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An Extension of the Internal Rate of Return to Stochastic Cash Flows

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The internal rate of return (IRR) is a venerable technique for evaluating deterministic cash flow streams. Part of the difficulty in extending this measure to stochastic cash flows is the lack of coherent methods for accounting for multiple or nonexistent internal rates of return in deterministic streams. Recently such a coherent theory has been developed, and we examine its implications for stochastic cash flows. We devise an extension of the deterministic IRR, which we call the stochastic *rate of return on mean investment*. It has significant computational and conceptual advantages over the stochastic internal rate. For instance, in the deterministic case, the standard result is that under proper conditions a cash flow stream is acceptable (in the sense of positive present value) if its internal rate exceeds the interest rate. We show that a stochastic cash flow stream is acceptable (in the sense of positive certainty equivalent expected value) if the rate of return on mean investment has a suitably defined certainty equivalent exceeding the risk-free interest rate.

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1. Introduction

The internal rate of return (IRR) is a widely used tool for evaluating deterministic cash flow streams, familiar to all students of finance and engineering economics (e.g., Brealey and Myers 1996). When used appropriately, it can be a valuable aid in project acceptance and selection. However, the method is subject to well-known difficulties: a cash flow stream can have multiple conflicting internal rates (both above and below the hurdle rate), or no real-valued internal rate at all and can appear to be inconsistent with net present value calculations. As Rothkopf (1965) and others note, it does not extend well to situations involving uncertainty because different realizations of an uncertain cash flow stream can have different numbers of internal rates or no real-valued internal rate at all. This fact makes calculation of the distribution of the internal rate very difficult (although there are approximations for special cases—see Fairley and Jacoby 1975). But whether one even needs that distribution is an open question—even under risk neutrality, there is no theoretical guidance as to whether one should use, for example, the mean internal rate versus the internal rate of the mean cash flow.

Recently, however, I have devised an approach that circumvents the problem of multiple or nonexisting internal rates (Hazen 2003) for deterministic cash flow streams, while retaining consistency with net present value. Here I explore an extension of this approach to stochastic cash flow streams. The result is a random rate-of-return measure, here termed the *return on mean investment*, that is consistent with expected utility of net present value. Even though it reduces to the standard internal rate for deterministic streams, this measure differs from the internal rate for stochastic streams. Nevertheless, it has important conceptual and computational advantages over the internal rate.

We consider here only cash flow streams whose payoffs cannot be hedged by market investments. The latter is treated, for example, in Smith and Nau (1995). The point of view here is the standard one—that net present value, or its expected utility, is the proper way to evaluate nonhedgeable cash flow streams.[1] However, the IRR has enduring intuitive appeal and may be a useful supplement to net present value when personal preference or institutional custom dictate. Indeed, a survey of 392 CFOs by Graham and Harvey (2001) found that IRR is employed in capital budgeting with at least as great a frequency as net present value.

In the next section, we review results from Hazen (2003), which are an essential basis for what follows.

[1] See Smith (1998) for a superior approach that resolves temporal risk issues that will not be addressed here.

In §3, we introduce the main results and present examples. Section 4 summarizes.

2. Deterministic Cash Flow Streams

The net present value $PV(\mathbf{x} \mid r)$ of a cash flow stream $\mathbf{x} = (x_0, x_1, \ldots, x_T)$ at interest rate r is given by

$$PV(\mathbf{x} \mid r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t}.$$

An IRR k for $\mathbf{x}$ is any value of r that makes net present value equal to zero. As is well known, for conventional cash flows $\mathbf{x}$ that are negative for the first few periods but positive thereafter, a unique proper ($k > -1$) IRR exists. Moreover, this internal rate is the largest interest rate r at which the cash flow shows a discounted net profit. So if k exceeds the available market rate of interest r, then $PV(\mathbf{x} \mid r) > 0$ and the investment that generates the cash flow $\mathbf{x}$ is worthwhile. This is the fundamental justification for the use of IRR. As is well known, this justification applies only to the accept/reject decision for a single proposed cash flow stream and not to the problem of comparing two candidate cash flows.

The results of this paper build on a *two-period intuition for rate of return*, which is as follows: If one invests an amount I now and receives return $(1+k)I$ one period hence—so k is the rate of return on I—then the net present value at interest rate r is

$$PV = -I + \frac{1+k}{1+r} \cdot I. \tag{1}$$

We say that *PV is the net of investing I at rate k*. The following result, derivable from Hazen (2003), provides an alternative interpretation of IRR as a rate of return on an underlying investment stream.

Theorem 1. *Suppose $\mathbf{x} = (x_0, \ldots, x_T)$ is a cash flow stream, and k is any real number. Then the following are equivalent*:

(a) *k is an IRR for $\mathbf{x}$.*

(b) *There is a unique stream $\mathbf{c} = (c_0, \ldots, c_{T-1})$ such that for any interest rate r, $\mathbf{x}$ in present value is the net of investing $\mathbf{c}$ at rate k; that is,*

$$PV(\mathbf{x} \mid r) = -PV(\mathbf{c} \mid r) + \frac{1+k}{1+r} PV(\mathbf{c} \mid r). \tag{2}$$

We call the stream $\mathbf{c}$ above the *investment stream* due to the intuitive two-period interpretation (1) of (2). Hazen (2003) shows that $\mathbf{c}$ is given by

$$c_t = -\big((1+k)^t x_0 + (1+k)^{t-1} x_1 + \cdots + x_t\big) \qquad t = 0, \ldots, T-1. \tag{3}$$

Rewriting (2) as $PV(\mathbf{x} \mid r) = ((k-r)/(1+r))PV(\mathbf{c} \mid r)$ yields the following result.

Theorem 2 (Hazen 2003). *Suppose k is an IRR on cash flow stream $\mathbf{x} = (x_0, \ldots, x_T)$, and $\mathbf{c} = (c_0, \ldots, c_{T-1})$ is the corresponding investment stream* (3) *guaranteed by* Theorem 1. *Suppose $PV(\mathbf{c} \mid r) > (<)\, 0$. Then*

$$PV(\mathbf{x} \mid r) \ge 0 \quad \text{if and only if } k \ge (\le)\, r. \tag{4}$$

What may escape initial notice is that Theorem 2 holds for *any* of the potentially multiple internal rates of return k for $\mathbf{x}$. The acceptance rules $k \ge (\le)\, r$ in (4) will never conflict for different internal rates k because they are all equivalent to the present value rule $PV(\mathbf{x} \mid r) \ge 0$. Therefore, it does not matter which internal rate k one uses as long as one examines the corresponding investment stream $\mathbf{c}$ and compares k to the market rate r in a manner consistent with the classification of $\mathbf{c}$ as having positive or negative present value—a so-called *net investment* or a *net borrowing* (see Hazen 2003). This result resolves the long-standing question of how to deal with multiple internal rates.

These results do not alter the warning against the common practice of using the IRR *by itself* to compare competing projects. Nevertheless, if one includes both internal rate *and* the underlying investment stream in the discussion, then Theorem 1 can assist in such comparisons. The following example illustrates.

Example 1. Consider the following two cash flow streams (in thousands of dollars).

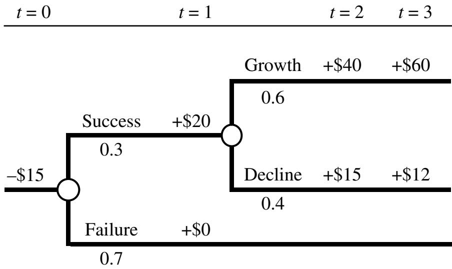
t	0	1	2	3
$\mathbf{x}$	−15	6	9	12.2
$\mathbf{y}$	−5	6	2	2

Take the interest rate r to be 5%. The stream $\mathbf{x}$ has present value \$9.42 and unique real internal rate 32.1%, whereas the stream $\mathbf{y}$ has present value \$4.26 and unique internal rate 60.5%. The stream $\mathbf{x}$ is superior in present value terms, but $\mathbf{y}$ has a higher IRR.

We know we should defer to the present value ranking, and this may in fact be justified by considering the underlying investment streams. Using (2), the underlying investment stream $\mathbf{c}_x$ for $\mathbf{x}$ has present value $PV(\mathbf{c}_x \mid r) = \$36.56$, whereas the underlying investment stream $\mathbf{c}_y$ for $\mathbf{y}$ has present value $PV(\mathbf{c}_y \mid r) = \$8.06$. So according to Theorem 1(b), $\mathbf{y}$ in present value is the net of investing \$8.06 at rate 60.5%, whereas $\mathbf{x}$ in present value is the net of investing \$36.56—over four times as much—at lower rate 32.1%. Investing a larger amount at a smaller rate of return can in fact net a higher profit, which is exactly what happens in this case. This logic provides an intuitive explanation for the conflict between IRR and present value as well as a rationale for accepting the present value recommendation.

Note that we did not have to actually calculate $\mathbf{c}_x$ or $\mathbf{c}_y$ to arrive at these conclusions, although we could have, using (3). The results are $\mathbf{c}_x = (15, 13.8, 9.24)$, $\mathbf{c}_y = (5, 2.02, 1.25)$.

Figure 1 **A Simple Investment with Uncertain Resulting Cash Flow $\tilde{\mathbf{x}}$**



	Prob	$\tilde{\mathbf{x}}$ 0	1	2	3	PV	$\tilde{k}$
	0.18	–$15	+$20	+$40	+$60	$92.2	270%
	0.12	–$15	+$20	+$15	+$12	$28.0	85%
	0.7	–$15	+$0	+$0	+$0	–$15.0	–38%
Certainty equivalents =						+$2.47	12.1%
Means =		–$15	+$6	+$9	+$12.2	$9.45	32.1%
c =		+$15.0	+$13.8	+$9.26		+$36.56	

Notes. The mean cash flow $\bar{\mathbf{x}} = (-\$15, \$6, \$9, \$12.2)$ has internal rate $\bar{k} = 32.1\%$ and underlying mean investment $PV(\mathbf{c} \mid r) = \$36.56$. The uncertain rates $\tilde{k} = 270\%, 85\%, -38\%$ on the mean investment \$36.56 are calculated to yield the present values \$92.2, \$28.0, −\$15.0. The underlying investment stream **c** may be calculated but is not needed to determine these quantities. The risk-free rate (not shown) is $r = 5\%$.

3. Stochastic Cash Flow Streams

One may in principle define an IRR of a stochastic cash flow stream $\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_T)$ as a random rate $\tilde{k}$ for which $PV(\tilde{\mathbf{x}} \mid \tilde{k}) = 0$ with probability one. Here we propose an alternative approach that avoids the computational and conceptual difficulties attached to this stochastic internal rate. We exploit the characterization from the previous section of an internal rate as a return on an underlying investment stream. The idea is to choose a deterministic underlying investment stream and force all uncertainty in $\tilde{\mathbf{x}}$ onto its rate of return.

Given a stochastic cash flow stream $\tilde{\mathbf{x}}$, let $\bar{k}$ be an IRR of the mean $\bar{\mathbf{x}} = E[\tilde{\mathbf{x}}]$, and let **c** be the corresponding investment stream, so that $\bar{\mathbf{x}}$ is the net of investing the stream **c** at rate $\bar{k}$, as in Theorem 1; that is,

$$PV(\bar{\mathbf{x}} \mid r) = -PV(\mathbf{c} \mid r) + \frac{1+\bar{k}}{1+r} PV(\mathbf{c} \mid r). \tag{5}$$

We assume $PV(\mathbf{c} \mid r) \neq 0$. It will not matter which internal rate $\bar{k}$ one uses if there are multiple internal rates, and we assume at least one such rate exists.[2] Now define $\tilde{k}$ to be the random rate of return on $PV(\mathbf{c} \mid r)$ that nets $PV(\tilde{\mathbf{x}} \mid r)$; that is,

$$PV(\tilde{\mathbf{x}} \mid r) = -PV(\mathbf{c} \mid r) + \frac{1+\tilde{k}}{1+r} PV(\mathbf{c} \mid r), \tag{6}$$

or, solving for $\tilde{k}$,

$$\tilde{k} = r + (1+r)\frac{PV(\tilde{\mathbf{x}} \mid r)}{PV(\mathbf{c} \mid r)}. \tag{7}$$

Comparing (5) with the expected value of (6), we see that $\tilde{k}$ has mean $\bar{k}$. Therefore, when $\tilde{\mathbf{x}}$ is deterministic, $\tilde{k}$ will be an IRR for $\tilde{\mathbf{x}}$. In general, however, this will not be the case; that is, the condition $PV(\tilde{x} \mid \tilde{k}) = 0$ will usually fail. To give it a distinctive name, call $\tilde{k}$ the *rate of return on the mean investment*, reflecting the definition (6). It appears one needs to know the mean investment $PV(\mathbf{c} \mid r)$ to find $\tilde{k}$, but in fact one may use (5) to obtain

$$\tilde{k} = r + (\bar{k} - r)\frac{PV(\tilde{\mathbf{x}} \mid r)}{PV(\bar{\mathbf{x}} \mid r)}, \tag{8}$$

so that one may obtain $\tilde{k}$ directly from $\tilde{\mathbf{x}}$, $\bar{\mathbf{x}}$, and $\bar{k}$. The following example illustrates.

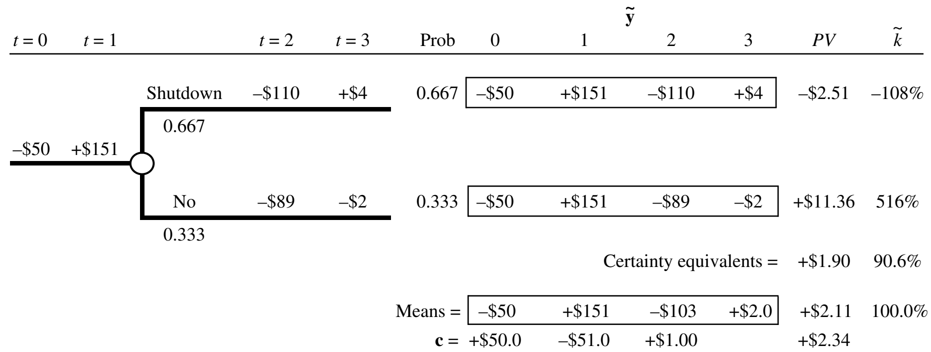
Example 2. Consider the simple investment depicted in the event tree of Figure 1, in which an initial investment of \$15 thousand can be followed by success or failure, and potential growth or decline if success occurs. Probabilities are indicated on the branches of the event tree. Three potential cash flow streams $\tilde{\mathbf{x}}$ can result, with corresponding probabilities 0.18, 0.12, 0.70 and present values \$92.2, \$28, −\$15 at rate $r = 5\%$. The mean cash flow stream $\bar{\mathbf{x}}$ is $(-\$15, \$6, \$9, \$12.2)$, which has present value \$9.45 and unique proper internal rate $\bar{k} = 32.1\%$. From this information, we can obtain the mean investment $PV(\mathbf{c} \mid r) = \$36.56$ using (5), and the rate of return $\tilde{k}$ on the mean investment using (8), equal to rates 270%, 85%, −38% with respective probabilities 0.18, 0.12, 0.70, as shown in Figure 1. Although one does not need to find the underlying mean investment stream **c**, it is also shown in Figure 1. The present values $PV(\tilde{\mathbf{x}} \mid r) = \$92.2, \$28.0, -\$15.0$ are the nets of investing the mean $PV(\mathbf{c} \mid r) = \$36.56$ at uncertain rates $\tilde{k} = 270\%, 85\%, -38\%$. The reader may verify, however, that $\tilde{k}$ is not an IRR; that is, $PV(\tilde{\mathbf{x}} \mid \tilde{k}) \neq 0$ for any realization of $\tilde{\mathbf{x}}, \tilde{k}$.

3.1. Preference over Stochastic Cash Flow Streams

The key feature of the rate of return $\tilde{k}$ on mean investment is that the certainty equivalent (CE), suitably

[2] If no real-valued internal rate exists, one can proceed with complex-valued $\bar{k}$ as in Hazen (2003). The results of this subsection extend as in Hazen (2003) but at the cost of complex-valued $\tilde{k}$ and **c**.

Figure 2 A Second Stochastic Cash Flow Stream $\tilde{\mathbf{y}}$ to Be Compared with $\tilde{\mathbf{x}}$ from Figure 1



Note. $\tilde{\mathbf{y}}$ has 90.6% certainty equivalent rate of return on mean investment \$2.34.

defined, of $\tilde{k}$ may be used to determine the desirability of the original cash flow stream $\tilde{\mathbf{x}}$ in a manner that is consistent with the expected utility of net present value of $\tilde{\mathbf{x}}$. In other words, Theorem 2 for deterministic streams may be extended to stochastic cash flow streams. This may be done as follows.

Given a utility function u over time-0 payoffs, consider the intuitive two-period scenario mentioned above in which a fixed amount I is invested at $t=0$ at a rate of return k to yield an amount $(1+k)I$ at time $t=1$, with net present value given by (1). The utility function u over time-0 payoffs PV therefore induces, via (1) for each I, a utility function u_I over rate of return k at a fixed investment level I:

$$u_I(k) = u\left(-I + \frac{1+k}{1+r} I\right). \qquad (9)$$

We denote certainty equivalent under u by CE and certainty equivalent under u_I by CE_I. They are related by

$$CE\left[-I + \frac{1+\tilde{k}}{1+r} I\right] = -I + \frac{1+CE_I[\tilde{k}]}{1+r} I \qquad (10)$$

(see the appendix for a derivation). The key result is as follows.

Theorem 3. *Let $\tilde{\mathbf{x}}$ be a stochastic cash flow stream whose mean $\bar{\mathbf{x}}$ is the net of investment stream $\mathbf{c}$ at IRR $\bar{k}$. If $\tilde{k}$ is the corresponding return on mean investment, then the following hold:*

(a) *$CE_I[\tilde{k}]$ is the return on mean investment $PV(\mathbf{c} \mid r)$ that nets $CE[PV(\tilde{\mathbf{x}} \mid r)]$; that is, with $I = PV(\mathbf{c} \mid r)$,*

$$CE[PV(\tilde{\mathbf{x}} \mid r)] = -PV(\mathbf{c} \mid r) + \frac{1+CE_I[\tilde{k}]}{1+r} PV(\mathbf{c} \mid r).$$

(b) *If $PV(\mathbf{c} \mid r) > (<)\, 0$, then with $I = PV(\mathbf{c} \mid r)$,*

$$CE[PV(\tilde{\mathbf{x}} \mid r)] \geq 0 \;\Leftrightarrow\; CE_I[\tilde{k}] \geq (\leq)\, r.$$

Example 2 (Continued). Suppose utility over time-0 payoffs x is exponential with risk tolerance $\rho = \$100$; that is, $u(x) = -\exp(-x/\rho)$. Then one may verify that $CE[PV(\tilde{\mathbf{x}} \mid r)] = \$2.47$ and, using (10), that $CE_I[\tilde{k}] = 12.1\%$. By construction, then, Theorem 3(a) holds; that is, \$2.47 is the net of the mean investment $PV(\mathbf{c} \mid r) = \$36.56$ at rate of return 12.1%. Moreover, because $CE_I[\tilde{k}]$ exceeds the risk-free rate $r = 5\%$, and the mean investment $PV(\mathbf{c} \mid r) = \$36.56$ is positive, the cash flow stream $\tilde{\mathbf{x}}$ is desirable. This is consistent with its positive certainty equivalent present value \$2.47, as must be the case by Theorem 3(b). It turns out that $\tilde{\mathbf{x}}$ remains desirable for risk tolerances ρ above \$67.0.

As in the deterministic case, conflicts between present value and rate of return may arise when uncertain cash flow streams are compared. Just as in the deterministic case (Example 1), the results here provide an intuitive rate-of-return rationale for resolving these conflicts in favor of present value. The following example illustrates.

Example 3. Suppose we wish to compare the cash flow stream $\tilde{\mathbf{x}}$ of Example 2 with a new stream $\tilde{\mathbf{y}}$ given by the scenario in Figure 2. Calculations are shown there under the same exponential utility function with risk tolerance $\rho = \$100$. First, note that $\tilde{\mathbf{y}}$ is desirable because its CE rate of return—90.6% on its mean investment \$2.34 > 0—exceeds the risk-free rate $r = 5\%$. This is consistent with its positive CE present value \$1.90 > 0.

Compared to $\tilde{\mathbf{x}}$, the stochastic stream $\tilde{\mathbf{y}}$ has a smaller CE present value \$1.90 but a higher CE rate of return 90.6%. This is because $\tilde{\mathbf{y}}$ in CE present value is the net of a smaller mean investment \$2.34 at a higher CE rate 90.6%. Investing a smaller amount at a larger rate need not net a better CE present value, and indeed it does not in this case, as $\tilde{\mathbf{y}}$ has a smaller CE present value than $\tilde{\mathbf{x}}$.

4. Summary and Discussion

We have introduced a stochastic *rate of return $\tilde{k}$ on mean investment* for stochastic cash flow streams $\tilde{\mathbf{x}}$, an extension of the IRR for deterministic streams.

This rate of return has several desirable properties:

- It is *deterministically consistent*: For deterministic cash flow streams $\tilde{\mathbf{x}} = \bar{\mathbf{x}}$, the return $\tilde{k}$ on mean investment is equal to the IRR for $\bar{\mathbf{x}}$. In general, $\tilde{k}$ will have mean equal to the internal rate of $E[\tilde{\mathbf{x}}]$.
- The certainty equivalent, suitably defined, of $\tilde{k}$ can be used to determine the desirability of $\tilde{\mathbf{x}}$ in a manner that is consistent with utility of net present value. This extends the results of Hazen (2003) for the deterministic case.
- Even though there may be multiple rates of return $\tilde{k}$ on mean investment, they are all consistent with each other in evaluating the acceptability of $\tilde{\mathbf{x}}$, as long as the characterization of the underlying mean investment stream as a net investment or net borrowing is observed.

However, the rate of return $\tilde{k}$ on mean investment is not *internal*, as it may change if the risk-free rate changes. Indeed, $\tilde{k}$ is not an IRR by the usual definition ($PV(\tilde{\mathbf{x}} \mid \tilde{k}) = 0$). However, $\tilde{k}$ has several advantages over the IRR for stochastic cash flows. The first is that there seems to be no way to avoid the problem of multiple or null real values associated with different realizations of a stochastic internal rate, a problem that arises in a usually harmless way for the return on mean investment.

Second, the computational burden for the stochastic internal rate is potentially large, essentially requiring polynomial root finding for each realization of the stochastic stream. It is true that the method of Fairley and Jacoby (1975) can provide an approximate distribution for the stochastic internal rate when it is unique with high probability, but the adequacy of the approximation depends on a high probability of uniqueness, a situation that may fail for important cases. In contrast, the return on mean investment requires the same amount of computation as in the deterministic case, a single root-finding operation to obtain one or more internal rates of the mean cash flow. Once this is done, the distribution of $\tilde{k}$ can be quickly derived from the distribution of the present value of the cash flow stream.

Third, there is a theoretical basis for the use of the return on mean investment, whereas the stochastic internal rate has none. There is no theoretical guidance available for whether one should even be computing the distribution of the stochastic internal rate—as opposed, say, to the internal rate of the mean cash flow stream—and no guidance as to what one should do with this distribution if one can obtain it. For the return on mean investment, there are specific answers: Yes, its distribution is relevant, and it is valid to compute suitably defined certainty equivalents. These can be used to determine the acceptability of a stochastic cash flow stream in a manner consistent with expected utility of net present value, via Theorem 3. Currently there is no corresponding, consistent method for the IRR. Under risk neutrality, the mean return on mean investment is the relevant statistic and is equal to the internal rate of the mean cash flow stream. This answers the question of whether one should compute the mean of the internal rate or the internal rate of the mean cash flow—one should do the latter under risk neutrality.

Theoretical considerations aside, some might view the return on mean investment as a convenient alternative to the "correct" internal rate for stochastic cash flow streams. However, there is no mathematical basis for labeling the standard IRR for stochastic streams as "correct." Both measures reduce to the IRR for deterministic streams.

Just as with the IRR, it is not valid to use return on mean investment in isolation to compare two stochastic cash flow streams. Nevertheless, in conjunction with the underlying mean investment, it can help rationalize net present value conclusions in rate-of-return language, as we have illustrated. In general, the approach is meant as a supplement to the net present value approach when there is a personal or institutional need for a rate-of-return interpretation.

Appendix. Proofs

PROOF OF (10). For strictly increasing u, Equation (10) holds if and only if

$$Eu\left(-I + \frac{1+\tilde{k}}{1+r}I\right) = u\left(-I + \frac{1+CE_I[\tilde{k}]}{1+r}I\right).$$

But by the definition of u_I, the left side is $Eu_I(\tilde{k})$ and the right side is $u_I(CE_I[\tilde{k}])$. The latter two quantities are equal by the definition of certainty equivalent. □

PROOF OF THEOREM 3.

(a) Note that due to the definition (6) of $\tilde{k}$, the quantity $PV = PV(\tilde{\mathbf{x}} \mid r)$ satisfies (1) with $I = PV(\mathbf{c} \mid r)$ and $k = \tilde{k}$. The claim then follows due to the relationship (10) between CE and CE_I.

(b) The result is immediate after rewriting part (a) as

$$CE[PV(\tilde{\mathbf{x}} \mid r)] = \frac{CE_I[\tilde{k}] - r}{1+r} PV(\mathbf{c} \mid r). \quad \square$$

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