A NEW PERSPECTIVE ON MULTIPLE INTERNAL RATES OF RETURN

GORDON B. HAZEN
Northwestern University

ABSTRACT

The most commonly cited drawback to using the internal rate of return to evaluate deterministic cash flow streams is the possibility of multiple conflicting internal rates, or no internal rate at all. We claim, however, that contrary to current consensus, multiple or nonexistent internal rates are not contradictory, meaningless or invalid as rates of return. There is, moreover, no need to carefully examine a cash flow stream to rule out the possibility of multiple internal rates, or to throw out or ignore "unreasonable" rates. What we show is that when there are multiple (or even complex-valued) internal rates, each has a meaningful interpretation as a rate of return on its own underlying investment stream. *It does not matter which rate is used* to accept or reject the cash flow stream, as long as one identifies the underlying investment stream as a *net investment* or *net borrowing*. When we say it does not matter which rate is used, we mean that regardless of which rate is chosen, the cash-flow acceptance or rejection decision will be the same, and consistent with net present value.

INTRODUCTION

The *internal rate of return* (IRR) is a widely used tool for evaluating deterministic cash flow streams, familiar to all students of finance and engineering economics. When used appropriately, it can be a valuable aid in project acceptance and selection. But the method is subject to well-known difficulties (Brealey and Myers 1996): It cannot be used to rank cash flows for mutually exclusive projects, except on an incremental basis. It does not extend well to situations involving uncertainty. But the most widely cited difficulty for IRR is the fact that a cash flow stream can have multiple conflicting internal rates, or no real-valued internal rate at all. The appeal of the IRR lies in its interpretation as a *rate of return*. However, multiple or nonexistent internal rates frustrate this interpretation. The consensus is that multiple internal rates constitute a severe drawback (White et al. 1998), are incorrect (Canada et al. 1996, Sullivan et al. 2000), difficult to explain or interpret properly (Blank and Tarquin 1989, White et al. 1998), invalid and not useful (Bussey and

Eschenbach 1992), meaningless (Stevens 1979, Steiner 1992, Young 1993, Eschenbach 1995, Sullivan et al. 2000), inaccurate (Park 1997), ambiguous and contradictory (Steiner 1992). The conclusion is that, "…when multiple rates of return are found, there is no rational means for judging which of them is most appropriate for determining economic desirability" (Thuesen and Fabrycky 1989). Worse, when there are no real-valued internal rates of return, then the IRR approach must be abandoned altogether.

The consensus is that to avoid this problem, one must examine one's cash flow stream carefully to rule out the possibility of multiple or nonexistent rates (Teichroew, Robichek, and Montalbano 1965a, 1965b, Bussey and Eschenbach 1992, Fleischer 1994, Eschenbach 1995, Park 1997). Failing this, one must either throw out or simply never compute "unreasonable" IRR values (Cannaday et al. 1986, Hajdasinski 1987, Blank and Tarquin 1989, Collier and Glagola 1998), or abandon the IRR approach altogether, reverting to net present value, the external rate of return approach, or other procedures.

The point of the present paper is that this consensus regarding multiple internal rates of return is incomplete and unnecessarily pessimistic. Even when there are multiple internal rates, their interpretation as rates of return can still be useful, valid, meaningful, correct, explainable, and non-contradictory. There is indeed a rational means for using multiple or even complex-valued internal rates of return to determine economic viability. And there is no need to try to rule out the existence of multiple rates, or to throw out "unreasonable" rates of return. We will demonstrate these facts in this paper.

What we show is that when there are multiple (or even complex-valued) internal rates, each has a meaningful interpretation as a rate of return on its own underlying investment stream. *It does not matter which rate is used* to accept or reject the cash flow stream, as long as one identifies the underlying investment stream as a *net investment* or *net borrowing* in a way that we will make precise below. When we say it does not matter which rate is used, we mean that regardless of which rate is chosen, the cash-flow acceptance or rejection decision will be the same, and consistent with net present value.

INTERNAL RATES AS RATES OF RETURN

Although internal rates of return are commonly regarded as *rates of return*, the latter notion is not part of their mathematical definition, namely as interest rates under which cash flow streams have present value zero. The key result of this section is that every internal rate of return is in fact a rate of return on some underlying investment stream. We present this result after introducing mathematical notation and terminology.

INTERNAL RATE OF RETURN

A *cash flow stream* is a finite or infinite sequence $\boldsymbol{x} = (x_0, x_1, \dots)$ of monetary values. The monetary amount received initially is x_0, and the amount received after period t is x_t. For a finite stream $\boldsymbol{x} = (x_0, x_1,\dots, x_T)$, we assume the horizon T is chosen so that $x_T \neq 0$. The *net present value* $PV(\boldsymbol{x} \mid r)$ of a cash flow stream x at interest rate r is given by

$$PV(\boldsymbol{x}|r) = \sum_t \frac{x_t}{(1+r)^t}$$

defined for *proper* interest rates $r > -1$. For a cash flow stream $\boldsymbol{x}$, let $IRR(\boldsymbol{x})$ be the set of all interest rates r which make $PV(\boldsymbol{x} \mid r) = 0$. (Note that $IRR(\boldsymbol{x})$ cannot contain -1 because $PV(x \mid r) = -1)$ is undefined.) For finite streams $\boldsymbol{x} = (x_0, x_1,\dots, x_T)$, the present value function $PV(\boldsymbol{x} \mid r)$ is a degree-T polynomial in $(1 + r)^{-1}$, so $IRR(x)$ can contain anywhere from 0 to T distinct values. If $r \in IRR(\boldsymbol{x})$, then we will call r *an internal rate of return* for $\boldsymbol{x}$.

As is well known, for conventional cash flows $\boldsymbol{x}$ that are negative for the first few periods but positive thereafter, the internal rate of return exists and is unique. Moreover, the internal rate of return is the largest interest rate at which the cash flow shows a discounted net profit. So if $IRR(\boldsymbol{x})$ exceeds the available market rate of interest r, then $PV(\boldsymbol{x} \mid r) > 0$ and the investment which generates the cash flow $\boldsymbol{x}$ is worthwhile. Conversely, if the internal rate of return is smaller than the market rate r, then one is better off investing at the market rate r. This is the fundamental justification for the use of internal rate of return.

CONSTANT PER-PERIOD RETURN ON INVESTMENT

If one invests an amount c now and receives an amount $(1 + k)\,c$ one period from now, then the rate of return on one's investment is k. The quantity k is also equal to the internal rate of return on the corresponding cash flow stream $\boldsymbol{x} = (-c, (1 + k)\,c)$ because

$$PV\left(\boldsymbol{x}|r\right) = c\left(-1+\frac{1+k}{1+r}\right),$$

and this function of the interest rate r is zero precisely when r is equal to k.

Similarly, if one invests an amount c now and receives $(1 + k)^3\, c$ three periods hence, then the rate of return is the constant value k per period, and this quantity is also equal to the internal rate of return of the corresponding cash flow stream $\boldsymbol{x} = (-c, 0, 0, (1 + k)^3\, c)$, because

$$PV\left(\boldsymbol{x}\middle|r\right) = c\left(-1 + \frac{(1+k)^3}{(1+r)^3}\right),$$

and this function of r is equal to zero when r is equal to k. To emphasize more strongly that the rate of return k is a constant per period, think of the cash flow stream $\boldsymbol{x}$ as a series of investments and returns:

	Receive	Invest	Net cash flow
$t = 0$:	–	$c_0 = c$.	$x_0 = -c$
$t = 1$:	$(1 + k)\, c$.	$c_1 = (1 + k)\, c$.	$x_1 = 0$.
$t = 2$:	$(1 + k)^2\, c$.	$c_2 = (1 + k)^2\, c$.	$x_2 = 0$.
$t = 3$:	$(1 + k)^3\, c$.	–	$x_3 = (1 + k)^3\, c$.

Here the investor receives in each period a rate of return k on the prior period's investment.

What if one invests an amount c now, then removes amount kc from the account in each period 1 through 3, and also receives the initial outlay c back in period 3? Then we can think of k as the constant per-period rate of return in the following way:

	Receive	Invest	Net cash flow
$t = 0$:	–	$c_0 = c$.	$x_0 = -c$
$t = 1$:	$(1 + k)\, c$.	$c_1 = c$.	$x_1 = kc$.
$t = 2$:	$(1 + k)\, c$.	$c_2 = c$.	$x_2 = kc$.
$t = 3$:	$(1 + k)\, c$.	–	$x_3 = (1+k)c$

Again, the investor receives in each period a rate of return k on the prior period's investment. And it is not hard to verify that k is an internal rate of return for the corresponding cash flow stream $\boldsymbol{x} = (-c, kc, kc, (1 + k)\, c)$.

This phenomenon is true in general: *If the constant per-period rate of return on an investment stream is k, then k will be an internal rate of return for the corresponding cash flow stream*. The converse holds as well: *If k is an internal rate of return for a cash flow stream* $\boldsymbol{x}$*, then there is an investment stream* *yielding* $\boldsymbol{x}$ *at constant per-period rate of return* k. We prove this claim after formally introducing the notion of *investment stream*.

INVESTMENT STREAMS

Technically, the *investment stream* is no more or less than the negative of what is known as the *unrecovered investment balance* stream (Bussey and Eschenbach 1992), the *project balance* (Teichroew et al. 1965a, 1965b), or the *capital invested* (Lohmann 1988). Our definition is as follows: We say that $\boldsymbol{c} = (c_0, c_1, \ldots, c_{T-1})$ is an *investment stream yielding cash flow stream* $\boldsymbol{x} = (x_0, x_1, \ldots, x_T)$, *at (constant per-period) rate of return* k if the following equations hold:

$$\begin{aligned} x_0 &= -c_0 \\ x_t &= (1+k)\cdot c_{t-1} - c_t \qquad t = 1, \ldots, T-1 \\ x_T &= (1+k)\cdot c_{T-1} \end{aligned} \tag{1}$$

In other words, $\boldsymbol{x}$ is the cash flow stream obtained when the investor sinks an amount c_t into a project at the end of each period $t < T$ and receives return $(1+k)\cdot c_t$ after period $t + 1$. The quantity c_t is the increment of capital invested at time t, and $(1+k)\cdot c_{t-1}$ is the increment of capital recovered at time t. We allow the quantities c_t to be negative as well. In this case, $-c_t$ is the increment of capital borrowed at time t and $(1+k)\cdot c_{t-1}$ is the increment of capital repaid at time t (Lohmann 1988).

Notice that the Eq. (1) can be written in vector form as

$$\boldsymbol{x} = -(\boldsymbol{c}, 0) + (1 + k)\cdot(0, \boldsymbol{c}). \tag{2}$$

The following result appears in Lohman (1988, equation (43)).

Theorem 1: If investment stream $\boldsymbol{c} = (c_0, \ldots, c_{T-1})$ yields $\boldsymbol{x} = (x_0, x_1, \ldots, x_T)$ at constant per-period rate of return k, then for $r \neq -1$,

$$PV(\boldsymbol{x}|r) = \frac{k-r}{1+r}\cdot PV(\boldsymbol{c}|r).$$

Proof: From Eq. (2) we obtain

$$
\begin{aligned}
PV(\boldsymbol{x}|r) &= -PV(\boldsymbol{c},0|r) + (1+k)PV(0,\boldsymbol{c}|r) \\
&= -PV(\boldsymbol{c}|r) + \frac{1+k}{1+r}PV(\boldsymbol{c}|r) \\
&= \left(-1 + \frac{1+k}{1+r}\right)PV(\boldsymbol{c}|r) = \frac{k-r}{1+r}PV(\boldsymbol{c}|r)
\end{aligned}
$$

QED

Theorem 1 has an intuitive interpretation: If we call $PV(\boldsymbol{c}|r)$ the *net investment*, then Theorem 1 states that the present value of a cash flow stream is the net investment times the rate of return $k - r$ in excess of the market rate, all discounted to the present. Next, we present the fundamental result of this section.

Theorem 2: The quantity k is an internal rate of return for a cash flow stream $\boldsymbol{x}$ if and only if there exists an investment stream $\boldsymbol{c}$ which yields $\boldsymbol{x}$ at constant per-period rate of return k.

Proof: If investment stream $\boldsymbol{c}$ yields $\boldsymbol{x}$ at constant per-period rate of return k, then k cannot equal -1, because with $k = -1$, Eq. (2) yields $x_T = 0$, a case we exclude by assumption. Then by Theorem 1, we have

$$PV(\boldsymbol{x} \mid r = k) = \frac{k-k}{1+k} \cdot PV(\boldsymbol{c} \mid r = k) = 0$$

so k is an internal rate of return for $\boldsymbol{x}$. Conversely, suppose k is an internal rate of return for $\boldsymbol{x}$. Then $k \neq -1$ since $PV(\boldsymbol{x}| \ r = -1)$ is undefined. The defining equation (1) for an investment stream $\boldsymbol{c}$ yielding $\boldsymbol{x}$ with rate of return k can be solved for $c_0, c_1, \ldots, c_{T-1}$ to obtain

$$
\begin{aligned}
c_0 &= -x_0 \\
c_1 &= -\left((1+k)\cdot x_0 + x_1\right) \\
c_2 &= -\left((1+k)^2 x_0 + (1+k)x_1 + x_2\right) \\
&\vdots \\
c_{T-1} &= -\left((1+k)^{T-1}\cdot x_0 + (1+k)^{T-2}\cdot x_1 + \cdots + (1+k)\cdot x_{T-2} + x_{T-1}\right) \\
c_{T-1} &= (1+k)^{-1}\cdot x_T.
\end{aligned}
\tag{3}
$$

The last two equations for c_{T-1} are consistent because the internal rate of return k by definition satisfies

$$(1+k)^{-1}\cdot x_T = -((1+k)^{T-1}\cdot x_0 + (1+k)^{T-2}\cdot x_1 + \ldots + (1+k)\cdot x_{T-2} + x_{T-1}).$$

Therefore, if k is an internal rate of return for $\boldsymbol{x}$, then the stream $\boldsymbol{c} = (c_0, \ldots, c_{T-1})$ given by Eq. (3) is an investment stream with constant per-period rate of return k yielding $\boldsymbol{x}$. QED

From Eq. (3), we see that in an investment stream $\boldsymbol{c}$, the t^{th} component c_t is actually the negative of the future value of the partial cash flow stream $(x_0, x_1,\ldots, x_t)$. Therefore, as we mentioned initially, an investment stream $\boldsymbol{c}$ yielding $\boldsymbol{x}$ at rate of return k is the negative of the unrecovered investment or project balance stream as it is usually defined.

As promised, Theorem 2 gives a rate-of-return characterization for internal rates: It states that for a cash flow stream $\boldsymbol{x}$, two distinct concepts are equivalent:

(1) the internal rate of return, equal to an interest rate which makes the present value of $\boldsymbol{x}$ equal to zero; and
(2) the constant per-period rate of return at which some investment stream yields $\boldsymbol{x}$.

PURE INVESTMENT AND BORROWING STREAMS

The theory of pure investment and borrowing streams follows easily from the results we have presented so far, and we pause briefly to present it. As we have mentioned, components c_t of an investment stream $\boldsymbol{c}$ may be negative. For an investment stream $\boldsymbol{c}$, a component $c_t < 0$ signifies that the investor *borrows* an amount $-c_t$ at time t from the project in question, and repays the loan at rate k one period hence. If one rules out this possibility, then one arrives at the notion of a *pure* investment (Bussey and Eschenbach 1992, Teichroew et al. 1965a, 1965b). If $\boldsymbol{c}$ is an investment stream corresponding to some cash flow stream $\boldsymbol{x}$, then we call $\boldsymbol{c}$ a *pure investment stream* if $\boldsymbol{c} \geq \mathbf{0}$ and $\boldsymbol{c}$ has at least one positive component. (This differs from the historical use of the terminology, in which the cash flow stream $\boldsymbol{x}$ is called the pure investment stream). Similarly, we may define a *pure borrowing stream* $\boldsymbol{c}$ as an investment stream having $\boldsymbol{c} \leq \mathbf{0}$ with at least one negative component. Here are the well-known results on pure investment (borrowing) streams.

Theorem 3: Suppose $\boldsymbol{x}$ is the yield of a pure investment or pure borrowing stream $\boldsymbol{c}$ at a proper internal rate of return k. Then k is the only proper internal rate of return for $\boldsymbol{x}$, and

a) If $\boldsymbol{x}$ is a pure investment stream, then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $k \geq r$
b) If $\boldsymbol{x}$ is a pure borrowing stream then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $k \leq r$.

Proof: Let $\boldsymbol{c}$ be the investment stream yielding $\boldsymbol{x}$ at rate of return k. From Theorem 1 we know

$$PV(\boldsymbol{x} | r) = \frac{k-r}{1+r} \cdot PV(\boldsymbol{c} | r).$$

If $\boldsymbol{c}$ is a pure investment stream, then because $\boldsymbol{c} \geq 0$ with at least one positive component, it follows that $PV(\boldsymbol{c} \mid r) > 0$ for any proper rate of return r. Therefore the only proper rate r which solves the equation $PV(\boldsymbol{x} \mid r) = 0$ is $r = k$. Moreover, because the function $r \to \dfrac{k-r}{1+r}$ is positive for proper values of r less than k and negative for proper values of r greater than k, the same may be said for $PV(\boldsymbol{x} \mid r)$ as a function of r. Therefore, $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $k \geq r$, as claimed. For pure borrowing streams, the proof is symmetric. QED

THE CONSISTENT USE OF MULTIPLE INTERNAL RATES

Investment streams $\boldsymbol{c}$ with both positive and negative components are known as *mixed* investments. The reader may note that in the proof above of Theorem 3, the restriction $\boldsymbol{c} \geq \boldsymbol{0}$ (respectively $\boldsymbol{c} \leq \boldsymbol{0}$) is stronger than what is needed. All the proof really requires is a restriction on the sign of $PV(\boldsymbol{c} \mid r)$. Accordingly, we distinguish two kinds of mixed investment streams. If the present value of the investment $PV(\boldsymbol{c} \mid r)$ is *positive* at the market rate of interest r, then at the net, the investor truly is investing funds in the project in question, and we call $\boldsymbol{c}$ a *net investment*. For net investments, the investor should like returns k on $\boldsymbol{c}$ *exceeding* the market rate r. However, if $PV(\boldsymbol{c} \mid r)$ is *negative*, then at the net, the investor is *borrowing* funds from the project, and we call $\boldsymbol{c}$ a *net borrowing*. An investor should like a net borrowing as long as the repayment rate k for the loan is *less than* the market rate r. This is the content of the following result.

Theorem 4: Suppose k is an internal rate of return for the cash flow stream $\boldsymbol{x}$, and let $\boldsymbol{c}$ be the investment stream yielding $\boldsymbol{x}$ at constant per-period rate of return k.
a) If $PV(\boldsymbol{c} \mid r) > 0$ (that is, $\boldsymbol{c}$ is a net investment) then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $k \geq r$.
b) If $PV(\boldsymbol{c} \mid r) < 0$ (that is, $\boldsymbol{c}$ is a net borrowing) then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $k \leq r$.
c) If $PV(\boldsymbol{c} \mid r) = 0$ then $PV(\boldsymbol{x} \mid r) = 0$.

This intuitive result follows immediately from Theorem 1. What may escape initial notice, however, is that the results hold for *any* of the internal rates of return k for $\boldsymbol{x}$. Each internal rate k is, by Theorem 2, a constant per-period rate of return on some investment stream $\boldsymbol{c}$ yielding $\boldsymbol{x}$. For those k for which the corresponding $\boldsymbol{c}$ is a net investment, one should accept the cash flow stream $\boldsymbol{x}$ if and only if the return on investment exceeds the market rate. For those k with corresponding $\boldsymbol{c}$ a net borrowing, one should accept $\boldsymbol{x}$ if and only the repayment rate k is below market. These acceptance rules will never conflict for different internal rates k, as they are all equivalent to the present value rule $PV(\boldsymbol{x} \mid r) \geq 0$. Therefore, *it does not matter which internal rate k one uses* as long as one examines the corresponding investment stream $\boldsymbol{c}$ and compares k to the market rate r in a manner consistent with the classification of $\boldsymbol{c}$ as a net investment or a net borrowing. In keeping with the term *mixed investment*, it is quite possible that for some internal rates k the corresponding $\boldsymbol{c}$ is a net investment, and for other internal rates k the corresponding $\boldsymbol{c}$ is a net borrowing, as the following examples illustrate.

EXAMPLE: HISTORICAL OBJECTIONS TO INTERNAL RATE OF RETURN

We consider the four hypothetical cash flow streams given in FIGURE 1. The first three were discussed by Hirshleifer (1958), and the fourth by Herbst (1978), as examples of cash flow streams for which internal rate of return is inadequate or misleading. FIGURE 1 also shows the net present values $PV_i = PV(\boldsymbol{x}^i \mid r)$ expressed as functions of the interest rate r. We take the market rate of interest to be $r = 10\%$, at which all four cash flows have negative present values. We see from FIGURE 1 that Project 1 has three internal rates of return $k = 0\%$, 100%, 200%, two of which exceed the market rate and one of which does not. Based on these alone, there is no way to discover that Project 1 is undesirable at the market rate of 10%. Project 2 has no internal rate of return, so no conclusion regarding economic desirability is evident. Project 3 has two internal rates of return both exceeding the market rate, but is nevertheless undesirable in present value terms. Project 4 has a unique proper internal rate of return 100%, but is nevertheless also undesirable in present value terms. For these reasons, internal rate of return was originally considered inappropriate to assist in economic evaluation of these projects.

	x_0	x_1	x_2	x_3	PV@10%
Stream 1	–1	6	–11	6	–0.128
Stream 2	–1	3	–2.5		–0.339
Stream 3	–1	5	–6		–1.413
Stream 4	–1	4	–4		–0.669

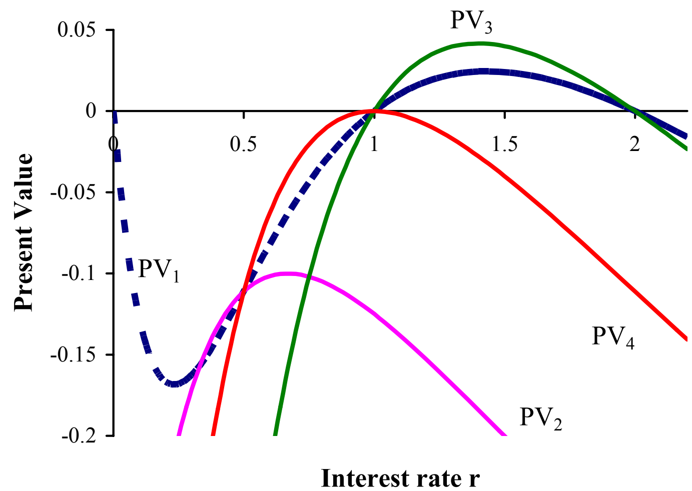

FIGURE 1: Present values of four cash flow streams as functions of the interest rate r. Cash flow stream 1 has multiple internal rates of return (0%, 100%, 200%), while cash flow stream 2 has no internal rate of return. Cash flow stream 3 has two internal rates of return (100%, 200%), both exceeding the market rate of 10%, but is not desirable. Stream 4 has unique proper internal rate of return 100% but is not desirable.

However, with our new perspective on internal rate of return, economic insight can indeed be gleaned. Consider Project 1 first. This cash flow stream has multiple internal rates of return, and can be obtained as the yield of multiple investment streams, one for each internal rate. We list them again, along with their present values at the market rate:

k	c_0	c_1	c_2	PV @ 10%
0%	1	–5	6	1.41
100%	1	4	3	–0.157
200%	1	–3	2	–0.0744

By checking the sign of its present value (1.41 > 0), we see that at internal rate k = 0%, the corresponding investment stream $\boldsymbol{c}$ is indeed a net investment.

However, since k = 0% falls short of the market rate, the project is undesirable. On the other hand, at internal rates k = 100% or k = 200%, the corresponding investment streams are net *borrowings* from the project, having present values –0.157 < 0 and –0.0744 < 0. Since the repayment rates k exceed the market rate, the project is undesirable. So regardless of which internal rate we use, the conclusion is the same and consistent with the present value evaluation at the market rate: Project 1 is undesirable.

Note also that these multiple internal rates are *meaningful, explainable and non-contradictory*: The cash flow stream $\boldsymbol{x}$ can be meaningfully regarded as the yield of the investment stream $\boldsymbol{c}$ = (1, –5, 6) at rate of return 0%, *and* as the yield of the borrowing stream $\boldsymbol{c}$ = (1, –4, 3) at repayment rate 100%, *and* as as the yield of the borrowing stream $\boldsymbol{c}$ = (1, –3, 2) at repayment rate 200%. There is no contradiction in these three interpretations, as they are all consistent with the present-value evaluation of $\boldsymbol{x}$ at the market rate.

Apparently, no such analysis can be carried out for Project 2, since it has no internal rate of return. We shall return to it below. The analysis for Project 3 is similar to that for Project 1. The investment streams and their present values are:

k	c_0	c_1	PV @ 10%
100%	1	–3	–1.73
200%	1	–2	–0.818

At the net, both investment streams are borrowings from the project, so are undesirable since the repayment rate k exceeds the market rate. Therefore Project 3 is undesirable. Project 4 runs into similar difficulties: Its investment stream for the unique internal rate k = 100% is $\boldsymbol{c}$ = (1, –2), which has present value –0.818 < 0. Therefore, Project 4 also acts like borrowing at above market rate, so is undesirable.

EXAMPLE: MINERAL EXTRACTION

Eschenbach (1995, Section 7.6) presents the following example. Consider the incremental cash flow stream $\boldsymbol{x}$ = (–4, 3, 2.25, 1.5, 0.75, 0, –0.75, – 1.5, –2.25) (in millions of dollars) obtained by adding oil wells to an oil field. The effect of adding wells is to shift production from the last three time periods to earlier periods. This incremental stream has two real internal rates of return, k = 10.4% and k = 26.3%, both exceeding the interest rate r = 5%. Yet its present value is $PV(\boldsymbol{x} \mid r = 5\%)$ = –\$0.338 million. This apparent paradox is easily resolved by calculating investment streams. We have:

k	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$PV(c \mid r)$
10.4%	4	1.417	−0.685	−2.256	−3.242	−3.58	−3.203	−2.037	−6.531
26.3%	4	2.052	0.342	−1.068	−2.098	−2.65	−2.598	−1.781	−1.665

Both investment streams $\boldsymbol{c}^{(1)}$ and $\boldsymbol{c}^{(2)}$ are net borrowings at rates k exceeding the interest rate r = 5%. Hence both cash flow streams are undesirable, a conclusion in accord with the present value calculation for the cash flow stream. The proper interpretation here is that the cash flow stream $\boldsymbol{x}$ is the yield of one borrowing stream $\boldsymbol{c}^{(1)}$ at repayment rate 10.4%, *and* is the yield of another borrowing stream $\boldsymbol{c}^{(2)}$ at repayment rate 26.3%. The analyst may safely choose either of these interpretations and neglect the other, as they will never give contradictory results.

It is instructive to consider what happens in this example if the market rate of interest takes on the higher value r = 12%. In this case, one internal rate exceeds the market rate and the other does not. But now calculations give $PV(\boldsymbol{c}^{(1)} \mid r = 12\%) = -3.523 < 0$, and $PV(\boldsymbol{c}^{(2)} \mid r = 12\%) = 0.386 > 0$, so $\boldsymbol{c}^{(1)}$ is a net borrowing at the below-market rate of 10.4%, and $\boldsymbol{c}^{(2)}$ is a net investment at the above-market rate of 26.3%. Either interpretation is allowable, and both yield the conclusion that the cash flow stream is desirable. This is consistent with present value, which gives $PV(\boldsymbol{x}|r = 12\%) = 0.0493 > 0$.

COMPARING COMPETING PROJECTS

It is well known that ranking cash flows of competing project by their internal rates of return may give results that conflict with net present value. The present-value formula of Theorem 1 makes it easy to see how this can occur: If competing cash flows $\boldsymbol{x}$ and $\boldsymbol{y}$ are the yields of investment streams $\boldsymbol{c}^x$ and $\boldsymbol{c}^y$ at respective internal rates k_x and k_y, then according the Theorem 1,

$$PV(\boldsymbol{x} \mid r) = \frac{k_x - r}{1+r} PV(\boldsymbol{c}^x \mid r)$$
$$PV(\boldsymbol{y} \mid r) = \frac{k_y - r}{1+r} PV(\boldsymbol{c}^y \mid r).$$

Clearly, if the net investments $PV(\boldsymbol{c}^x \mid r)$ and $PV(\boldsymbol{c}^y \mid r)$ are equal, then the present value ranking of $\boldsymbol{x}$ and $\boldsymbol{y}$ will be determined by the ranking of the internal rates k_x, k_y. On the other hand, if the net investments $PV(\boldsymbol{c}^x \mid r)$ and $PV(\boldsymbol{c}^y \mid r)$ are very different, then comparing the internal rates k_x, k_y will tell us little about the relative desirability of $\boldsymbol{x}$ and $\boldsymbol{y}$ in present value terms.

Consider, for example, two competing 5-year investments with cash flows

t	0	1	2	3	4	5
$\boldsymbol{x}$	–20	14	10	6	2	-2
$\boldsymbol{y}$	–20	-6	1.1	8.2	15.3	22.4

discussed in Example 9.2 of Eschenbach (1995). These conventional investments have unique proper internal rates $k_x = 28.3\%$, $k_y = 16.0\%$. Does this indicate that $\boldsymbol{x}$ is nearly twice as good as $\boldsymbol{y}$? No, because the net investment $PV(\boldsymbol{c}^x \mid r) = 33.87$ for $\boldsymbol{x}$ is less than one-third the net investment $PV(\boldsymbol{c}^y \mid r) = 109.24$ for $\boldsymbol{y}$. Net present values at $r = 10\%$ are $PV(\boldsymbol{x} \mid r) = 5.624$, $PV(\boldsymbol{y} \mid r) = 5.974$, so ranking by net present value conflicts with ranking by internal rate of return. The larger rate of return for project $\boldsymbol{x}$ is not enough to compensate in present value terms for the smaller net investment.

If two project cash flows $\boldsymbol{x}$, $\boldsymbol{y}$ each have multiple internal rates, it is important to note that the remarks above apply to *any* of the multiple internal rates k_x, k_y of $\boldsymbol{x}$, $\boldsymbol{y}$. If there are internal rates k_x, k_y corresponding to the same or approximately the same net investments then the analyst may properly single out k_x and k_y to compare – the results will be the same or approximately the same as comparing present values. If there is a third project cash flow $\boldsymbol{z}$ to compare with $\boldsymbol{x}$, the analyst may properly compare a *different* internal rate k_x' for $\boldsymbol{x}$ with an internal rate k_z for $\boldsymbol{z}$, if they correspond to approximately the same net investments. The results will be consistent with net present value.

Consider, for example, Projects 3 and 4 in our initial example above, with cash flows $\boldsymbol{x}^{(3)} = (-1, 5, -6)$ and $\boldsymbol{x}^{(4)} = (-1, 4, -4)$, and multiple internal rates $k_3 = 100\%$, $k_3' = 200\%$ for $\boldsymbol{x}^{(3)}$ and $k_4 = 100\%$ for $\boldsymbol{x}^{(4)}$. Neither project is desirable, but if we are forced to rank them, we can take advantage of the fact that k_3' and k_4 each correspond to the same net investment $PV(\boldsymbol{c}^{(3)\prime} \mid r) = PV(\boldsymbol{c}^{(4)} \mid r) = -0.818$. (Indeed, $\boldsymbol{c}^{(3)\prime} = \boldsymbol{c}^{(4)} = (1, -2)$.) Since these investment streams are net borrowings, the smaller internal rate $k_4 = 100\%$ is better, and Project 4 is preferred, a conclusion consistent with ranking by net present value. Note that it would not be valid to compare k_3 and k_4, as these correspond to different net investments.

COMPLEX-VALUED INTERNAL RATES OF RETURN

Surprisingly, the procedure of Theorem 4 for handling multiple real internal rates of return extends in the natural way to complex-valued internal rates of return as well. This unexpected conclusion arises as a consequence of the fact that neither the definition (1) for investment stream, nor the Eqs. (3) for obtaining the investment stream $\boldsymbol{c}$ depend on the internal rate k being real-valued. Complex-valued internal rates k work equally well, as long as one is willing to accept complex-valued investment streams $\boldsymbol{c}$. Not only that, but Theorems 1 and 2 hold as well for the case of complex-valued internal rates k.

In the following, $\mathrm{Re}(k)$ denotes the real part of the possibly complex-valued internal rate k, and $\mathrm{Re}(\boldsymbol{c})$ denotes the real part of the possibly complex-valued investment stream $\boldsymbol{c}$.

Theorem 5: Suppose k is a (possibly complex-valued) internal rate of return for $\boldsymbol{x}$, and let $\boldsymbol{c}$ be the corresponding (possibly complex-valued) investment stream yielding $\boldsymbol{x}$ at return k.

a. If $PV(\mathrm{Re}(\boldsymbol{c}) \mid r) > 0$ (that is, $\mathrm{Re}(\boldsymbol{c})$ is a net investment) then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Re}(k) \geq r$.
b. If $PV(\mathrm{Re}(\boldsymbol{c}) \mid r) < 0$ (that is, $\mathrm{Re}(\boldsymbol{c})$ is a net borrowing) then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Re}(k) \leq r$.

The proof of this result may be found in the APPENDIX. For simplicity here, we have omitted the case $PV(\mathrm{Re}(\boldsymbol{c}) \mid r) = 0$, which is more involved and also less important due to its rarity. However, the interested reader may find this case discussed in the appendix as well.

What is remarkable about Theorem 5 is it has nearly the same form as its real-valued counterpart Theorem 4, with $\mathrm{Re}(k)$ replacing k and $\mathrm{Re}(\boldsymbol{c})$ replacing $\boldsymbol{c}$. This is so in spite of the fact that $\mathrm{Re}(k)$ need not be an internal rate of return for $\boldsymbol{x}$ when the complex-valued k is, and $\mathrm{Re}(\boldsymbol{c})$ need not be an investment stream for $\boldsymbol{x}$ when the complex-valued $\boldsymbol{c}$ is.

EXAMPLE: NO REAL-VALUED RATE OF RETURN

We return to Project 2 in FIGURE 1. For its cash flow stream $\boldsymbol{x} = (-1, 3, -2.5)$, we obtain two complex-valued internal rates of return $k = 0.5 \pm 0.5i$. The corresponding complex-valued investment streams are $\boldsymbol{c} = (1, -1.5 \pm 0.5i)$. We have $PV(\mathrm{Re}(\boldsymbol{c}) \mid r = 10\%) = -0.364 < 0$, so $\mathrm{Re}(\boldsymbol{c})$ is a net borrowing and by Theorem 5, $\boldsymbol{x}$ will be desirable when $\mathrm{Re}(k) \leq r$. However, the real part of k is 50%, which exceeds the market rate 10%, so $\boldsymbol{x}$ is undesirable. This is consistent with the present value calculation $PV(\boldsymbol{x} \mid r = 10\%) = -0.339 < 0$. So even though all internal rates are complex-valued in this case, one can still use them to make an economic recommendation, albeit in a mechanical fashion. We are currently unaware of an economic interpretation of complex-valued rates of return or complex-valued investment streams, and without such an interpretation, it would be hard to justify any economic recommendation without resort to other performance measures such as present value.

EXAMPLE: REAL AND COMPLEX INTERNAL RATES

The cash flow stream $\boldsymbol{x}$ = (500, 0, −1000, 250, 250, 250) is considered by Sullivan et al. (2000, Example 4-A-1, p. 186). It has five distinct internal rates of return, two of which are complex-valued. These rates k and their corresponding investment streams $\boldsymbol{c}$ are as follows:

k	c_0	c_1	c_2	c_3	c_4	$PV(\mathrm{Re}(\boldsymbol{c})\|r)$
−1.618	−500	1309	−809.0	250	−404.5	−67.05
−1.149 + 0.603i	−500	1074 − 301.4i	22.075 + 692.4i	−670.7 − 89.565i	−96.365 − 391.0i	−74.82
−1.149 −0.603i	−500	1074 + 301.4i	22.075 − 692.4i	−670.7 + 89.565i	−96.365 + 391.0i	−74.82
0.618	−500	191.0	309.0	250	154.5	222.4
0.297	−500	351.4	455.8	341.3	192.7	584.3

The first three internal rates have corresponding investment streams whose real parts are net borrowings at the market rate r = 10%. Because Re(k) < 10% for these k, Theorem 5 indicates the project is desirable. The same conclusion results (as it must) from examining the last two internal rates, which are real and proper and whose investment streams are net investments at the market rate. Therefore, because these rates k exceed the market rate, the project is desirable. Of course, we present these results for expository purposes − there is really no point in performing five separate IRR analyses, since they must all give the same result. Alternately, we could have confirmed the result by simply calculating the present value $PV(\boldsymbol{x} \mid r = 10\%) = 104.72 > 0$.

INTERPRETING MULTIPLE INTERNAL RATES

According to Theorem 2, every internal rate of a cash flow stream can be regarded as a rate of return on some investment stream. But how, the reader may ask, can a cash flow stream *meaningfully* have several different rates of return? For instance, in the last example, leaving aside the complex roots, there are still three different internal rates k = −161.8%, 61.8%, and 29.7%. In what meaningful sense can these *all* be regarded as rates of return for the cash flow stream?

The short answer is: Just as it is not generally meaningful to compare internal rates of return *between* mutually exclusive projects, it is also not meaningful to compare internal rates of return *within* a single project. All that should matter about an internal rate of return is whether it exceeds the market rate r. The *magnitude* of the internal rate by itself carries no further information.

To elaborate on this point, the formula in Theorem 1 substantiates the widely accepted maxim that internal rates should not in general be compared between projects because they may be rates of return on different investment streams. The same advice holds for different internal rates *within the same project*, as they will also correspond to different investment streams. In the last example, the 61.8% internal rate is the rate of return on an investment stream having present value \$222.4, whereas the internal rate 29.7% is the rate of return on an investment stream having the higher present value \$584.3. Because both investment streams yield the *same* cash flow stream $\boldsymbol{x}$ at their internal rates, it only makes sense that the higher net investment must have a lower rate of return. *So the fact that one internal rate has greater magnitude than another for the same cash flow only indicates that the corresponding net investment has smaller magnitude. There are no other economic implications.*

To further emphasize this point, consider a more meaningful comparison of internal rates, a comparison that involves *both* the internal rate k and its investment stream $\boldsymbol{c}$, namely, the discounted net return $(1 + k)\, PV(\boldsymbol{c} \mid r)/(1 + r)$ less the net investment $PV(\boldsymbol{c} \mid r)$. Here is this comparison for the three internal rates in the last example:

Internal rate k	Net Investment $PV(\boldsymbol{c} \mid r)$	Discounted Net Return $(1+ k)\, PV(\boldsymbol{c} \mid r)/(1 + r)$	Difference
61.8%	\$222.4	\$327.1	\$104.7
29.7%	\$584.3	\$688.9	\$104.6
−161.8%	−\$67.1	\$37.7	\$104.7

We see that the difference is *identical* (modulo rounding error) for all three internal rates. In fact, the difference is *equal to the present value of the cash flow stream* $\boldsymbol{x}$. That this conclusion will always hold is an immediate consequence of Theorem 1, as

$$\frac{(1+k)PV(\boldsymbol{c}|r)}{1+r} - PV(\boldsymbol{c}|r) = \left(\frac{1+k}{1+r} - 1\right)PV(\boldsymbol{c}|r)$$
$$= \frac{k-r}{1+r}PV(\boldsymbol{c}|r) = PV(\boldsymbol{x}|r).$$

The point is that meaningful comparison of different internal rates, either across projects or within a project, should incorporate the associated investment streams, and when this is done, the comparison reduces to examining present value of the cash flow streams. For internal rates within a project, there is only one cash flow stream to examine, and the conclusion is that all internal rate/ investment stream combinations are equivalent.

By implication, there is no need to discard "unreasonable" or "extreme" internal rates – all are equally valid. For instance, the incremental cash flow $\boldsymbol{x}$ = (−1600, 10^4, -10^4) for the Lorie–Savage "pump problem" (e.g., Chapter 7 in Bussey and Eschenbach 1992) has two internal rates k_1 = 25% and k_2 = 400%. The latter value k_2 seems extreme, but its large size compared to k_1 in fact *reflects only a small corresponding net investment*. The two investment streams in this case are $\boldsymbol{c}^1$ = (1600, −8000) and $\boldsymbol{c}^2$ = (1600, −2000), with corresponding net investments $PV(\boldsymbol{c}^1 \mid r)$ = −5673 and $PV(\boldsymbol{c}^2 \mid r)$ = −218.2 at r = 10%. Both investment streams are net borrowings at above–market rates; hence $\boldsymbol{x}$ is undesirable. This conclusion can be reached using either of the internal rates, which are both valid.

CONCLUSION

Let the reader not misunderstand our purpose: We do not here advocate the use of IRR in preference to other methods such as net present value. Computationally, the procedure we presented above is rather roundabout: Solve for an internal rate of return, compute the corresponding investment stream, check the sign of its net present value to determine whether it is a net investment or net borrowing, then treat the internal rate of return accordingly. Why not instead simply compute the NPV of the original cash flow stream and be done with it? Obviously, the latter is much easier, both conceptually and computationally.

Our point is merely this: The problem of multiple or nonexistent internal rates of return – universally regarded as a fatal flaw for the IRR method – is not really a flaw at all, and can be easily dealt with conceptually and procedurally.

Procedurally, we provide a method for handling multiple internal rates of return that is consistent with net present value. Conceptually, our insight is that a cash flow stream can always be interpreted as the yield of any of several investment streams at several different internal rates. Which investment stream/ internal rate combination to use does not matter as long as the investment stream is properly categorized as a net investment or net borrowing. The analyst may choose to emphasize any one internal rate to the neglect of the others, as each will lead to conclusions consistent with net present value. For analysts who are institutionally or personally committed to using internal rates of return, perhaps to supplement the conclusions from NPV or other measures, our results provide a coherent, consistent conceptual and computational approach.

The impact of this new perspective on related methods such as external rate of return is, we think, slight. There is also no change in the advice that mutually exclusive projects must be compared on an incremental basis, and our IRR procedure applies in that setting, as the mineral extraction example above illustrates. Moreover, our results provide an intuitive understanding of when this incremental approach may be bypassed in favor of direct comparison of internal rates.

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APPENDIX: PROOFS

We prove here a broader version of Theorem 5, extended to include the case in which the present value of the real part of the investment stream is zero. Theorem 4, which deals with real internal rates only, follows as a special case. We begin with a key lemma.

Lemma: Suppose u and v are complex numbers with $\mathrm{Im}(uv) = 0$. Then:
(a) If $\mathrm{Re}(v) > 0$, then $uv \geq 0 \Leftrightarrow \mathrm{Re}(u) \geq 0$.
(b) If $\mathrm{Re}(v) < 0$, then $uv \geq 0 \Leftrightarrow \mathrm{Re}(u) \leq 0$.
(c) If $\mathrm{Re}(v) = 0$ and $\mathrm{Im}(v) > 0$, then $uv \geq 0 \Leftrightarrow \mathrm{Im}(u) \leq 0$.
(d) If $\mathrm{Re}(v) = 0$ and $\mathrm{Im}(v) < 0$, then $uv \geq 0 \Leftrightarrow \mathrm{Im}(u) \geq 0$.

Proof: Suppose $u = a + bi$, $v = c + di$. Then $uv = \mathrm{Re}(uv) = ac - bd$, and $\mathrm{Im}(uv) = ad + bc$. Moreover, because $\mathrm{Im}(uv) = 0$, we have

$$ad + bc = 0 \tag{1}$$

Consider part (a). We have $c = \mathrm{Re}(v) > 0$. Then

$$\begin{aligned} uv \geq 0 &\Leftrightarrow ac - bd \geq 0 & \\ &\Leftrightarrow ac^2 - bcd \geq 0 & \text{because } c > 0 \\ &\Leftrightarrow ac^2 + ad^2 \geq 0 & \text{using (1)} \\ &\Leftrightarrow a(c^2 + d^2) \geq 0 & \\ &\Leftrightarrow a \geq 0. & \end{aligned}$$

Because $a = \mathrm{Re}(u)$ this completes the proof of (a).

The proof of (b) is symmetric. To show (c), suppose $c = \mathrm{Re}(v) = 0$ and $d = \mathrm{Im}(v) > 0$. Then

$$\begin{aligned} uv \geq 0 &\Leftrightarrow ac - bd \geq 0 & \\ &\Leftrightarrow bd \leq 0 & \text{since } c = 0 \\ &\Leftrightarrow b \leq 0 & \text{since } d > 0. \end{aligned}$$

Because $b = \mathrm{Im}(u)$, this completes the proof of (c). The proof of (d) is symmetric. QED

Theorem 5′: Suppose k is a (possibly complex–valued) internal rate of return for $\boldsymbol{x}$, and let $\boldsymbol{c}$ be the corresponding (possibly complex–valued) investment stream. Let r be a proper interest rate.

a. If $PV(\mathrm{Re}(\boldsymbol{c}) \mid r) > 0$, then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Re}(k) \geq r$.
b. If $PV(\mathrm{Re}(\boldsymbol{c}) \mid r) < 0$, then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Re}(k) \leq$ r.
c. If $PV(\mathrm{Re}(\boldsymbol{c})\mid r) = 0$ and $PV(\mathrm{Im}(\boldsymbol{c}) \mid r) > 0$, then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Im}(k) \leq 0$.

d. If $PV(\mathrm{Re}(\boldsymbol{c})|\ r) = 0$ and $PV(\mathrm{Im}(\boldsymbol{c}) \mid r) < 0$, then $PV(\boldsymbol{x} \mid r) \geq 0$ if and only if $\mathrm{Im}(k) \geq 0$.

Proof: From Theorem 1 we have

$$PV(\boldsymbol{x} \mid r) = \frac{k-r}{1+r}.PV(\boldsymbol{c} \mid r).$$

Let $u = \dfrac{k-r}{1+r}$ and $v = PV(\boldsymbol{c} \mid r)$. Then

$$uv = PV(\boldsymbol{x} \mid r)$$

$$\mathrm{Re}(u) = \frac{\mathrm{Re}(k)-r}{1+r} \qquad \mathrm{Im}(u) = \frac{\mathrm{Im}(k)}{1+r}$$

$$\mathrm{Re}(v) = PV(\mathrm{Re}(\boldsymbol{c}) \mid r).$$

Because $PV(\boldsymbol{x} \mid r)$ is real-valued, we have $\mathrm{Im}(uv) = 0$, and because r is proper, we have $1 + r > 0$. Therefore $\mathrm{Re}(u) \geq 0$ if and only if $\mathrm{Re}(k) \geq$ r, and $\mathrm{Im}(u) \leq 0$ if and only if $\mathrm{Im}(k) \leq 0$. The claims of the theorem now follow by applying the lemma. QED.

BIOGRAPHICAL SKETCH

GORDON B. HAZEN (HAZEN@IEMS.NORTHWESTERN.EDU) is professor of Industrial Engineering and Management Sciences at Northwestern University, where he teaches an economics/ decision analysis course that motivated this article. He received his B.S. in mathematics from the University of North Dakota, his M.S. in statistics from Purdue University, and his Ph.D. in industrial engineering from Purdue University. His research interests include decision analysis, utility and preference theory, and medical decision-making. He currently serves as associate editor for *Operations Research* and *Naval Research Logistics*, and is a member of the editorial board of *Medical Decision Making*.
