T = 4$, while the OEM’s order size follows a $\text{Uniform}(1, 15)$ distribution with an average of $E[q] = 8$. We study the value of information under the SP profit levels of $R = 3, 9, \text{ and } 15$, which results in the ratio $R/C$ taking values of $1/6, 1/2, \text{ and } 5/6$. The result is shown in Figure 6.

By studying several cases similar to that in Figure 6 we found that, regardless of the manufacturer’s capacity, as $R$ approaches $C$, the value of information declines. The reason is that, as $R$ approaches $C$, the service policy becomes trivial: serve the SP customer to gain the profit now and accept the risk of losing the same amount later.

5. CONCLUSION

In this paper we study a manufacturer who produces to stock and serves two types of customers: an OEM customer who orders periodically but his order size varies and SP customers who order an individual item, but arrive randomly. The manufacturer receives advance information regarding the OEM’s order some time before its due date. The manufacturer is obligated to satisfy the entire OEM’s order, but can reject a SP customer order. With the objective of maximizing the manufacturer’s profit, we formulated the problem as a Markov Decision Process and we found that the manufacturer’s optimal production and service (stock rationing) policies for both discounted-profit and average profit models are threshold-type policies. We also showed that these thresholds are monotone with respect to the OEM’s order size.

We also performed a numerical study in order to study the value of information under different scenarios. We observed the following:

- We found exception to the reports in the existing literature that the value of information always increases with the manufacturer’s capacity. We find that, if the manufacturer also serves a secondary market, the value of advance order information can sometimes decrease as the manufacturer’s capacity increases. We explained that this happens when the SP market is significantly larger than the OEM market.
- If the manufacturer does not invest in increasing her capacity as the SP market grows, the value of the OEM’s information decreases and then remains constant. This suggests that, if the manufacturer attempts to increase the SP market without investing in her capacity, her profit becomes less dependent on the OEM’s demand information. However, there is a limit on the size of the SP market that can be served without increasing the manufacturer’s capacity. If the SP market grows beyond this point, the manufacturer’s profit and the value of OEM information will not be affected, since the manufacturer will not be able to take advantage of the additional profit, nor will she be able to take additional advantage of the advance order information.
- If the manufacturer increases her capacity to correspond to an increase in the size of the SP market, the value of the OEM’s order information increases and then decreases as the SP market becomes large compared to the OEM market. This implies that having a larger secondary market and having additional capacity to serve that market does not always reduces the manufacturer’s dependence on


![Figure 6. The effects of the relative values of SP and OEM customers on the benefit of advance demand information.](image-url)
the OEM’s information. In fact, an increase in secondary market may increase the value of the OEM’s information.

One avenue of future research would be to develop easy-to-implement but cost-effective policies to replace the complicated structure optimal production and stock-reservation policies. Another line of research would be to study similar cases but in a centralized supply chain or in multi-echelon supply chains with advance demand information and secondary customers. In those cases it would also be useful to determine how the advance information effects other stages of the supply chain when there are two markets with different characteristics. One can also study cases where the manufacturer receives two types of information in an order cycle. First she receives a signal that specifies what the relative demand in this order cycle is, for example, high or low. At a later point in the cycle she receives information about the exact order size of the OEM. All these studies will provide valuable insights regarding the effects of a secondary market in managing supply chains.

APPENDIX A

Proofs of Analytical Results

Proof of Lemma 1

Using induction, we prove that each condition in $F$ is preserved by $H_0$, $H_1$, $H_2$, $H_3$, and $H_4$.

Period $T$. Operator $H_T^3$, $i = \mu, \lambda, 0$, can be seen as a composite operator $H_T^3 = H_T^i$, where $H_T f(x, t, q) = -C(q - x)^+ + f(x, q -1, 1, \cdot)$, $C = 1$. Thus, in order to prove that $H_T^3, H_T^4, H_T^5$ preserve the optimality conditions, we need to show: (i) operator $H_T^3$ preserves the optimality conditions and (ii) operators $H_T^1, H_T^2, H_T^3$ preserve the optimality conditions. Here we prove that operator $H_T^3$ preserves the optimality conditions. The proof that operators $H_T^1, H_T^2,$ and $H_T^3$ preserve the optimality conditions in period $T$ is the same as that in the interval from period $T - 1$ to period $T_n + 1$, which we will discuss in the next part of the proof.

For Operator $H_T^3$ we have

$$\Delta_x H_T^3 f(x, t, q) = H_T^3 f(x, t, q) - H_T^3 f(x, T, q)$$

$$= \begin{cases} -C(q - x)^+ + f(x, q - 1, 1, \cdot) = C, & \text{if } x < q \\ f(x + 1 - q, 1, \cdot) - f(x + q - 1, 1, \cdot) = \Delta_x f(x - q, 1, 1, \cdot), & \text{if } x \geq q. \end{cases}$$

(11)

By $F_1$, $\Delta_x f(x - q, 1, \cdot) \leq C$, and by $F_2$, $\Delta_x f(x - q, 1, \cdot)$ is non-increasing in $x$. Therefore, $\Delta_x H_T^3 f(x, t, q) \leq C$ and $\Delta_x H_T^3 f(x, T, q)$ is non-increasing in $x$, and Conditions $F_1$ and $F_2$ are preserved by $H_T^3$ at period $T$.

Since the operator function $H_T^3$ preserves the concavity condition $F_2$, $\Delta_x H_T^3 f(x, T, q)$ increases as $x$ decreases. As $x$ falls smaller than the threshold level $P(t, q)$, $\Delta_x H_T^3 f(x, T, q)$ becomes larger than 0, and therefore $H_T^3$ preserves Condition $F_3$. Similarly, we can show that $H_T^3$ also preserves Condition $F_4$.

To show that operator $H_T^3$ preserves Condition $F_5$, we have

$$\Delta_x H_T^3 f(x, T, q + 1) - \Delta_x H_T^3 f(x, T, q)$$

$$= \begin{cases} 0, & \text{if } x < q \\ C - \Delta_x f(x - q, 1, \cdot), & \text{if } x = q \\ \Delta_x f(x - q - 1, 1, \cdot) - \Delta_x f(x - q, 1, \cdot), & \text{if } x > q. \end{cases}$$

By $F_1$, $C - \Delta_x f(x - q, 1, \cdot) \geq 0$, and by $F_2$, $\Delta_x f(x - q - 1, 1, \cdot) - \Delta_x f(x - q, 1, \cdot) \geq 0$. Overall, $\Delta_x H_T^3 f(x, T, q + 1) - \Delta_x H_T^3 f(x, T, q) \geq 0$, so $\Delta_x H_T^3 f(x, T, q)$ is non-decreasing in $q$ at period $T$. Therefore, Condition $F_5$ is preserved by $H_T^3$ at period $T$.

It is easy to show $\Delta_x H_T^3 f(x + u, T, q + u) = \Delta_x H_T^3 f(x, T, q)$ with (11). Therefore, Condition $F_6$ is preserved by $H_T^3$ at period $T$.

Periods $T - 1$ to $T_n + 1$. We first derive three differentiation equations that will be used later. For $H_T^3 f(x, t, z)$ we have

$$\Delta_x H_T^3 f(x, t, z) = H_T^3 f(x, t, z) - \Delta_x f(x, t, z)$$

$$= f(x + 1, t + 1, z) - f(x - 1, t + 1, z) = \Delta_x f(x, t + 1, z).$$

(12)

For $H_T^4 f(x, t, z)$ we have

$$\Delta_x H_T^4 f(x, t, z) = H_T^4 f(x, t, z) - \Delta_x f(x, t, z)$$

$$= \max(f(x + 2, t + 1, z), f(x + 1, t + 1, z)) \cdot - \max(f(x + 1, t + 1, z), f(x + 1, t, z)).$$

We consider three cases: (i) $x + 1 < P(t, z)$, (ii) $x + 1 = P(t, z)$, and (iii) $x + 1 > P(t, z)$.

In case (i), since $x + 1 < P(t, z)$, then according to Condition $F_3$, $f(x + 2, t + 1, z) - f(x + 1, t + 1, z) = \Delta_x f(x + 1, t + 1, z) \geq 0$, so $\max(f(x + 2, t + 1, z), f(x + 1, t + 1, z)) = f(x + 2, t + 1, z)$. On the other hand, since $x + 1 < P(t, z)$, then $x + 1 < P(t, z)$, and thus according to Condition $F_3$, $\max(f(x + 1, t + 1, z), f(x, t + 1, z)) = f(x + 1, t + 1, z)$. Hence, for case (i) we have

$$\Delta_x H_T^4 f(x, t, z) = f(x + 2, t + 1, z) - f(x + 1, t + 1, z) = \Delta_x f(x + 1, t + 1, z).$$

In case (ii), since $x + 1 = P(t, z)$, then according to Condition $F_3$, we have $\max(f(x + 2, t + 1, z), f(x + 1, t + 1, z)) = f(x + 1, t + 1, z)$. On the other hand, since $x + 1 = P(t, z)$, then $x + 1 < P(t, z)$, and thus according to Condition $F_3$, $\max(f(x + 1, t + 1, z), f(x, t + 1, z)) = f(x + 1, t + 1, z)$. Thus, for case (ii) we have

$$\Delta_x H_T^4 f(x, t, z) = f(x + 1, t + 1, z) - f(x + 1, t + 1, z) = 0.$$

In case (iii), $x + 1 > P(t, z)$, and therefore, $x \geq P(t, z)$, and we will have

$$\Delta_x H_T^4 f(x, t, z) = f(x + 1, t + 1, z) - f(x + 1, t + 1, z) = \Delta_x f(x, t + 1, z).$$

Thus, in summary,

$$\Delta_x H_T^4 f(x, t, z) = \begin{cases} \Delta_x f(x, t + 1, z), & \text{if } x + 1 < P(t, z) \\ 0, & \text{if } x + 1 = P(t, z) \\ \Delta_x f(x, t + 1, z), & \text{if } x + 1 > P(t, z). \end{cases}$$

(13)
Using a similar approach with Condition F4, one can find that

\[
\Delta_i H_i f(x,t,z) = \begin{cases} 
 f(x + 1,t + 1,z) - f(x,t + 1,z) & \text{if } x < Q(t,z) \\
 f(x,t + 1,z) - f(x,t + 1,z) & \text{if } x = Q(t,z) \\
 f(x + 1,t + 1,z) - f(x - 1,t + 1,z) & \text{if } x > Q(t,z).
\end{cases}
\]

**Proof for Condition F1.** By F1, \(\Delta_i f(x,t + 1,z) \leq C\), and since \(R \leq C\), we have \(\Delta_i H_i f(x,t,z) \leq C\), and \(\Delta_i H_i f(x,t,z) \leq C\), and \(\Delta_i H_i f(x,t,z) \leq C\).

**Proof for Condition F2.** We must prove that \(\Delta_i H_i f(x,t,z)\), \(\Delta_i H_i f(x,t,z)\), and \(\Delta_i H_i f(x,t,z)\) are non-increasing in \(x\).

- **Proof for \(\Delta_i H_i f(x,t,z)\):** Please refer to (12) for \(\Delta_i H_i f(x,t,z)\).
  By F2, \(\Delta_i f(x,t + 1,z)\) is non-increasing in \(x\), so \(\Delta_i H_i f(x,t,z)\) is non-increasing in \(x\).

- **Proof for \(\Delta_i H_i f(x,t,z)\):** Please refer to (13) for \(\Delta_i H_i f(x,t,z)\).
  By F2, \(\Delta_i f(x,t + 1,z)\) is non-increasing in \(x\), so \(\Delta_i H_i f(x,t,z)\) is non-increasing in \(x\) within each of the three sub-condition intervals listed in (13). In the following, we will show that \(\Delta_i H_i f(x,t,z)\) is non-increasing in \(x\) across any two adjacent intervals.
  To show \(\Delta_i H_i f(x,t,z)\) is non-increasing in \(x\) across the intervals, let us denote \(x^0 = \bar{P}(t,z) - 1\). By the results above, we have

\[
\Delta_i H_i f(x^0 - 1,t,z) = \Delta_i f(x^0,t + 1,z),
\]

\[
\Delta_i H_i f(x^0,t,z) = 0,
\]

\[
\Delta_i H_i f(x^0 + 1,t,z) = \Delta_i f(x^0 + 1,t + 1,z).
\]

By F3, we have \(\Delta_i f(x^0 + 1,t + 1,z) > 0 \geq \Delta_i f(x^0 + 1,t + 1,z)\). So \(\Delta_i H_i f(x,t,z)\) is non-increasing in \(x\) across the adjacent intervals at \(x^0\).

**Proof for Condition F3 and F4.** The proof is similar to that for Period \(T\).

**Proof for Condition F5.** By the results of (12), (13), (14), using the approach to prove F2, it is easy to show \(\Delta_i H_i f(x,t,q + 1) \geq \Delta_i H_i f(x,t,q)\), \(\Delta_i H_i f(x,t,q + 1) \geq \Delta_i H_i f(x,t,q)\), and \(\Delta_i H_i f(x,t,q + 1) \geq \Delta_i H_i f(x,t,q)\).

**Proof for Condition F6.** By F6, we have \(\Delta_i f(x + u,t + 1,q + u) = \Delta_i f(x,t + 1,q), P(t,q + u) = P(t,q) + u, and Q(t,q + u) = Q(t,q) + u\).

With the results of (12), (13), (14), it is easy to show \(\Delta_i H_i f(x + u,t,q + u) = \Delta_i H_i f(x,t+1,q)\), \(\Delta_i H_i f(x + u,t,q + u) = \Delta_i H_i f(x,t,q)\), and \(\Delta_i H_i f(x + u,t,q + u) = \Delta_i H_i f(x,t,q)\).

**Proof for Proposition 1**

We compare the profits under actions production and idling at period \(T\). When \(x < q\), we compare the two terms in Eq. (6):

\[
- C[q-x-1]^+ + V(\{(x+1-q)^+,1,-\}) - [ - C[q-x]^+ + V(\{(x-q)^+,1,-\}) ]
\]

\[
= - C(q-x-1) + V(0,1,-) + C(q-x) - V(0,1,-) = C \geq 0.
\]

Therefore, at period \(T\) it is optimal to produce if inventory level is less than \(q\).

To study the optimal service policy at period \(T\) when \(x \leq q\), we compare the two terms in Eq. (7):

\[
- C[q-x]^+ + V(\{(x-q)^+,1,-\}) - [ - C[q-x+1]^+ + V(\{(x-1-q)^+,1,-\}) ]
\]

\[
= - C(q-x) + V(0,1,-) - R + C(q-x+1) - V(0,1,-) = C - R \geq 0,
\]

which implies that, at Period \(T\), it is optimal to reject SP demand if the inventory level is less than or equal to \(q\).

**Proof of Theorem 1**

We first prove the existence of an optimal stationary policy under the discounted-profit criterion by following the approaches in [8]. For this purpose, we need to show: (i) the set of structured functions \(F\) is complete and (ii) \(V(x,t,z) \in F\).

By Lemma 1, the structural properties from Condition F1 to F6 are all preserved by the operators. Because the limit of any converging sequences of functions in \(F\) will be in \(F\) as well, the set of structured functions \(F\) is complete. On the other hand, since function \([q-x]^-\) \( \in F\), Lemma 1 implies that \(V(x,t,z) \in F\). So the optimal expected profit function \(V\) is structured and satisfies all conditions in \(F\). Hence, the existence of an optimal stationary policy under the discounted-profit criterion follows from the corollaries of Theorem 5.1 of [17].

The existence and convergence of the optimal profit functions and the optimal stationary policies under the average-profit criterion are more complicated since there are infinite number of states. Moreover, the cost in a period can be unbounded from above when \(x \to \infty\). We will show that our MDP model satisfies the three conditions (SEN 1), (SEN2), (SEF) in [19] so that the average-profit exists by letting \(\alpha \to 0^+\).

Note that our MDP model is unichain because all the states lead to a single recurrent class for any deterministic stationary policy. Let \(\pi_0\) be a policy in which the manufacturer never produces. It is not hard to show that \(\pi_0\) induces a positive recurrent class and \(\pi_0\) is a standard policy defined in [19] with a distinguished set of states \((0,t,z)\). (It is a distinguished single state \(0\) if we do not include the time and order quantity, which have no impact on policy \(\pi_0\).) By Proposition 7.5.3 and 7.2.4 in [19], (SEN1) and (SEN2) hold for every state in \(\Omega\).

Let \(b\) be the upper bound of an OEM order in one cycle, then we can show that the one-cycle expected profit function is bounded, i.e., \(-C(\tilde{q} - x) - xhT \leq V_{\pi_0}(x,t,z) \leq Rx\). As a result, the total discounted-profit under the
optimal policy starting at state \((x, t, z)\) is bounded, i.e., \[ \frac{1}{1-\alpha} (-Cq - x) - xhT) \leq V_{a}(x, t, z) \leq \frac{1}{1-\alpha} Rx. \]

Let \(M\) be the optimal total discounted-profit associated with the distinguished set of states \((0, t, z)\); then we get \[ \frac{1}{1-\alpha} (-C(q - x) - xhT) - M \leq V_{a}(x, t, z) - V_{a}(0, t, z) \leq \frac{1}{1-\alpha} Rx - M, \]

for all \((x, t, z) \in \Omega\). Thus, (SEN3) holds and the average profit exists by Theorem 7.2.3 in [19]. By the same theorem, any limit point of the optimal stationary policies under total discount-profit criterion is average-profit optimal. The limit point can be obtained by appropriately choosing a sequence \(\alpha \to 0\).

**Part 1.** From Eq. (3) it is clear that the optimal production control only depends on the sign of \(\Delta_{c} V(x, t, z)\) to produce if the sign is positive and to idle otherwise. By Conditions F2 and F3, it is optimal to produce if \(x < P(t, z)\).

**Part 2.** From Eq. (4), the service control for an arriving SP order depends on the sign of \(\Delta_{s} V(x - 1, t, z)\); accept the demand if the sign is positive, reject otherwise. So by Conditions F2 and F4, the optimal service policy is to accept an arriving SP order if \(x > Q(t, z)\) and to reject otherwise.

**Part 3.** By Conditions F5, at any time \(T_{m} < t \leq T\), the switching curves \(P(t, q)\) and \(Q(t, q)\) are non-decreasing in \(q\) for \(T_{m} < t \leq T - 1\).

**Part 4.** By Conditions F6, at any time \(T_{0} < t \leq T\) and integer \(u\), the optimal policy at states \((x, t, q)\) and \((x + u, t, q + u)\) is the same.

**Proof of Theorem 2**

By contradiction, assume \(Q(t, z) > P(t, z)\), which is equivalent to \(Q(t, z) \geq P(t, z) + 1\). By Condition F4, \(\Delta_{c} V(P(t, z) + 1 - 1, t + 1, z) > R\). Noting that \(\Delta_{c} V(P(t, z), t + 1, z) < 0\) by Condition F3, it contradicts with the assumption of \(R \geq 0\).

**Discussion on the Special Case of \(P(t, z) = Q(t, z)\)**

In Figure A1, the solid curve represents \(\Delta_{c} V(x, t + 1, z)\), and the dotted curve represents \(\Delta_{s} V(x - 1, t + 1, z)\), shifting the solid curve to the right by one unit. Note that the horizontal axis represents the inventory level at period \(t\), while the vertical axis corresponds to the value of \(\Delta_{c} V(x, t, z)\).

Let us denote \(x_{0}(t, z) = \arg\{\Delta_{c} V(x, t + 1, z) = 0\}\) and \(x_{g}(t, z) = \arg\{\Delta_{c} V(x - 1, t + 1, z) = R\}\). Then by Conditions F3 and F4, we have \(P(t, z) = (x_{0}(t, z))\) and \(Q(t, z) = (x_{g}(t, z))\). As shown in Figure A1 (left), when \(R\) is relatively small, it may happen that \(\{x_{0}(t, z)\} < \{x_{g}(t, z)\}\) for some value of \(t\) and \(z\), and as a result, we will have \(P(t, z) \neq Q(t, z)\). The optimal policy at this inventory level is \((idle, Reject)\).

In Figure A1 (right), when \(R\) is relatively large, we will have \(Q(t, z) < P(t, z)\), and the optimal policy at inventory level \(x\), \(Q(t, z) < x < P(t, z)\), is \((produce, serve)\).

**APPENDIX B**

**Design of Experiments for the Numerical Study**

Due to the large number of states in our MDP model (since \(T\) is a very large number, i.e., \(T = 4000\)) we had to truncate the maximum inventory levels \(x\) and the order size \(q\) in our numerical study. For each case of our numerical study we truncated \(x\) at a level that does not significantly affect the profit if it is increased by any amount.

We considered uniform distributions for the OEM’s order size, since it provides a finite number of values for \(q\). Specifically, we considered cases where \(q\) follows uniform distribution with parameters \([7.9\), \((.5, 11), (3.13), \((1.15). We also considered Bernoulli distribution in which the \(q\) can take the values of \(q_{1}\) or \(q_{2}\), with probability \(p\) and \(1 - p\), respectively.

Our numerical study includes Bernoulli distributions with \((q_{1}, q_{2}, p) \in \{(1, 0.7, 1), (2.2, 0.8)\}\). Our Uniform and Bernoulli distributions result in an average order size \(E[q] = 8\) for the OEM, with coefficient of variation ranging from 0.0625 to 1.5.

For other parameters of the problem we considered \(h \in \{1, 3\}, \lambda \in \{0, 1, 2, 4, 6, 8\}, \mu \in \{2, 3, 4, 6, 8, 10, 12, 16\}, R \in \{3, 6, 9, 12, 15\}, \) and \(C = 18\). In our set of experiments, the ratio \(\mu/\lambda + E[q]/T\) (which is an indication of the manufacturer’s capacity relative to demand) takes values in \([0.33, 0.67, 1.33, 2.26, 6.7]\). The ratio \(T/E[q]\) (which is an indication of the relative size of the SP market compared to the OEM demand) has values of \([0, 0.5, 1, 2, 3, 4]\). The ratio of \(R/C\), which represents the relative value of the SP and OEM customers, has values of \([1/6, 2/6, 3/6, 4/6, 5/6]\). The ratio of \(h/C\) in our numerical study has values of 1/18 and 1/6, which represents the cases of low and high inventory costs compared to the shortage cost.

**REFERENCES**


