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Multiattribute Structure for QALYs

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Health status is inherently a multiattribute construct. We examine multiattribute utility decompositions for the *quality-adjusted life year* (QALY) utility model commonly employed in medical decision and cost-effectiveness analyses. We consider several independence conditions on preference, including the classical notions of preferential independence and utility independence, as well as new related notions of *standard-gamble independence* and *time-tradeoff independence*. The latter conditions are helpful in simplifying standard-gamble utility assessment procedures and time-tradeoff assessment procedures in the presence of multiple health attributes. Under the QALY model, all these conditions are equivalent and result in a purely multiplicative decomposition of utility over health states.

Key words: multiattribute utility; QALYs; medical decision analysis; standard gamble; time tradeoff; multiplicative
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1. Introduction

Methods for evaluating health quality are central to medical decision analyses and cost-effectiveness analyses. The most important such method is the quality-adjusted life year (QALY) model, in which a patient's survival duration is given weight proportional to the quality of health the patient experiences. The recommendation of the Panel on Cost Effectiveness in Health and Medicine (Gold et al. 1996) is that medical cost-effectiveness studies should incorporate morbidity and mortality consequences into a single measure using QALYs. QALYs have indeed become ubiquitous in these and other analyses: A *Medline* search on *Quality-Adjusted Life Years* for the five-year period ending December 2002 produced 1,070 articles. Assumptions under which quality-adjusted lifetime constitutes a von Neuman-Morgenstern utility function over health pathways are given by Pliskin et al. (1980), Miyamoto et al. (1998), and Miyamoto (1999). Under such assumptions, the utility $U(y, t)$ assigned to a duration-t sojourn in health state y is given by

$$U(y, t) = u(y) \cdot m(t). \quad (1)$$

Here $u(y)$ represents the health quality or health utility associated with state y, and $m(t)$ is the overall utility of a duration-t sojourn in a unit-quality health state, with $m(0) = 0$.

In practice, methods for assessing the QALY utility function $U(y, t)$ take $m(t)$ to be determined exogenously (e.g., the linear function $m(t) = t$) and focus on eliciting the health utility $u(y)$ by subjectively querying patients, physicians, or community members. A variety of methods are available for this purpose (e.g., Gold et al. 1996, Hunink et al. 2001). Listed in order of increasing level of cognitive burden for subjects, these include rating scale, time-tradeoff, standard gamble, and multiattribute health indexes such as the Health Utilities Index (HUI, e.g., Feeny et al. 1995) or the EuroQol (e.g., Dolan 1997).

Eliciting a multiattribute health index is too challenging to be done for individual decision or cost-effectiveness analyses. However, analysts may obtain health utilities $u(y)$ from in-depth studies in which utilities are carefully assessed over combinations of attributes representing most morbidities encountered in medical interventions. For example, the HUI Mark III (Feeny et al. 2002) is an eight-attribute system with attributes describing five to six levels each of vision, hearing, speech, ambulation, dexterity, emotion, cognition, and pain. The EuroQol is a five-attribute system with three levels each describing mobility, self-care, usual activity, pain/discomfort, and anxiety/depression. Nevertheless, for some analyses, these attributes are not specific enough to address important issues and analysts must assess health utilities directly.

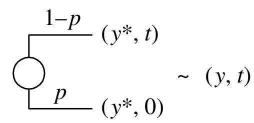
Direct elicitation of health utilities is often hampered by the natural multiattribute structure of health status. A multiattribute description of health status is cognitively complex and can yield more health status combinations than can be feasibly elicited. One expedient is to apply time-tradeoff or standard-gamble techniques along each health attribute separately, and subsequently combine the results in some fashion. The first contribution of this paper is to indicate for what forms of utility function $u(y)$ this single-attribute assessment strategy is valid.

A second common expedient is to *multiply* single-attribute health utilities, however elicited, to obtain overall health utility. For example, Roach et al. (1988) obtain an overall utility of 0.30 for the simultaneous presence of AIDS and metastatic cancer by multiplying the utility 0.50 for AIDS by the utility 0.60 for metastatic cancer. The same strategy is used by Sonnenberg and Pauker (1986), Plante et al. (1987), Fleming et al. (1988), and Eckman et al. (2002). The second contribution of this paper is to indicate what preference assumptions are required to justify this procedure.

This paper is organized as follows. In §2, we discuss single-attribute versus multiattribute elicitation using time-tradeoff and standard-gamble procedures. Basic results involving single-attribute elicitation using standard-gamble and time-tradeoff are given in §3. In §4, we delineate relationships of the assumptions we introduce to other conditions such as preferential independence and utility independence. The appendix contains proofs of these and related results under a general utility structure. The conclusion is given in §5 and discusses limitations, extensions, and related research.

2. Single-Attribute vs. Multiattribute Elicitation

The *standard-gamble procedure* in the medical literature is a specific form of the traditional probability-equivalent approach for utility assessment (e.g., Clemen 1996). In this approach, a subject is asked to specify the largest chance p of immediate death she would be willing to incur to raise her health status from a given state y to full health y^*. In other words, the subject must be indifferent between duration t in health state y, and a gamble yielding immediate death with probability p and duration t in full health y^* with probability $1-p$:



Here the symbol $\sim$ denotes indifference and immediate death is represented by $(y^*, 0)$. Equating expected utilities and using (1) with $m(0)=0$ gives

$$(1-p)u(y^*)m(t) = u(y)m(t)$$

and using $u(y^*)=1$ gives the desired utility $u(y) = 1-p$.

In the *time-tradeoff procedure*, a subject is asked what reduced survival duration she would accept to improve her health from y to y^*. In other words, the subject is asked what duration t' should be so that

$$(y^*, t') \sim (y, t).$$

Under the linear QALY model, equating utilities gives

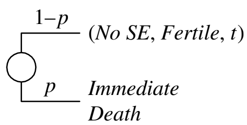
$$u(y^*) \cdot t' = u(y) \cdot t,$$

from which the desired utility $u(y) = t'/t$ can be obtained.

In this paper, we are interested in multiattribute health states $y = (y_1, \ldots, y_n)$. It has long been recognized that directly assessing a utility function over multiattribute outcomes can be a daunting task due to both a subject's potential cognitive overload in thinking about multiattribute health states or to the sheer number of indifference responses required. Consider, for example, a health state y that might be appropriate for an analysis of screening or treatment options for ovarian cancer. Treatment outcomes include side effects due to radiation therapy, and infertility should ovaries be surgically removed. A health-state descriptor incorporating these issues might be $y = (y_1, y_2)$, where y_1 = radiation side-effects (SE) (none, mild, or severe[1]), and y_2 = fertility (fertile or infertile).

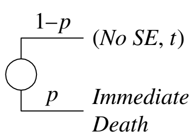
[1] In practice, each level of side effects would be described in detail.

The state $(y_1, y_2) = (No\ SE, Fertile)$ corresponds to full health and would be assigned utility one. There remain five combinations (y_1, y_2) to assess. The standard-gamble approach would require subjects to choose five probabilities p under which the gamble

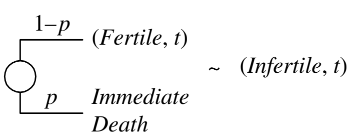


is respectively indifferent to the five health scenarios: (*No SE, Infertile, t*), (*Mild SE, Fertile, t*), (*Mild SE, Infertile, t*), (*Severe SE, Fertile, t*), and (*Severe SE, Infertile, t*). Such assessments would require potentially complex cognitive tradeoffs between level of side effects, fertility, and chance of death.

On the other hand, if preferences for side-effect severity are somehow *independent* of fertility, then one could perform standard-gamble assessments on side-effect severity without reference to fertility. The subject would need to provide two probabilities p under which the gamble



is respectively indifferent to the two health scenarios (*Mild SE, t*) and (*Severe SE, t*). Similarly, if preferences for fertility are somehow independent of side-effect severity, then the subject would need to provide a single probability p yielding the indifference



Here we have only three cognitively simpler indifference responses required, trading off side-effect severity against chance of death or fertility against chance of death. Of course, questions remain. When (if ever) is it valid to proceed in this way? How can one recover overall utilities $u(y_1, y_2)$ from these single-attribute assessments and what additional information (if any) is needed to do so? In the next section, we answer these questions for the standard-gamble procedure and for the corresponding single-attribute time-tradeoff procedure.

Those familiar with multiattribute utility theory may suspect that what is required here is some form of *utility independence*, and that the utility function $u(y_1, y_2)$ must have the additive form with multiplicative interaction terms

$$u(y_1, y_1) = k_1 u_1(y_1) + k_2 u(y_2) + k k_1 k_2 u_1(y_1) u_2(y_2)$$

specified by Keeney and Raiffa (1976). Indeed, the additive/multiplicative utility function has been employed by Torrance, Feeny, and their colleagues to evaluate health states arising in their HUI models (Torrance et al. 1982, 1996; Feeny et al. 2002). Utility independence is in fact one version of the required condition, as we will discuss further below. However, due to the special multiplicative structure of the QALY model (1), the additive/multiplicative form collapses to the special *purely multiplicative* instance

$$u(y_1, y_1) = u_1(y_1) u_2(y_2).$$

For this simpler form, *no assessments* are required to obtain attribute weights k_i or interaction parameter k. We elaborate on these points in the following sections.

3. Standard-Gamble and Time-Tradeoff Independence for Nonlinear QALY Preferences

We assume preference over gambles involving state-duration pairs (y, t) is represented by a von Neuman-Morgenstern utility function $U(y, t)$ having the *nonlinear QALY* form (1), where $t \geq 0$ and the health descriptor y is contained in some set Y of possible health states. We assume that the function $m(t)$ is a continuous, increasing[2] function of $t \geq 0$, with $m(0) = 0$. We assign maximum utility $u(y^*) = 1$ to the state y^* of *full health*, as defined by problem context. The quantity $m(t)$ is the utility associated with a duration-t sojourn in y^*, anchored by $m(0) = 0$. In the simplest version of this model, it is assumed that $m(t) = t$. Known as the *linear QALY* model, this represents the case in which there is no time discounting and preferences for survival duration are risk neutral. The more general nonlinear QALY model (1) allows

[2] Throughout this paper, increasing is meant in the strict sense, that is $m(t_1) < m(t_2)$ whenever $t_1 < t_2$.

time discounting and/or risk-averse preferences for survival duration. For example, the function

$$m(t) = \int_0^t e^{-\alpha \cdot s}\, ds = \frac{1}{\alpha}(1 - e^{-\alpha t}) \quad (\alpha > 0)$$

can account for constant risk aversion with coefficient of risk aversion equal to α (e.g., see Pratt 1964; Keeney and Raiffa 1976, Chapter 4; Clemen 1996, Chapter 13) or time discounting at rate α.

The set Y of possible health states y need not include a *death* state, as that is accounted for by duration $t = 0$. However, the nonlinear QALY model does allow states y to be *worse-than-death* ($u(y) < 0$) or *equivalent-to-death* ($u(y) = 0$). We will discuss such possibilities later. For the present, we note that the results in this paper assume that all states $y \in Y$ are better-than-death ($u(y) > 0$).

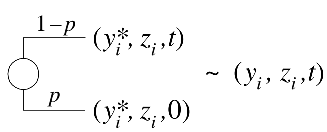
We consider the case in which Y has multiattribute structure, that is, $Y \subset Y_1 \times \cdots \times Y_n$ for sets Y_i of health attributes. If $y \in Y$, we write $y = (y_1, \ldots, y_n)$. If we wish to focus on component i of y, then we may write $y = (y_i, z_i)$, where $z_i = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ is the vector of complementary components to y_i, and we let $Z_i = Y_1 \times \cdots \times Y_{i-1} \times Y_{i+1} \times \cdots \times Y_n$ be the set of all such z_i. We assume the most preferred state $y^* = (y_i^*, z_i^*) \in Y$ has the *partial closure property:*

$$(y_i, z_i) \in Y \text{ implies } (y_i^*, z_i) \in Y \text{ and } (y_i, z_i^*) \in Y \text{ for all } i.$$

Following Keeney and Raiffa (1976), we abuse notation slightly and use $u(y_i)$ to mean $u(y_i, z_i^*)$, and similarly, $u(z_i)$ to mean $u(y_i^*, z_i)$. Because $u(y^*) = 1$, it follows that $u(y_i^*) = 1$ and $u(z_i^*) = 1$.

Standard-Gamble Independence

We now formalize the independence notion discussed above in the context of standard-gamble assessment. We say that Y_i is *standard-gamble independent* of Z_i if for all $t > 0$, whenever the standard-gamble indifference



holds for one pair $(y_i, z_i) \in Y$, it holds for *all* values of z_i with $(y_i, z_i) \in Y$. Under the nonlinear QALY model, this standard-gamble indifference becomes

$$(1 - p)u(z_i)m(t) = u(y_i, z_i)m(t).$$

Suppose that health quality $u(y_i, z_i)$ decomposes into the *purely multiplicative form*

$$u(y_i, z_i) = u(y_i)u(z_i).$$

Then, the standard-gamble indifference is equivalent to

$$(1 - p)u(z_i)m(t) = u(y_i)u(z_i)m(t).$$

The quantities $u(z_i)$ and $m(t)$ are positive and may be cancelled. Therefore, the last equality is equivalent to

$$1 - p = u(y_i).$$

Therefore, under the purely multiplicative form $u(y_i, z_i) = u(y_i)u(z_i)$, the standard-gamble equivalence above determines the single-attribute utility $u(y_i)$ *independently* of the health state z_i. It follows that Y_i is standard-gamble independent of Z_i. The converse holds as well, as we shall see below.

Time-Tradeoff Independence

In this section, we show how a similar notion of independence for time tradeoffs also leads to the purely multiplicative utility form. First, we give some additional comments on the time-tradeoff procedure.

In the time-tradeoff procedure, a subject is given a health state y and a duration t, and is asked to provide a survival duration t' so that the indifference

$$(y^*, t') \sim (y, t)$$

holds. In practice, this task may be difficult for subjects. To ease the cognitive burden, strict preferences $(y^*, s) \succ (y, t)$ or $(y^*, s) \prec (y, t)$ may be elicited for various durations s until the desired indifference is obtained for some $s = t'$. For the multiple-attribute case $y = (y_i, z_i)$, we require that these intermediate preferences as well as the final indifference not depend on z_i. In particular, we say that Y_i is *time-tradeoff independent*[3] of Z_i provided that for all $s, t > 0$, whenever

$$(y_i, z_i, t) \succeq (y_i^*, z_i, s)$$

holds for one pair $(y_i, z_i) \in Y$, it holds for all z_i with $(y_i, z_i) \in Y$. Note that for the purely multiplicative utility form $u(y_i, z_i) = u(y_i)u(z_i)$, the latter time tradeoff is equivalent to

$$u(y_i)u(z_i)m(t) \geq u(z_i)m(s).$$

[3] This should not be confused with constant proportional tradeoffs as discussed by Pliskin et al. (1980).

Because $u(z_i) > 0$, the latter is equivalent to

$$u(y_i)m(t) \geq m(s),$$

a statement that is independent of z_i. Therefore, Y_i is time-tradeoff independent of the complementary attributes Z_i under the multiplicative form $u(y_i, z_i) = u(y_i)u(z_i)$. The converse of this assertion holds as well, as we shall see next.

Multiplicative Decomposition

We have seen that the purely multiplicative form for $u(y)$ implies both standard-gamble independence and time-tradeoff independence. In fact, these conditions are equivalent to each other and to the purely multiplicative form, as the next result states.

THEOREM 1. *Suppose that the nonlinear QALY model holds over $Y \subset Y_1 \times \cdots \times Y_n$. Then, the following statements are equivalent.*

(a) *Y_i is standard-gamble independent of the complementary attributes $Z_i = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$.*

(b) *The complementary attributes Z_i are standard-gamble independent of Y_i.*

(c) *Y_i is time-tradeoff independent of the complementary attributes Z_i.*

(d) *The complementary attributes Z_i are time-tradeoff independent of Y_i.*

(e) *The purely multiplicative model $u(y_i, z_i) = u(y_i)u(z_i)$ holds over $(y_i, z_i) \in Y$.*

A surprising and nonobvious part of this result is that Y_i standard-gamble independent (or time-tradeoff independent) of Z_i implies, under the nonlinear QALY model, that Z_i is standard-gamble independent (or time-tradeoff independent) of Y_i. The proof of this and all subsequent results may be found in the appendix, where this result follows from Theorem 6 and Corollary 8a. The following result gives necessary and sufficient conditions for a complete multiplicative decomposition over all attributes $Y_1, \ldots, Y_n$. This combines Theorem 7 and Corollary 9a in the appendix.

THEOREM 2. *Suppose that the nonlinear QALY model holds over $Y \subset Y_1 \times \cdots \times Y_n$. Then, the following statements are equivalent.*

(a) *Y_i is standard-gamble independent over Y of the complementary attributes $Z_i = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ for at least $n-1$ of the n attributes Y_i.*

(b) *Y_i is time-tradeoff independent of the complementary attributes Z_i for at least $n-1$ of the n attributes Y_i.*

(c) *The purely multiplicative model $u(y) = u(y_1) \cdot u(y_2) \cdots u(y_n)$ holds over $y = (y_1, \ldots, y_n) \in Y$.*

(d) *Y_i is standard-gamble independent over Y of the complementary attributes Z_i for $i = 1, \ldots, n$.*

(e) *Y_i is time-tradeoff independent of the complementary attributes Z_i for $i = 1, \ldots, n$.*

4. Relationship to Other Utility Models

There is a close relationship between the notions of standard-gamble independence and time-tradeoff independence that we have introduced here and the traditional concepts of preferential independence and utility independence found in the literature (e.g., Keeney and Raiffa 1976). In this section, we discuss and clarify these relationships.

Preferential Independence and Utility Independence

Let $T = [0, \infty)$ be the time attribute of the utility function $U(y, t)$, and let $T^+ = \{t \mid t > 0\}$ be the set of positive times. Y_iT^+ is said to be *preferentially independent of the complementary attributes Z_i* provided for all positive t, t', whenever

$$(y_i, z_i, t) \succeq (y'_i, z_i, t')$$

holds for one value of z_i with $(y_i, z_i), (y'_i, z_i) \in Y$, it holds for all such z_i. The following result holds without any restriction on the utility model (see Theorem 5 in the appendix).

THEOREM 3. *Y_i is time-tradeoff independent of the complementary attributes Z_i if and only if Y_iT^+ is preferentially independent of Z_i.*

Y_iT^+ is said to be *utility independent* of the complementary attributes Z_i provided for all nonnegative, nonzero random durations $\tilde{s}, \tilde{t}$, whenever $(\tilde{y}_i, z_i, \tilde{t}) \succeq (\tilde{y}'_i, z_i, \tilde{s})$ holds for one value of z_i with $(\tilde{y}_i, z_i), (\tilde{y}'_i, z_i) \in Y$, it holds for all such z_i. Here $(\tilde{y}_i, z_i, \tilde{t})$ and $(\tilde{y}'_i, z_i, \tilde{s})$ are gambles in which there may be joint uncertainty involving both Y_i and T.

Even without any assumptions on the utility model, it is easy to see that Y_iT^+ utility independent of Z_i implies that Y_i is standard-gamble independent of Z_i over Y: Simply take $(\tilde{y}_i, \tilde{t})$ to be the gamble giving a $1-p$ chance at (y_i^*, t) and a p chance at $(y_i^*, t = 0)$, and

$(\tilde{y}_i', \tilde{s})$ equal to (y_i, t) to obtain the standard-gamble indifference

$$(y_i^*, z_i, \tilde{t}) \sim (y_i, z_i, t).$$

Then, $Y_i T^+$ utility independent of Z_i implies that this indifference holds for all values of z_i if it holds for one, which is precisely the statement that Y_i is standard-gamble independent of Z_i. Therefore, $Y_i T^+$ utility independent of Z_i over Y implies that Y_i is standard-gamble independent of Z_i. Moreover, under the nonlinear QALY model, the two notions are equivalent to each other and to both time-tradeoff independence and preferential independence. The following result follows from the corollaries to Theorems 8 and 10 in the appendix.

Theorem 4. *Suppose that the nonlinear QALY model holds. Then, the following are equivalent.*

(a) *$Y_i T^+$ is utility independent of Z_i.*
(b) *Y_i is standard-gamble independent of Z_i.*
(c) *$Y_i T^+$ is preferentially independent of Z_i.*
(d) *Y_i is time-tradeoff independent of Z_i.*

Conditional Utility Independence

What then is the relationship between the purely multiplicative utility decompositions discussed in this paper and the Keeney/Raiffa additive/multiplicative decomposition for $u(y)$ employed by Torrance, Feeny, and their colleagues in the HUI? We address this question in this subsection. The validity of the additive/multiplicative decomposition for health utility depends on a weaker form of utility independence, which we now introduce.

We say that Y_i is *conditionally utility independent* of Z_i given T^+ provided for each $t > 0$ and all lotteries $\tilde{y}_i, \tilde{y}_i'$, if

$$(\tilde{y}_i, z_i, t) \succeq (\tilde{y}_i', z_i, t)$$

holds for one value of z_i with $(\tilde{y}_i, z_i), (\tilde{y}_i', z_i) \in Y$, then it holds for all such z_i. Conditional utility independence of Y_i from Z_i given T^+ is a weaker assumption than utility independence of $Y_i T^+$ from Z_i, as the latter implies the former.

When there are n health attributes $Y_1, \ldots, Y_n$, the assumption that every subset $\{Y_i \mid i \in I\}$ of attributes is conditionally utility independent of the complementary set given T^+ is known as *mutual conditional utility independence* given T^+. Under this assumption, we can invoke Keeney and Raiffa in the following way: Let y^0 be a "least desirable"[4] health state, and for each $t > 0$, let $U^0(y \mid t)$ be the positive linear transformation of $U(y \mid t) = u(y)m(t)$ having $U^0(y^0 \mid t) = 0$, $U^0(y^* \mid t) = 1$. We have

$$U^0(y \mid t) = \frac{U(y \mid t) - U(y^0 \mid t)}{U(y^* \mid t) - U(y^0 \mid t)} = \frac{u(y) - u(y^0)}{u(y^*) - u(y^0)} = \frac{u(y) - u(y^0)}{1 - u(y^0)} \equiv u^0(y);$$

that is, $U^0(y \mid t)$ equals $u^0(y)$ and is therefore independent of t. Mutual utility independence holds for the utility function $u^0(y)$, and we may invoke Keeney and Raiffa's fundamental Theorem 6.1 (1976) to obtain the additive/multiplicative form

$$u^0(y) = \frac{1}{k}\Big(-1 + \prod_{i=1}^{n}(1 + kk_i u_i(y_i))\Big).$$

Overall, using the definition of $u^0(y)$, we have $U(y, t) = (u_0 + (1 - u_0)u^0(y))m(t)$, where $u_0 = u(y^0)$. This derivation of the additive/multiplicative form of multiattribute utility from mutual conditional utility independence is new, as far as we know.

Torrance et al. (1982) use the disutility form of this model,

$$\bar{u}^0(y) = \frac{1}{c}\Big(-1 + \prod_{i=1}^{n}(1 + cc_i \bar{u}_i(y_i))\Big) \tag{2}$$

where the disutility $\bar{u}^0(y)$ relates to the nonlinear QALY utility $u(y)$ by

$$\bar{u}^0(y) = 1 - u^0(y) = \frac{1 - u(y)}{\bar{u}_0}, \qquad \bar{u}_0 = 1 - u(y^0). \tag{3}$$

Moreover, the c_i are the disutilities of the complementary corner points,

$$c_i = \bar{u}^0(y_1^*, \ldots, y_{i-1}^*, y_i^0, y_{i+1}^*, \ldots, y_n^*) = \bar{u}^0(y_i^0)$$

and $\bar{u}_i(y_i)$ is $\bar{u}^0(y_i)$ normalized so that $\bar{u}_i(y_i^0) = 1$:

$$\bar{u}_i(y_i) = \frac{\bar{u}^0(y_i)}{\bar{u}^0(y_i^0)} = \frac{\bar{u}^0(y_i)}{c_i}. \tag{4}$$

Note that we may invert (3) to obtain

$$u(y) = 1 - \bar{u}_0 \bar{u}^0(y) \tag{5}$$

from which we conclude, using (2), that

$$u(y) = 1 - \frac{\bar{u}_0}{c}\Big(-1 + \prod_{i=1}^{n}(1 + cc_i \bar{u}_i(y_i))\Big). \tag{uCUI}$$

[4] Actually, any fixed health state inferior to y^* will do.

This is the form of u implied by mutual conditional utility independence given T^+.

To compare this with the purely multiplicative form $u(y) = \prod_{i=1}^n u(y_i)$ implied by standard-gamble independence or time-tradeoff independence, we need an expression for the factors $u(y_i)$. We have, from (5) and (4),

$$u(y_i) = 1 - \bar{u}_0 \bar{u}^0(y_i) = 1 - \bar{u}_0 c_i \bar{u}_i(y_i).$$

Substitute this to obtain

$$u(y) = \prod_{i=1}^{n} (1 - \bar{u}_0 c_i \bar{u}_i(y_i)). \qquad (u\text{SGI})$$

This is the form of u implied by mutual standard-gamble independence.

Comparing (uCUI) with (uSGI), we note that the former is obtained from the latter by setting $c = -\bar{u}_0$. Therefore, one measure of the similarity of specific instances of the two forms is the difference between the quantities c and $-\bar{u}_0$. The values obtained by Torrance et al. (1996) for their HUI Mark II were $c = -0.967$ and $\bar{u}_0 = 1.03$ for the utility function formed from mean subject responses. For their HUI Mark III (Feeny et al. 2002), the values were $c = -0.991$ and $\bar{u}_0 = 1.36$. Thus, the Mark II utility $u(\cdot)$ is very near our purely multiplicative form, whereas the Mark III $u(\cdot)$ is not. Note, however, that the utility (uCUI) can be defined over worse-than-death states y as well as better-than-death states, whereas the utility (uSGI) is defined only over better-than-death states. This is an advantage of the conditional utility independence assumption compared to standard-gamble independence or time-tradeoff independence.

5. Conclusion

The results of this paper provide answers to two questions concerning the nonlinear QALY model $U(y, t) = u(y)m(t)$ when the health state $y = (y_1, \ldots, y_n)$ has multiattribute structure. First, for what functions $u(\cdot)$ is it valid to apply the standard-gamble or time-tradeoff techniques one attribute at a time to elicit health utilities? That is, when are standard-gamble independence and time-tradeoff independence valid? The answer is that $u(y)$ must have the purely multiplicative form $u(y) = u(y_1) \cdots u(y_n)$. It is only under the purely multiplicative form that one can consistently use standard-gamble assessments or time-tradeoff assessments over single health attributes. For example, under the Keeney-Raiffa additive/multiplicative form used in the HUI, it is not valid to attempt to assess the single-attribute utility functions via standard-gamble or time-tradeoff procedures.

Second, when is the common practice of obtaining overall health utility $u(y)$ by *multiplying* single-attribute health utilities valid? The answer is that this procedure is valid when and only when for each i the combination $Y_i T^+$ of attribute i and durations T^+ is *preferentially independent* (or equivalently, *utility independent*) of the complementary attributes $Z_i = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$.

Should these independence assumptions be acceptable, the assessment burden for multiattribute health states is substantially eased. Both the number of health states and their cognitive complexity are substantially less for single-attribute standard-gamble or time-tradeoff assessments as compared to direct multiattribute assessment. Moreover, because results from single-attribute assessment can be *directly combined by multiplying*, no importance weights for the attributes need be assessed.

Standard-gamble or time-tradeoff independence as we have defined them cannot, however, be invoked when there are both positive (better-than-death) and negative (worse-than-death) health states. The purely multiplicative form for health utility therefore is not a viable candidate for health classification systems such as the HUI, in which some combinations of health attributes are ranked worse-than-death by many subjects. Fortunately, most effectiveness and cost-effectiveness analyses examine health impacts along only some of the wide range of possible health attributes, and worse-than-death states are rare in these settings. For example, Bell et al. (2001) found *no* such states in 228 cost-utility analyses published from 1976 to 1997. The purely multiplicative form could be appropriate here, and in fact has already been used in this context, as we pointed out in the introduction.

Our standard-gamble independence condition is superficially similar to the standard-gamble invariance condition introduced by Miyamoto et al. (1998). The latter states that gambles over duration t are independent of health state y, and assumes no multiattribute structure for y, so the two conditions are not really comparable.

We note that we have previously provided formal conditions for the purely multiplicative form (Hazen 2000). However, the results in that paper apply only to the two-attribute case and only to so-called Markovian utility, which includes the linear QALY model—but not the nonlinear QALY model—as a special case. Results in the appendix of this paper apply to utility functions that are more general than Markovian, however.

For utility structures $U(y, t)$ that are more general than nonlinear QALY, three of the four equivalent conditions we have treated above—time-tradeoff independence, standard-gamble independence, preferential independence, and utility independence—do not remain equivalent. The appendix details their implications for utility structure in the absence of the nonlinear QALY model.

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Appendix. General Results and Proofs

If $I \subset \{1, \ldots, n\}$ and $y \in Y$, then we let y_I be the vector $(y_i \mid i \in I)$ of components of y for $i \in I$ and we let Y_I be the set of all such y_I. Let $\bar{I}$ be the complement of I in $\{1, \ldots, n\}$. Then, the complementary component Z_i used above is merely $Y_{\bar{i}}$. We shall use the two terms interchangeably.

Utility Models

The proofs below are presented for the following general class of utility models that includes the nonlinear QALY case. We will assume that the von Neuman-Morgenstern utility function $U(y, t)$ has the following properties.

Assumption 1. *$U(y, 0) = 0$ for all states y (the zero condition).*

Assumption 2. *$U(y, t)$ is a continuous increasing function of t for all states y.*

Miyamoto et al. (1998) discuss the zero condition. These assumptions imply that $U(y, t) > 0$ for all $t > 0$, that is, all states are preferred to death. Define

$$u(y) = U(y, 1) \quad m(t \mid y) = U(y, t)/U(y, 1).$$

Then, we have for all y

$$U(y, t) = u(y)m(t \mid y).$$

Under Assumptions 1 and 2 above, the function $m(\cdot)$ has the following properties.

Property 1. *$m(0 \mid y) = 0$ for all states y.*

Property 2. *$m(1 \mid y) = 1$ for all states y.*

Property 3. *$m(t \mid y)$ is a continuous increasing function of $t \geq 0$ that is positive for all $t > 0$ and all states y.*

If the function $m(t \mid y)$ is independent of health state y, then we obtain the nonlinear QALY model $U(y, t) = u(y)m(t)$. If $m(t \mid y)$ takes the form $m(t \mid y) = e^{-a(y)t}$, then we obtain *Markovian utility* (Hazen and Pellissier 1996) and the utility independence results in this appendix generalize those for Markovian utility in Hazen (2000), which apply to only two attributes. In the main part of this paper, we assumed that the special health state $y^* \in Y$ had $u(y^*) = 1$, but here we let $u(y^*) = u^*$ be an arbitrary positive quantity.

Time-Tradeoff Independence and Preferential Independence

Theorem 5. *Y_i is time-tradeoff independent of Z_i if and only if $Y_i T^+$ is preferentially independent of Z_i.*

Proof. Time-tradeoff independence is a special case of preferential independence, hence is implied by it. Conversely, suppose time-tradeoff independence holds. To show preferential independence holds, suppose we have pairs $(y_i, z_i), (y_i', z_i) \in Y$ for which $(y_i, z_i, t) \succeq (y_i', z_i, t')$. We wish to show for $(y_i, z_i'), (y_i', z_i') \in Y$ that we have $(y_i, z_i', t) \succeq (y_i', z_i', t')$.

Because $(y_i, z_i) \in Y$, it follows by partial closure that $(y_i^*, z_i) \in Y$ also. Because of Property 3, we can find durations s, s' such that $(y_i, z_i, t) \sim (y_i^*, z_i, s)$ and $(y_i', z_i, t') \sim (y_i^*, z_i, s')$. Because $(y_i, z_i, t) \succeq (y_i', z_i, t')$, we have by transitivity $(y_i^*, z_i, s) \succeq (y_i^*, z_i, s')$. Then, by time-tradeoff independence, the last three preference conclusions imply

$$(y_i, z_i', t) \sim (y_i^*, z_i', s), \quad (y_i', z_i', t') \sim (y_i^*, z_i', s'),$$
$$(y_i^*, z_i', s) \succeq (y_i^*, z_i', s').$$

Therefore, by transitivity we have $(y_i, z_i', t) \succeq (y_i', z_i', t')$, which is the desired conclusion. □

Theorem 6. *Suppose the nonlinear QALY model holds over $Y \subset Y_i \times Z_i$. Then, the following are equivalent.*

(a) *Y_i is time-tradeoff independent of Z_i.*
(b) *Z_i is time-tradeoff independent of Y_i.*
(c) *$u(y_i, z_i) = u(y_i)u(z_i)/u^*$ for all $(y_i, z_i) \in Y$.*

Proof. (a) ⇒ (c). Let $Y_i(z_i)$ be the set of all y_i for which $(y_i, z_i) \in Y$. If $(y_i, z_i) \in Y$, then by partial closure $(y_i, z_i^*) \in Y$ also, so $Y_i(z_i) \subset Y_i(z_i^*)$. Therefore, by the preferential independence assumption, for each z_i for which $Y_i(z_i) \neq \varnothing$, the utility functions $U(y_i, z_i, t)$ and $U(y_i, z_i^*, t)$ are ordinally equivalent as functions over $Y_i(z_i) \times T^+$. Therefore, there is a function $f(u, z_i)$ increasing in u with $f(u, z_i^*) = u$ such that

$$U(y_i, z_i, t) = f(U(y_i, z_i^*, t), z_i)$$

for all y_i, z_i, t with $(y_i, z_i) \in Y$. That is,

$$u(y_i, z_i)m(t) = f(u(y_i)m(t), z_i).$$

Then, because

$$u(y_i, z_i) = u(y_i, z_i)m(1) = f(u(y_i)m(1), z_i) = f(u(y_i), z_i)$$

we have, substituting this into the last equation,

$$f(u(y_i), z_i)m(t) = f(u(y_i)m(t), z_i).$$

Set $y_i = y_i^*$ to get

$$f(u^*, z_i)m(t) = f(u^*m(t), z_i),$$

that is, quantities $m(t)$ may be factored from the first component of f. In the defining equation for f

$$u(y_i, z_i)m(t) = f(u(y_i)m(t), z_i).$$

Let t' be such that $u(y_i)m(t) = u^*m(t')$. This is possible because $u(y_i) \leq u^*$ and by Property 3, $m(\cdot)$ is continuous, positive, and increasing. Then, the right-hand side of the last equation becomes

$$\begin{aligned} f(u(y_i)m(t), z_i) &= f(u^*m(t'), z_i) = m(t')f(u^*, z_i) \\ &= (u(y_i)m(t)/u^*)f(u^*, z_i). \end{aligned}$$

Cancel the positive value $m(t)$ to get

$$u(y_i, z_i) = (u(y_i)/u^*)f(u^*, z_i).$$

Set $y_i = y_i^*$ to conclude $u(z_i) = f(u^*, z_i)$. Substitute this conclusion into the last equation to get $u(y_i, z_i) = u(y_i)u(z_i)/u^*$, as claimed in (c).

(c) $\Rightarrow$ (a) Suppose (c) holds. Then, for states (y_i, z_i), $(y_i, z_i') \in Y$,

$$U(y_i, z_i, t) = u(y_i)u(z_i)m(t)/u^*,$$
$$U(y_i, z_i', t) = u(y_i)u(z_i')m(t)/u^*.$$

With respect to y_i both these utility functions are ordinally equivalent to $u(y_i)m(t)$. Hence, Y_i is preferentially independent of Z_i, and therefore (a) holds, by Theorem 5.

The arguments above may be repeated with Y_i and Z_i interchanged to show that (b) and (c) are equivalent. Therefore, all three statements are equivalent, and the theorem is proved. □

Theorem 7. *Suppose the nonlinear QALY model holds over $Y \subset Y_1 \times \cdots \times Y_n$. Then, the following statements are equivalent.*

(a) *Y_i is time-tradeoff independent over Y of its complementary attributes $Y_{\bar{i}}$ for at least $n-1$ of the n attributes Y_i.*

(b) *For $y \in Y$,*

$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*}.$$

(c) *Y_i is time-tradeoff independent over Y of its complementary attributes $Y_{\bar{i}}$ for $i = 1, \ldots, n$.*

Proof. (b) implies (c) as in Theorem 6, and (c) implies (a) trivially. So it remains to show that (a) implies (b). Suppose Y_i is time-tradeoff independent of $Y_{\bar{i}}$ over Y for $i = 1, \ldots, n-1$. Then, by Theorem 6 we have for $y \in Y$,

$$u(y) = u(y_i, y_{\bar{i}}) = u(y_i)u(y_{\bar{i}})/u^*, \quad i = 1, \ldots, n-1. \tag{6}$$

Consider the induction hypothesis

$$\frac{u(y)}{u^*} = \frac{u(y_{\bar{I}})}{u^*} \prod_{i=1}^{k} \frac{u(y_i)}{u^*}, \quad I = \{1, \ldots, k\}, \tag{7}$$

for $y \in Y$. By (6), we know this holds for $k = 1$. We suppose it holds for some $k < n$, and show that it then must hold for $k+1$. So suppose (7) holds. Then, invoke (6) with $i = k+1$ and $y_I = y_I^*$ to get

$$u(y_{\bar{I}}) = u(y_{k+1})u(y_{\overline{I \cup (k+1)}})/u^*.$$

Substitute back into (7) to get for $y \in Y$,

$$\begin{aligned} \frac{u(y)}{u^*} &= \frac{u(y_{k+1})}{u^*} \frac{u(y_{\overline{I \cup (k+1)}})}{u^*} \prod_{i=1}^{k} \frac{u(y_i)}{u^*} \\ &= \frac{u(y_{\overline{I \cup (k+1)}})}{u^*} \prod_{i=1}^{k+1} \frac{u(y_i)}{u^*}, \quad I = \{1, \ldots, k\}. \end{aligned}$$

Therefore, the induction hypothesis holds for $k+1$. By induction it holds for all $k = 1, \ldots, n$, so (b) is proved. □

Standard-Gamble Independence

Let $\langle p, (y, t) \rangle$ denote a gamble in which outcome (y, t) occurs with probability p and immediate death (that is, the state-duration pair $(y, 0)$) occurs with probability $1-p$. Using this notation, Y_i is standard-gamble independent of Z_i if for all $t > 0$ and all y_i, whenever

$$\langle p, (y_i^*, z_i, t) \rangle \sim (y_i, z_i, t)$$

holds for one level z_i with $(y_i, z_i) \in Y$, then it holds for all such z_i.

Theorem 8. *The following are equivalent.*

(a) *Y_i is standard-gamble independent of Z_i.*

(b) *Z_i is standard-gamble independent of Y_i.*

(c) *For all $(y_i, z_i) \in Y$, we have*

$$u(y_i, z_i) = u(y_i)u(z_i)/u^*,$$
$$m(t \mid y_i, z_i) = m(t \mid y_i, z_i^*)m(t \mid y_i^*, z_i)/m(t \mid y_i^*, z_i^*).$$

Proof. (a) $\Rightarrow$ (c). Suppose Y_i is standard-gamble independent of Z_i. If p satisfies the standard-gamble indifference $\langle p, (y_i^*, z_i, t) \rangle \sim (y_i, z_i, t)$ for some $(y_i, z_i) \in Y$, then we have

$$pu(z_i)m(t \mid y_i^*, z_i) = u(y_i, z_i)m(t \mid y_i, z_i)$$

and therefore

$$p = \frac{u(y_i, z_i)m(t \mid y_i, z_i)}{u(z_i)m(t \mid y_i^*, z_i)}.$$

Denote the right-hand side of this equality by $p(y_i, z_i, t)$. The hypothesized standard-gamble independence asserts that for $(y_i, z_i) \in Y$, $p(y_i, z_i, t)$ does not depend on z_i. Therefore, $p(y_i, z_i, t) = p(y_i, z_i^*, t)$, that is,

$$\frac{u(y_i, z_i)m(t \mid y_i, z_i)}{u(z_i)m(t \mid y_i^*, z_i)} = \frac{u(y_i)m(t \mid y_i, z_i^*)}{u^* m(t \mid y_i^*, z_i^*)}.$$

Set $t = 1$ and rearrange to conclude $u(y_i, z_i) = u(y_i)u(z_i)/u^*$. Substitute this back into the last equality and cancel to obtain the second assertion in (c).

(c) $\Rightarrow$ (a). Under (c), the quantity $p(y_i, z_i, t)$ defined above is given by

$$p(y_i, z_i, t) = \frac{u(y_i, z_i)m(t \mid y_i, z_i)}{u(z_i)m(t \mid y_i^*, z_i)} = \frac{u(y_i)m(t \mid y_i, z_i^*)}{u^* m(t \mid y_i^*, z_i^*)}$$

for $(y_i, z_i) \in Y$. Because $p(y_i, z_i, t)$ does not depend on z_i, it follows that Y_i is standard-gamble independent of Z_i.

The arguments above may be repeated with Y_i and Z_i interchanged to show that (b) and (c) are equivalent. Therefore, all three statements are equivalent, and the theorem is proved. □

Corollary 8a. *Suppose the nonlinear QALY model holds for all states $y \in Y$. Then, the following are equivalent.*
(a) *Y_i is standard-gamble independent of Z_i.*
(b) *Z_i is standard-gamble independent of Y_i.*
(c) *For all $(y_i, z_i) \in Y$, we have*

$$u(y_i, z_i) = u(y_i)u(z_i)/u^*.$$

Theorem 9. *The following statements are equivalent.*
(a) *Y_i is standard-gamble independent of $Y_{\bar{i}}$ for at least $n-1$ of the n attributes Y_i.*
(b) *For all $y \in Y$,*

$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*},$$

$$\frac{m(t \mid y)}{m(t \mid y^*)} = \prod_{i=1}^{n} \frac{m(t \mid y_i, y_{\bar{i}}^*)}{m(t \mid y^*)}.$$

(c) *Y_i is standard-gamble independent over Y of $Y_{\bar{i}}$ for $i = 1, \ldots, n$.*

Proof. We show (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b). First, suppose (b) holds. If p satisfies the standard-gamble indifference $\langle p, (y_i^*, y_{\bar{i}}, t)\rangle \sim (y, t)$ for some $y \in Y$, then under (b) we have

$$p = p_i(y, t) = \frac{u(y)m(t \mid y)}{u(y_i^*, y_{\bar{i}})m(t \mid y_i^*, y_{\bar{i}})} = \frac{u(y_i)m(t \mid y_i, y_{\bar{i}}^*)}{u^* m(t \mid y^*)}.$$

It follows that for $y \in Y$, $p_i(y, t)$ does not depend on $y_{\bar{i}}$. Therefore, Y_i is standard-gamble independent of $Y_{\bar{i}}$ over Y, and (b) implies (c).

(c) implies (a) trivially. To show (a) implies (b), suppose (a) holds for attributes $i = 1, \ldots, n-1$. By Theorem 8 we have for $i = 1, \ldots, n-1$ and all $y \in Y$,

$$u(y) = u(y_i)u(y_{\bar{i}})/u^*,$$
$$m(t \mid y) = m(t \mid y_i, y_{\bar{i}}^*)m(t \mid y_i^*, y_{\bar{i}})/m(t \mid y^*).$$

The first of these implies

$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*} \quad \text{for all } y \in Y$$

as in the proof of Theorem 7. Rewrite the second equality as

$$m^*(t \mid y) = m^*(t \mid y_i, y_{\bar{i}}^*)m^*(t \mid y_i^*, y_{\bar{i}}), \quad i = 1, \ldots, n-1, \quad (8)$$

where $m^*(t \mid y) = m(t \mid y)/m(t \mid y^*)$. For $i = 1$, it follows that

$$\begin{aligned} m^*(t \mid y) &= m^*(t \mid y_1, y_{\bar{1}}^*)m^*(t \mid y_1^*, y_{\bar{1}}) \\ &= m^*(t \mid y_1, y_{\bar{1}}^*)m^*(t \mid y_1^*, y_2, y_{\overline{12}}). \end{aligned}$$

If $n = 2$, we are done. For $n > 2$, apply the $i = 2$ version of (8) to the last factor to get

$$m^*(t \mid y) = m(t \mid y_1, y_{\bar{1}}^*)m^*(t \mid y_2, y_{\bar{2}}^*)m^*(t \mid y_1^*, y_2^*, y_{\overline{12}}).$$

For $n = 3$, we are done. If $n > 3$, then continue in this fashion, applying (8) for $i = 3, \ldots, n-1$ to get

$$m^*(t \mid y) = \prod_{i=1}^{n} m^*(t \mid y_i, y_{\bar{i}}^*)$$

for all $y \in Y$. Therefore (b) holds. □

Corollary 9a. *Suppose the nonlinear QALY model holds. Then, the following statements are equivalent.*
(a) *Y_i is standard-gamble independent of $Y_{\bar{i}}$ for at least $n-1$ of the n attributes Y_i.*
(b) *For all $y \in Y$,*

$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*}.$$

(c) *Y_i is standard-gamble independent of $Y_{\bar{i}}$ for $i = 1, \ldots, n$.*

Utility Independence

Theorem 10. *$Y_i T^+$ is utility independent of Z_i if and only if*
(i) *$u(y_i, z_i) = u(y_i)u(z_i)/u^*$ for all $(y_i, z_i) \in Y$, and*
(ii) *for $(y_i, z_i) \in Y$, $m(t \mid y_i, z_i)$ does not depend on z_i.*

Proof. ($\Rightarrow$) If $Y_i T^+$ is utility independent of Z_i, then Y_i is standard-gamble independent of Z_i over Y, so by Theorem 8,

$$u(y_i, z_i) = u(y_i)u(z_i)/u^*, \quad (9)$$

$$m(t \mid y_i, z_i) = m(t \mid y_i, z_i^*)m(t \mid y_i^*, z_i)/m(t \mid y_i^*, z_i^*). \quad (10)$$

The condition that Y_iT^+ is utility independent of Z_i means that for some $\alpha(z_i) > 0$, $\beta(z_i)$, we have

$$U(y_i, z_i, t) = \alpha(z_i)U(y_i, z_i^*, t) + \beta(z_i).$$

The zero condition forces $\beta(z_i) = 0$. Substitute $U(y_i, z_i, t) = u(y_i, z_i)m(t \mid y_i, z_i)$ and use (9) and (10) to obtain

$$u(y_i)u(z_i)/u^* m(t \mid y_i, z_i^*)m(t \mid y_i^*, z_i)/m(t \mid y_i^*, z_i^*)$$
$$= \alpha(z_i)u(y_i)m(t \mid y_i, z_i^*).$$

Substitute $y_i = y_i^*$ and $t = 1$ to get

$$u(z_i) = \alpha(z_i)u^*.$$

Substitute this into the penultimate equation and cancel to get

$$m(t \mid y_i^*, z_i)/m(t \mid y_i^*, z_i^*) = 1.$$

From this it follows using (10) that

$$m(t \mid y_i, z_i) = m(t \mid y_i, z_i^*).$$

Therefore, $m(t \mid y_i, z_i)$ does not depend on z_i.

($\Leftarrow$) Suppose conditions (i) and (ii) hold, and let $m(t \mid y_i) = m(t \mid y_i, z_i^*)$. Then, for states $(y_i, z_i), (y_i, z_i') \in Y$,

$$U(y_i, z_i, t) = u(y_i)u(z_i)m(t \mid y_i)/u^*,$$
$$U(y_i, z_i', t) = u(y_i)u(z_i')m(t \mid y_i)/u^*.$$

Because $u(z_i)/u^* > 0$ and $u(z_i')/u^* > 0$, both utility functions are cardinally equivalent to $u(y_i)m(t \mid y_i)$. Hence, utility independence holds. □

Corollary 10a. *The following are equivalent.*

(a) *Y_iT^+ is utility independent of Z_i and Z_iT^+ is utility independent of Y_i.*

(b) *The nonlinear QALY model holds for $y \in Y$, that is, for all $(y_i, z_i) \in Y$, $m(t \mid y_i, z_i)$ does not depend on (y_i, z_i), and*

$$u(y_i, z_i) = u(y_i)u(z_i)/u^*.$$

Corollary 10b. *Suppose the nonlinear QALY model holds for all states $y \in Y$. Then, the following are equivalent.*

(a) *Y_iT^+ is utility independent of Z_i.*

(b) *Z_iT^+ is utility independent of Y_i.*

(c) *$u(y_i, z_i) = u(y_i)u(z_i)/u^*$ for all $(y_i, z_i) \in Y$.*

Theorem 11. *Suppose $n \geq 3$. Then, the following statements are equivalent.*

(a) *Y_iT^+ is utility independent of the complementary attributes $Y_{\bar{i}}$ for at least $n-1$ of the n attributes Y_i.*

(b) *For $y \in Y$, the nonlinear QALY model holds with*

$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*}.$$

(c) *Y_iT^+ is utility independent over Y of the complementary attributes $Y_{\bar{i}}$ for $i = 1, \ldots, n$.*

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are easily shown. It remains to demonstrate (a) $\Rightarrow$ (b). So suppose (a) holds. Specifically, suppose Y_iT^+ is utility independent over Y of the complementary attributes $Y_{\bar{i}}$ for $i = 1, \ldots, n-1$. Then, by Theorem 10, $m(t \mid y)$ does not depend on $y_{\bar{i}}$ for $i = 1, \ldots, n-1$, that is, $m(t \mid y)$ does not depend on $y_{\bar{1}\cup\cdots\cup\overline{n-1}}$. However, for $n \geq 3$, we have $\bar{1} \cup \cdots \cup \overline{n-1} = \{1, \ldots, n\}$, so we conclude that $m(t \mid y)$ does not depend on y. Therefore, the nonlinear QALY model holds. To show the remainder of (b), invoke Theorem 10(i) to get

$$u(y) = u(y_i, y_{\bar{i}}) = u(y_i)u(y_{\bar{i}})/u^*, \quad i = 1, \ldots, n-1.$$

Now proceed as in the proof of Theorem 7 to reach the desired conclusion. □

Corollary 11a. *Suppose $n \geq 2$. Then, under the nonlinear QALY model, the following statements are equivalent.*

(a) *Y_iT^+ is utility independent over Y of the complementary attributes $Y_{\bar{i}}$ for at least $n-1$ of the n attributes Y_i.*

(b)
$$\frac{u(y)}{u^*} = \prod_{i=1}^{n} \frac{u(y_i)}{u^*}.$$

(c) *Y_iT^+ is utility independent over Y of the complementary attributes $Y_{\bar{i}}$ for $i = 1, \ldots, n$.*

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