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Path Planning in an Anisotropic Medium

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Abstract

Many of the optimal path finding problems studied to-date are restricted to a direction-independent metric. In this paper we discuss path planning in an anisotropic medium illustrated by the fastest-path problem where speed is direction-dependent. Such problems arise in vessel routing, robotics, and aircraft navigation, where the agent's speed is affected by the direction of waves, winds or slope of the terrain. The difficulty of optimal-path finding in a direction-dependent medium comes from the fact that our travel-time function is asymmetric, and in general, violates the triangle inequality. We present an analytical form solution for the fastest-path finding problem in an obstacle-free domain without making any assumptions on the structure of the speed function. Subsequently, we merge these results with *visibility graph search* methods to develop an obstacle-avoiding fastest-path finding algorithm for a direction-dependent speed function. Our results provide computationally fast techniques for finding a closed form solution to a very large class of applied problems.

1 Introduction

In this paper, we address a broad class of optimal path finding in anisotropic environment problems where the cost function is direction-dependent. For ease of exposition, we focus our discussion on fastest-path finding problems for direction-dependent speed functions; however, our analysis and results can be easily extended to any anisotropic cost function. We are given the points of origin and destination, and time and space homogenous speed function of heading. Our objective is to find a path that minimizes the total travel time. Problems of optimal path finding in an obstacle-free domain as well as in the presence of polygonal obstacles are addressed.

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The difficulty of optimal-path finding in an anisotropic medium comes from the fact that our travel-time function is asymmetric; that is, the time it takes to travel along a straight line path from a to b does not necessarily equal the time required to traverse the reversed path ba. Therefore, our cost function is not a metric, which prevents us from using more traditional and established approaches to solving optimal-path finding problems. Furthermore, the anisotropic cost, in general, violates the triangle inequality, which is another key property exploited in Euclidean shortest-path finding problems. In particular, it is not guaranteed that one of the 'taut-string' paths will be an optimal obstacle-avoiding path. Thus, the traditional approach of searching among a finite number of taut-string paths may fail to deliver an optimal solution.

1.1 Related Work

Optimal path planning problems have been studied for a very long time. However, the majority of the work to date concentrates on determining Euclidean shortest paths (see an extensive survey by Mitchell [16]). Even though a number of extensions to optimal path planning have been considered (e.g., traversing through polygonal constantly-weighted regions in [18, 27]), most work is restricted to isotropic metrics, where the cost function is assumed to be independent of the traveling direction. Some shortest path finding problems discussed in the literature [28, 15] introduce direction dependency by restricting the feasible paths to a fixed set of orientations; however the resulting cost function retains its metric properties.

Optimal path finding problems in anisotropic media have been addressed for a few specific applications, however the solution approach and results are often customized to the application at hand. Furthermore, the presence of obstacles is not commonly addressed in the published studies. For example, [25, 24] study optimal path finding for a mobile agent (e.g., robot or vehicle) across hilly terrains, where a simple and specific physical model of friction and gravity forces is used to compute the anisotropic cost function for the agent.

In the area of optimal yacht sailing, [21] created a mathematical programming model that evaluates the vessel speed for a specified range of wind speeds and yacht heading angles. The resulting velocity prediction data is used to find the yacht fastest path by applying dynamic programming algorithms [1, 22, 20]. Alternatively, Sellen [26] studies the optimal sailing routing problems for a more abstract scenario, and presents results similar to ours by heuristically arguing that an optimal path in an obstacle-free domain consists of at most two line segments. He also introduces a set of polygonal obstacles and extends his discussion to this restricted domain. However, Sellen's analysis is limited to problems with very specific speed functions represented by piecewise-linear reciprocal functions (i.e., for a direction-dependent speed function denoted by $V(\theta)$, the function $1/V(\theta)$ is assumed to be piecewise-linear). Unlike in the aforementioned work, we make absolutely no assumptions on the structure of the speed function, and find closed form solutions for *any* time and space homogeneous medium.

Some researchers have employed the calculus of variations and optimal control theory for optimal vessel routing problems. References [9, 10, 19] employ Euler's equations to characterize an optimal path; while [12] establishs an analogy between a traveling light ray and an optimal path seeking sailboat, and extend the use of optical notions such as Fermat's principle, Huygens' principle and Hamilton's optics to sailing strategies. These optimal-path finding methods reduce to solving systems of differential equations, which can present a difficult and challenging task. Moreover, researchers typically use a simplified form of the speed function in order to make the analysis more manageable. From our experience of working on vessel routing problems [8], it is clear that analytical functions cannot accurately describe vessel movement through waves, thus obliging us to look for alternative methods to solve the problem.

Reif and Sun [23] investigate a problem of time-optimum movement planning through a set of polygonal regions, where anisotropy is introduced as a uniform flow assigned to each region. The actual velocity of an object is defined to be the sum of a flow vector and a chosen control velocity. While the resulting speed function does display the direction-dependent property, its structure is very specific, and Reif and Sun's analysis does not extend to more general problems addressed in this paper.

In the most recent work on anisotropic movement, [4] generalizes the problem studied by Reif and Sun, and looks at shortest path finding in anisotropic regions where the direction-dependency of the speed is not restricted to the effect of the uniform flow. However, Cheng et al. still limit their research to the speed function with a very specific structure, referred to as a 'convex distance function' (first discussed by [5]). Their convex distance function is equivalent to our case of a convex *linear path attainable region*, however the results presented in our work subsequently relax the convexity assumption and deliver a closed form fastest path among obstacles for a general anisotropic speed function. In addition, we provide rigorous proofs previously absent in the published work on convex distance functions.

1.2 Overview of the Results

This paper presents an analytical form solution to the fastest-path finding problem for any given anisotropic speed function. We demonstrate that an optimal path in a general obstacle-free, time and space homogeneous medium is piecewise-linear with at most two line segments (i.e., one waypoint). Consequently, we merge these results with the visibility graph search methods developed for Euclidean shortest path problems [13, 2], to develop an obstacle-avoiding fastest-path finding algorithm for an anisotropic speed function. Our results provide computationally fast techniques for finding a closed form solution to the very large class of applied problems discussed earlier.

While our main results make no assumptions about the structure of the speed function, we first consider a special case of the problem where the speed polar plot (or the *linear path attainable region*) encloses a convex region. This restricted scenario provides important insight and intuition to the structure of an optimal path for the more general case. Subsequently, we relax the convexity assumption to consider a case for a very general speed function. One of our main results is presented in Theorem 10, which characterizes a fastest path for an arbitrary speed function in an obstacle-free domain. Algorithm 1 describes a step-by-step procedure to construct such an optimal path. In addition to characterizing a fastest path, we also compute a bound on the improvement in travel time were one to choose to follow an optimal path as opposed to traversing the simpler linear path between the two points. This bound is an important tool for evaluating tradeoffs, as well as for proving our key theorem.

We employ our findings for fastest path in an obstacle-free domain to the problems that consider the presence of polygonal obstacles. For the speed functions corresponding to convex linear path attainable regions, the straight line path is a fastest path in $\mathbb{R}^2$, and the triangle inequality holds true in an obstacle-free domain. Consequently, fastest-path finding in a polygonal domain can be restricted to a modified visibility graph, similarly to Euclidian shortest-path finding problems. The triangle inequality might not hold true for a general speed function. In that case, an augmented speed function corresponding to the convex hull of the original speed polar plot is used to find a lower bound on the minimum travel time for our problem. We use the results for an optimal path in the obstacle-free domain to construct an obstacle-avoiding path that achieves this lower bound, thus establishing its optimality.

The rest of the paper is organized as follows. Subsection 2 provides the key notation used throughout the paper and gives a more rigorous statement of the problem. Section 3 develops and presents fastest paths for an anisotropic speed function in an obstacle-free domain. This section includes the analysis for a convex linear path attainable region (Subsection 3.1); the construction of a bound on the optimal travel time for the general speed function (Subsection 3.3); and the later employment of this bound to prove Theorem 10 that characterizes a fastest path in a general anisotropic medium (Subsection 3.4). In Subsections 3.1 through 3.4, we assume that a speed function takes on only positive values, where as in Subsection 3.5 we discuss the problem of feasibility and fastest paths for the case where speed can be zero for some headings, such as in the cases of stalling or infeasible headings. Subsection 3.6 concludes the section with the description of Algorithm 1 that facilitates the implementation of the presented results.

The following Section 4 extends our analysis and results to the obstacle-avoiding fastest-path finding problems in anisotropic domain. Similarly to our discussion of the obstacle-free domain, we first find an optimal path for the case of a convex linear path attainable region (Subsection 4.1) and then relax the convexity assumption to find a path for an arbitrary speed function (Subsection 4.2). Algorithms 2 and 3 describe the fastest-path finding procedures corresponding to each of the cases.

Section 5 concludes this paper with an example of application of our results to the vessel routing problem and summarizes the findings and contribution of the presented work.

2 Notation and Problem Statement

In this section, we introduce the notation and a precise description of the fastest-path finding problem that we analyze in this paper.

The problem of interest is to find a fastest path from one given point to another for a direction-dependent speed function. We consider two separate scenarios of the problem: (i) an obstacle-free domain where all the feasible paths must lie in $\mathbb{R}^2$; and (ii) a domain containing a set of polygonal obstacles that must be avoided. We are given a direction-dependent speed function, which characterizes the movement within the domain. The speed is assumed to only depend on the heading direction, implying a time and space homogeneous domain (with the exception of obstacles). Next, we introduce the notation to be used throughout the paper.

Let $\mathcal{P}$ denote a set of open polygonal obstacles, such that their closures do not intersect, or in other words, the distance between any two obstacles is assumed to be greater than zero. Note that since each obstacle is assumed to be an open set, the movement along its edges is permitted. All the feasible paths, including a starting points s and a target point t, are assumed to lie in the *free space*, denoted by $\mathcal{F}$, which we define as the compliment of the obstacles, or $\mathcal{F} := \mathbb{R}^2 \backslash \mathcal{P}$. For consistency, we use this notation and terminology for both aforementioned scenarios, including the obstacle-free case where we set $\mathcal{P} = \emptyset$.

We define P_{st} to be the set of all continuous and rectifiable feasible paths from the start point s to the target point t. That is, $P_{st} = \{p : [0,1] \to \mathcal{F}$ such that $p(0) = s,\ p(1) = t,\ p$ is continuous and rectifiable$\}$. Then, for any $p \in P_{st}$, let $t(p)$ denote the travel time required to traverse the path p.

Let $V(\theta)$ for $0 \leq \theta \leq 2\pi$ denote the maximum attainable speed for a given heading θ. Unless otherwise specified, we assume that $V(\theta) > 0$ for all $\theta \in [0, 2\pi]$. Allowing the speed function to take on a value of zero for some headings might result in an infeasible problem. Consequently, this case requires special attention and is discussed separately in Section 3.5. It is worth noting, that time and space homogeneous nature of our problem, allows us to assume, without loss of optimality, that one always travels at the *maximum* attainable speed, since voluntary speed reduction would never result in a faster path.

We define $L_\delta(x)$ to be the *linear path attainable region* (LPAR) for a given point $x \in \mathbb{R}^2$ and time $\delta > 0$. That is, $L_\delta(x)$ is the set of all points that can be reached in a fixed time period, $\delta > 0$, from point x along a straight line path. In other words, $L_\delta(x) = \{y \in \mathbb{R}^2 : \|y - x\| \leq \delta V(\theta_{y-x})\}$, where θ_{y-x} and $\|y - x\|$ denote the angle and length of a vector $y - x$, respectively. Note that $V(\theta)$ uniquely defines $L_\delta(x)$ for a given x and δ, and vise versa. In the presence of obstacles (i.e., $\mathcal{P} \neq \emptyset$) it is assumed that $x \notin \mathcal{P}$, and δ is too small to reach any obstacles from x. An alternative way to define $L_\delta(x)$, is to introduce an *elementary* LPAR, L, uniquely defined by $V(\theta)$ as $L = \{y \in \mathbb{R}^2 : \|y\| \leq V(\theta_y)\}$. Then from time and space homogeneity we have $L_\delta(x) = x + \delta L$. Note that L is equivalent to a region enclosed by a polar graph of the speed function $V(\theta)$, and is often referred to as the 'speed polar plot' in some literature.

Let $A_\delta(x)$ be the *attainable region* (AR). That is, $A_\delta(x)$ is the set of all points that can be reached in a fixed time period, $\delta > 0$, from point $x \in \mathbb{R}^2$. We can give a more precise definition of $A_\delta(x)$ as follows, $A_\delta(x) = \{y : \exists p \in P_{xy} \text{ such that } t(p) \leq \delta\}$. Note that in the definition of $A_\delta(x)$, we do not restrict a path to be the straight line path, that is, $A_\delta(x)$ represents the set of all points that can be reach in time δ following *any* rectifiable path from point x. Similarly to the definition of LPAR, we assume that $x \notin \mathcal{P}$, and δ is too small to reach any obstacles from x, if $\mathcal{P} \neq \emptyset$.

Finally, we let the function $\tau(x, y) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ denote the travel time from point x to point y following the straight line path connecting these two points. We assume that $\tau(x, y)$ is only defined if the straight line segment xy does not intersect the set of obstacles $\mathcal{P}$. Then $\tau(x, y) = \min\{\delta : y \in L_\delta(x),\ \delta > 0\}$. The value of $\tau(x, y)$ can be also computed explicitly using the speed function $V(\theta)$ as $\tau(x, y) = \|y - x\|/V(\theta_{y-x})$. Note that $\tau(x, y)$ is not well defined if $V(\theta_{y-x}) = 0$, in which case we set $\tau(x, y) = \infty$.

Now, we can give the formal statement of our problem.

Problem statement: For a given speed function $V(\theta) : [0, 2\pi] \to \mathbb{R}^+$, a starting point $s \in \mathcal{F}$, and a target point $t \in \mathcal{F}$, find a fastest path from s to t that lies in $\mathcal{F}$. That is, our objective is to find $p^* \in P_{st}$ such that $t(p^*) \leq t(p)$ for all $p \in P_{st}$.

3 Fastest-Path Finding for an Anisotropic Speed Function in an Obstacle-Free Domain

In this section we study the fastest-path finding problems in an obstacle-free domain, that is, $\mathcal{P} = \emptyset$ and $\mathcal{F} = \mathbb{R}^2$ for the entire Section 3.

3.1 Fastest Path for a Convex Linear Path Attainable Region

We first analyze a problem restricted to the convex linear path attainable region (LPAR), as this special case gives an intuitive and insightful analysis of the problem, and it then can be extended to a general case. Therefore, throughout Section 3.1, we assume that $L_\delta(x)$ is convex for all x and δ; in other words, the convex combination of any two points in the set $L_\delta(x)$ is contained by that set (see Figure 1). Note that from the time and space homogeneity of the speed function $V(\theta)$, we know that if $L_\delta(x)$ is convex for some specific x and δ, then it is convex for all $x \in \mathbb{R}^2$ and all positive δ.

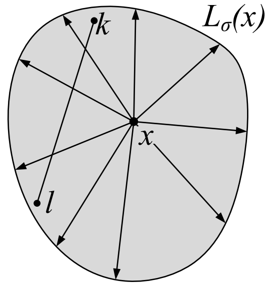

Figure 1: An example of a convex linear path attainable region $L_\delta(x)$.

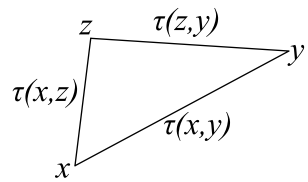

Figure 2: Illustration of the inequality from Lemma 1, $\tau(x, y) \leq \tau(x, z) + \tau(z, y)$.

Let $m(x)$ denote the smallest non-negative scalar such that $L_{m(x)}(0, 0)$ contains point x. We can also write $m(x) := \inf\{r : \frac{x}{r} \in L_1(0, 0), r > 0\}$. Observe that since $L_1(0, 0)$ is a closed set, the infimum is achieved and the definition can we rewritten as $m(x) := \min\{r : \frac{x}{r} \in L_1(0, 0), r > 0\}$ as long as $x \neq (0, 0)$. Also, note that since $L_1(0, 0) = L_{m(x)}(0, 0)/m(x)$ is a convex set in $\mathbb{R}^2$ and $(0, 0)$ is its interior point, we conclude that $m(x)$ is the Minkowski functional. We then know from [14] that the Minkowski functional $m(.)$ satisfies the inequality $m(x_1 + x_2) \leq m(x_1) + m(x_2)$ for all $x_1, x_2 \in \mathbb{R}^2$. A couple of algebraic manipulations lead to the fact that $m(x)$ reduces to the straight line travel time function $\tau(.)$, that is, $\tau(x, y) = m(y - x)$ for all $x, y \in \mathbb{R}^2$. We now show that the equivalent inequality holds true for the travel time function τ.

Lemma 1. *For any $x, y, z \in \mathbb{R}^2$, we have $\tau(x, y) \leq \tau(x, z) + \tau(z, y)$. That is, traveling time along the straight line path from x to y is never greater than the time it takes to travel along straight lines from x to z and then from z to y. (See Figure 2)*

Proof. see A.1. □

We now can use the inequality from Lemma 1 to show that the straight line path between two points is a fastest path for a convex LPAR, $L_\delta(x)$. Recall that $p \in P_{st}$ is an arbitrary continuous and rectifiable path from s to t, and that $t(p)$ denotes the travel time along the path p. Then we can state the following lemma.

Lemma 2. *For a convex $L_\delta(x)$ and an arbitrary continuous and rectifiable path $p \in P_{st}$, we have $\tau(s,t) \leq t(p)$. In other words, in the case of a convex LPAR, the travel time along the straight line path is never greater than that of any other path.*

Proof. [sketch] Since the length of any continuous and rectifiable path p, in the limit, equals to the length of a piecewise-linear approximation, we can iteratively apply the inequality from Lemma 1 to obtain the desired result. See A.2 for the complete proof. $\square$

Lemma 2 above provides the fastest path between two points for a convex LPAR. Furthermore, the following theorem adds that convexity of an LPAR is also a necessary condition for the straight line path to be optimal.

Theorem 3. *A fastest path in $\mathbb{R}^2$ from an arbitrary start point $s \in \mathbb{R}^2$ to any other point in $\mathbb{R}^2$ is a path along the straight line connecting the two points **if and only if** the linear path attainable region $L_\delta(x)$ is a convex set for all $x \in \mathbb{R}^2$.*

Proof. Lemma 2 concludes that a fastest path from an arbitrary start point $s \in \mathbb{R}^2$ to any other point in $\mathbb{R}^2$ is a path along the straight line connecting the two points **if** the linear path attainable region $L_\delta(x)$ is a convex set for all $x \in \mathbb{R}^2$.

Now, we prove the **only if** statement of the theorem by contradiction.

Select an arbitrary start point $s \in \mathbb{R}^2$ and assume that $L_\delta(s)$ is not convex. Then, there exist $x_1, x_2 \in L_\delta(s)$ and $\lambda \in [0,1]$ such that $\lambda x_1 + (1-\lambda)x_2 \notin L_\delta(s)$, and we set point $y = \lambda x_1 + (1-\lambda)x_2$.

Since $x_1, x_2 \in L_\delta(s)$, we have $\tau(s,x_1) \leq \delta$ and $\tau(s,x_2) \leq \delta$. Then, consider the following path p: from point s, we first travel following the vector $\lambda(x_1 - s)$ and then continue on following the vector $(1-\lambda)(x_2-s)$ (Figure 3). Our path p starts at point s and ends at point $s+\lambda(x_1-s)+(1-\lambda)(x_2-s) = \lambda x_1+(1-\lambda)x_2 = y$. Note that time and space homogeneity give us that traveling time for this path, $t(p) = \lambda\tau(s,x_1) + (1-\lambda)\tau(s,x_2) \leq \lambda \cdot \delta + (1-\lambda) \cdot \delta = \delta$. However, $\tau(s,y) > \delta$ since $y \notin L_\delta(s)$. We reach a contradiction that the straight line path from s to any point in $\mathbb{R}^2$ is not necessarily the fastest path if the attainable region is not convex. Thus, $L_\delta(s)$ has to be convex. $\square$

It is important to acknowledge that earlier work on 'convex distance functions' [3, 5, 4] have stated some results similar to Lemma 1 and Theorem 3. However, none of the found literature provides a rigorous proof in its entirety, compelling us to include our proofs developed independently of the cited literature.

Next, we analyze optimal path finding for a general LPAR, which may or may not be convex.

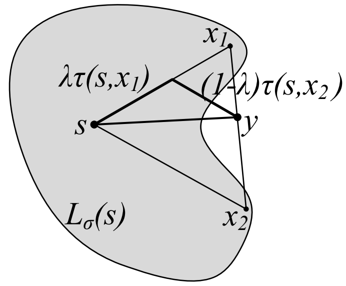

Figure 3: Theorem 3 counter example for a non-convex linear path attainable region.

3.2 Properties of an Attainable Region and the Corresponding Linear Path Attainable Region

From this point on, we relax the convexity assumption for the linear path attainable region, $L_\delta(x)$, and analyze the problem for a general time and space homogeneous speed function. In this section, we provide a series of lemmas, theorems and propositions stating supporting properties of LPARs and the corresponding attainable regions, ARs. Lemmas represented here are the building blocks for our main results presented in the following sections.

Proposition 4. *$L_\delta(x) = A_\delta(x)$ **if and only if** $L_\delta(x)$ is convex.*

Proof. [sketch] From Theorem 3 we know that a convex $L_\delta(x)$ implies the optimality of a straight line path, therefore $A_\delta(x) = L_\delta(x)$. On the other hand, if $L_\delta(x)$ were not convex, then there would exist a point $y \notin L_\delta(x)$ as constructed in the proof of Theorem 3 (see Figure 3). Consequently, $y \in A_\delta(x)$, which would imply a contradiction $L_\delta(x) \neq A_\delta(x)$. See A.3 for the complete proof. □

3.2.1 Comparison of Two Distinct LPARs and Their Corresponding ARs.

Consider two arbitrary maximum attainable speed functions $V^1(\theta)$ and $V^2(\theta)$ defined on $\theta \in [0, 2\pi]$. Recall that each speed function uniquely defines the linear path attainable region for a given time interval, and vise versa. Thus, let $L^1_\delta(x)$ and $L^2_\delta(x)$ be the linear path attainable regions corresponding to the speed functions $V^1(\theta)$ and $V^2(\theta)$, respectively. Then we can make the following observations about $L^1_\delta(x)$ and $L^2_\delta(x)$.

Lemma 5. *$L^1_\delta(x) \subseteq L^2_\delta(x)$ **if and only if** $V^1(\theta) \leq V^2(\theta)$ for all θ.*

Proof. see A.4. □

Lemma 6. *Let $A^1_\delta(x)$ and $A^2_\delta(x)$ be the attainable regions corresponding to linear path attainable regions $L^1_\delta(x)$ and $L^2_\delta(x)$, respectively. Then, $L^1_\delta(x) \subseteq L^2_\delta(x)$ implies $A^1_\delta(x) \subseteq A^2_\delta(x)$.*

Proof. see A.5. □

3.2.2 Attainable Region Corresponding to a Given Linear Path Attainable Region.

The problem discussed in our work assumes that a maximum attainable speed function $V(\theta)$ is given for all $\theta \in [0, 2\pi]$. Since the speed function uniquely defines the linear path attainable region for a given x and δ, $L_\delta(x)$, one can always use the definition of the LPAR to find $L_\delta(x)$ corresponding to a given function $V(\theta)$. Theorem 3 establishes the fact that a straight line is not necessarily the fastest path for a non-convex $L_\delta(x)$, and from Proposition 4 we know that $L_\delta(x) \neq A_\delta(x)$ if $L_\delta(x)$ is not convex. Thus, finding the attainable region, $A_\delta(x)$, corresponding to a given speed function is not always a straight forward task. In this section, we establish how one can find the attainable region corresponding to a given $L_\delta(x)$.

Lemma 7. *The convex combination of any two points in $L_\delta(x)$ is contained in $A_\delta(x)$, i.e., $\forall x_1, x_2 \in L_\delta(x)$ and $\forall \lambda \in [0, 1]$, $\lambda x_1 + (1 - \lambda)x_2 \in A_\delta(x)$.*

Proof. For any point $y = \lambda x_1 + (1 - \lambda)x_2$ we can construct a path p analogous to the path described in the proof of Theorem 3. Since the travel time for such path p is less than or equal to δ, we conclude that $y \in A_\delta(x)$. See A.6 for the complete proof. □

Theorem 8. *Attainable region, $A_\delta(x)$, is the convex hull of the corresponding linear path attainable region, $L_\delta(x)$, i.e., $A_\delta(x) = conv(L_\delta(x))$.*

Proof. The statement $conv(L_\delta(x)) \subseteq A_\delta(x)$ follows directly from Lemma 7. It is worth noting, that path p constructed in the lemma's proof is not necessarily a fastest path from x to y, it is just a path that reaches point y in time less than or equal to δ.

Next, we show that $A_\delta(x) \subseteq conv(L_\delta(x))$. Consider a new linear path attainable region $L'_\delta(x) = conv(L_\delta(x))$. Since our linear path attainable region $L'_\delta(x)$ is convex, from Proposition 4 we know that the corresponding attainable region $A'_\delta(x) = L'_\delta(x)$. Since $L_\delta(x) \subseteq L'_\delta(x)$, from Lemma 6 it follows that $A_\delta(x) \subseteq A'_\delta(x)$. And since $A'_\delta(x) = conv(L_\delta(x)) \Rightarrow A_\delta(x) \subseteq conv(L_\delta(x))$. Hence, $A_\delta(x) = conv(L_\delta(x))$. □

3.3 Bound on the Optimal Travel Time

From Theorem 3, we know that sometimes a straight line path is not necessarily a fastest path for a given speed function $V(\theta)$. In particular, a straight line is the fastest path for a convex linear path attainable region, but not necessarily so for a non-convex region. Here, we calculate a bound on the shortest travel time error if the straight line path is implemented for a non-convex LPAR. A lower bound on the minimum travel time is not only important

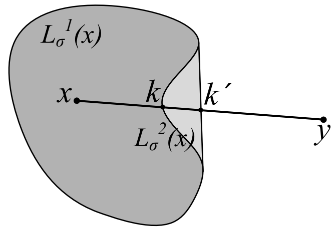

Figure 4: Computing a bound on the decrease in travel time for a non-convex linear path attainable region, $L^1_\delta(x)$.

for assessing the penalty for deviating from the optimal path by following a straight line, but the bound also plays a significant role in the proof of our key result: by showing that in some cases the travel time for our proposed path is equal to the lower bound, we prove its optimality.

Consider a non-convex linear path attainable region, $L^1_\delta(x)$, corresponding to some speed function $V^1(\theta)$. Then, we can calculate a bound on a decrease in the travel time from point x to point y by following an optimal path instead of the straight line path, without actually knowing the optimal path. Consider a new linear path attainable region defined as the convex hull of the original LPAR, that is, $L^2_\delta(x) = conv(L^1_\delta(x))$. And let $V^2(\theta)$ be the maximum attainable speed function associated with the new LPAR. From Theorem 3, we know that for $L^2_\delta(x)$, the fastest path from x to y is along the straight line segment connecting these two points, l_{xy}, with the total travel time $\tau_2(x,y) = \|y-x\|/V^2(\theta_{y-x})$. Since $L^1_\delta(x) \subset L^2_\delta(x)$, from Lemma 5 we know that $V^1(\theta) \leq V^2(\theta)$ for all θ. Then, the smallest travel time from x to y for the linear path attainable region $L^1_\delta(x)$, denoted by $t^*_{L^1_\delta(x)}(x,y)$, is at least as much $\tau_2(x,y)$, i.e., $t^*_{L^1_\delta(x)}(x,y) \geq \tau_2(x,y)$.

Define k to be the point of intersection of the line connecting points x and y, and the boundary of the linear path attainable region $L^1_\delta(x)$, i.e., $k := l_{xy} \cap bd(L^1_\delta(x))$. Similarly, we define $k' := l_{xy} \cap bd(L^2_\delta(x))$ (see Figure 4). Note that the sets $L^1_\delta(x)$ and $L^2_\delta(x)$ are closed and therefore contain their boundaries. Also note that the travel time along the straight line path from x to y corresponding to the linear path attainable region $L^1_\delta(x)$ is $\tau_1(x,y) = \delta\frac{\|y-x\|}{\|k-x\|}$. Set $\beta := \frac{\|k-x\|}{\|k'-x\|} \leq 1$. Then, we have the following bounds on $t^*_{L^1_\delta(x)}(x,y)$.

$$\begin{aligned}
\tau_2(x,y) \leq \quad & t^*_{L^1_\delta(x)}(x,y) & \leq \tau_1(x,y) \\
\delta\frac{\|y-x\|}{\|k'-x\|} \leq \quad & t^*_{L^1_\delta(x)}(x,y) & \leq \delta\frac{\|y-x\|}{\|k-x\|} \\
\beta\delta\frac{\|y-x\|}{\|k-x\|} \leq \quad & t^*_{L^1_\delta(x)}(x,y) & \leq \delta\frac{\|y-x\|}{\|k-x\|} \\
\beta\tau_1(x,y) \leq \quad & t^*_{L^1_\delta(x)}(x,y) & \leq \tau_1(x,y)
\end{aligned} \tag{1}$$

From inequalities (1), we deliver the following proposition.

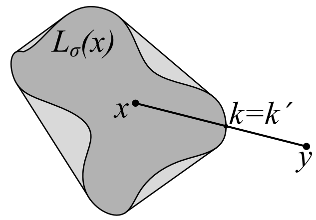

Figure 5: Illustration of Theorem 10 scenario 1: $k = k'$.

Proposition 9. *The optimal travel time for a non-convex LPAR is at most β times shorter than following a straight line path from x to y, where $\beta := \frac{\|k-x\|}{\|k'-x\|}$. That is, the traveling time would at most decrease by $100(1-\beta)$ percent, if one were to follow an optimal path instead of traveling along the straight line.*

This lower bound is next used to show that a proposed path is, in fact, optimal.

3.4 Fastest Path for an Arbitrary Linear Path Attainable Region

In the earlier Section 3.1 we solved an instance of our fastest-path finding problem for a convex linear path attainable region (LPAR). In this section, we describe a closed form solution to our fastest path problem in $\mathbb{R}^2$ for any speed function, even when corresponding $L_\delta(x)$ fails to be convex. The following theorem is one of the key results of this paper.

Theorem 10. *Consider a linear path attainable region $L_\delta(x)$. For two arbitrarily given points $x, y \in \mathbb{R}^2$, let k denote the intersection point of the line connecting x and y, l_{xy}, and the boundary of the set $L_\delta(x)$, i.e., $k := l_{xy} \cap bd(L_\delta(x))$. Similarly, let $k' := l_{xy} \cap bd(conv(L_\delta(x)))$. Then, the fastest path from x to y is described by one of the following two scenarios.*

1. *If $k = k'$, the fastest path from x to y is the straight line segment connecting these two points (Figure 5).*
2. *If $k \neq k'$, the fastest path from x to y consists of two line segments: the straight line segment from point x to point $z = x + \alpha\lambda^*(x_1 - x)$ and the second line segment from point z to point y, where $\alpha = \frac{\|y-x\|}{\|k'-x\|}$ and $x_1, x_2 \in L_\delta(x)$ s.t. $\exists \lambda^* \in [0,1] : k' = \lambda^* x_1 + (1-\lambda^*)x_2$. (See Figure 6, and note that $(y-z)\|(x_2 - x)$).*

Proof. 1. Consider the case where $k = k'$. From inequalities (1), we have $\beta\tau(x,y) \leq t^*_{L_\delta(x)}(x,y) \leq \tau(x,y)$, where $\beta := \frac{\|k-x\|}{\|k'-x\|}$ and $t^*_{L_\delta(x)}(x,y)$ is the minimum travel time from x to y. Since $k = k'$, we have $\beta = 1$, and $\tau(x,y) \leq t^*_{L_\delta(x)}(x,y) \leq \tau(x,y) \Rightarrow t^*_{L_\delta(x)}(x,y) = \tau(x,y)$. This means that the travel time from x to y along the straight

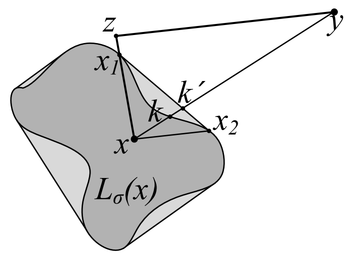

Figure 6: Illustration of Theorem 10 scenario 2: $k \neq k'$.

line path equals the minimum travel time, and hence, straight line path is a fastest path from x to y.

2. Now, we consider the case where $k \neq k'$. From definition of k', we have that $k' \in conv(L_\delta(x))$. Then, $\exists \lambda^* \in [0,1]$ and $\exists x_1, x_2 \in L_\delta(x)$, such that $\lambda^* x_1 + (1-\lambda^*)x_2 = k'$. Note, that since $x_1, x_2 \in L_\delta(x)$, we know that $\tau(x, x_1) \leq \delta$ and $\tau(x, x_2) \leq \delta$.

   From inequalities (1), we have $t^*_{L_\delta(x)}(x,y) \geq \delta \frac{\|y-x\|}{\|k'-x\|} = \delta\alpha$, where $t^*_{L_\delta(x)}(x,y)$ is the minimum travel time from x to y. Now, consider the following path p: from point x we follow vector $\alpha\lambda^*(x_1 - x)$, and then, continue on following vector $\alpha(1-\lambda^*)(x_2 - x)$. Note, that the first part of the path is equivalent to following a straight line segment from point x to point $x + \alpha\lambda^*(x_1 - x) = z$. And the second part of the path ends at point $x + \alpha\lambda^*(x_1 - x) + \alpha(1-\lambda^*)(x_2 - x) = x + \alpha((\lambda^*)(x_1 - x) + (1-\lambda^*)(x_2 - x)) = x + \alpha(k' - x) = y$. Hence, the proposed path p is the same path as in the statement of the theorem. This proves the existence of the path described in the theorem.

   Next, we want to find the travel time along this path p, $t(p)$. From the space and time homogeneity property, we have $t(p) = \alpha\lambda^*\tau(x, x_1) + \alpha(1-\lambda^*)\tau(x, x_2) \leq \alpha\lambda^* \cdot \delta + \alpha(1 - \lambda^*) \cdot \delta = \alpha\delta$. Since travel time for path p is less than or equal to the lower bound on the minimum travel time from x to y (i.e., $t(p) \leq t^*_{L_\delta(x)}(x,y)$), $t(p)$ must be equal to the minimum travel time from x to y. Hence, our path p is, in fact, a fastest path from x to y.

□

It is worth noting that in the case when $k \neq k'$ (corresponding to scenario 2 of Theorem 10) the fastest path constructed in the theorem is not uniquely optimal. It is only one of the infinitely many feasible paths with the same minimum travel time. Note that any zigzag path from x to y restricted to the traveling directions of the vectors $x_1 - x$ and $x_2 - x$ would correspond to the same minimum travel time. Furthermore, the straight line path in the case of $k = k'$ might also not be uniquely optimal. Depending on the structure of the speed function, it is possible that a piecewise-linear path would have the same optimal travel time as the straight line.

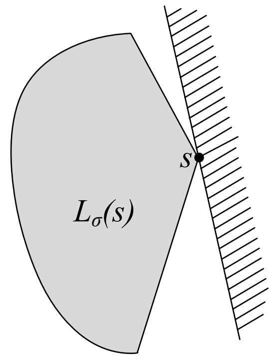

Figure 7: Example of a convex LPAR where $V(\theta_{st}) = 0$; there is no feasible path from s to t.

3.5 Problem Feasibility and Fastest-Path Finding for a Non-negative Speed Function

All the analysis and results presented above assume that $V(\theta) > 0$ for all $\theta \in [0, 2\pi]$. However in practice, the speed function $V(\theta)$ can take on the value of zero for some headings, i.e., leading to a 'stall'. For example, a vehicle traveling across some hilly terrain might encounter impermissible headings due to overturn danger or power limitations [25]. On another hand, a sailing boat can not travel in head sea corresponding to a zero speed for that heading [22]. In this section, we discuss how allowing $V(\theta)$ to take on zero values for some headings can change the results presented in the previous sections and in particular, its possible effects on problem feasibility. Consequently, we will allow $V(\theta) \geq 0, \forall\theta \in [0, 2\pi]$ for the discussion in this section. To avoid the trivial case, we assume that there always exists some θ such that $V(\theta) > 0$.

3.5.1 Feasibility and Optimal Path Finding for a Convex LPAR.

Similar to a positive speed function case, convexity of a linear path attainable region (LPAR) is a useful property that simplifies the optimal path finding task. Therefore, we first analyze the case where LPAR, $L_\delta(s)$, is convex. In the following section, we look at a more general case where $L_\delta(s)$ does not have to be convex. Lemma 11 below helps establish the existence of a feasible path from s to t.

Lemma 11. *If $V(\theta) = 0$ for some $\theta \in [0, 2\pi]$, and the corresponding linear path attainable region, $L_\delta(s)$, is convex, then there exists a line passing through the starting point s such that none of the points belonging to one of the half-planes created by this line can be reached. See Figure 7.*

Proof. see A.7. □

Next, Theorem 12 describes an optimal path from s to t for a convex LPAR.

Theorem 12. *Assume that LPAR, $L_\delta(s)$, corresponding to some speed function $V(\theta) \geq 0$, is convex. Then,*

1. *if $V(\theta_{st}) = 0$, a feasible path from s to t does not exist; and*
2. *if $V(\theta_{st}) > 0$, a fastest path from s to t is along the straight line path st.*

Proof. see A.8. □

3.5.2 Feasibility and Optimal Path Finding for an Arbitrary LPAR.

We now relax the convexity assumption for a linear path attainable region and analyze optimal paths for a general $L_\delta(s)$. Note that results presented below apply to a convex as well as a non-convex LPAR cases. However, if one knows that the linear path attainable region is convex, application of Theorem 12 would be more straight forward.

Recall that θ_{st} denotes the heading angle of the vector $t-s$. It is apparent that if $V(\theta_{st}) > 0$, then the optimal path finding problem is feasible. We are interested in describing necessary and sufficient conditions for the problem to be infeasible. Assuming $V(\theta_{st}) = 0$, we can define $\underline{\theta}$ and $\bar{\theta}$ as given below.

$$\underline{\theta} = \inf\{\theta^* : V(\theta) = 0, \forall\theta \in [\theta^*, \theta_{st}]\} \quad (2)$$
$$\bar{\theta} = \sup\{\theta^* : V(\theta) = 0, \forall\theta \in [\theta_{st}, \theta^*]\} \quad (3)$$

Note that *infimum* and *supremum* in equations (2) and (3) might not be actually achieved. Also note that in defining $\underline{\theta}$ and $\bar{\theta}$ we extend the domain of the speed function to $[-\pi, 3\pi]$, by observing that $V(\theta) = V(\theta + 2\pi), \forall\theta$. This extension is necessary to guaranty the continuity of the interval at the boundary points $\theta = 0$ and $\theta = 2\pi$. Now, we are ready to state our problem feasibility theorem.

Theorem 13. *A feasible path from s to t does not exist* ***if and only if*** *$V(\theta_{st}) = 0$ and $\bar{\theta}-\underline{\theta} \geq \pi$.*

Proof. see A.9. □

Next, Theorem 14 delivers an optimal path from s to t for a general linear path attainable region.

Theorem 14. *Consider $V(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$, and let $L_\delta(s)$ be the corresponding linear path attainable region. Then,*

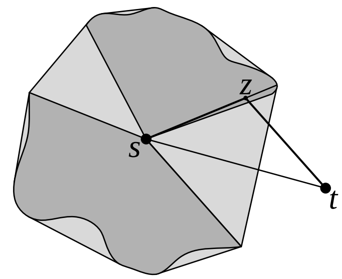

Figure 8: An optimal path from s to t, szt.

1. *if* $V(\theta_{st}) = 0$ *and* $\bar{\theta} - \underline{\theta} \geq \pi$*, a feasible path from* s *to* t *does not exist; and*
2. *if* $V(\theta_{st}) > 0$ *or* $\bar{\theta} - \underline{\theta} < \pi$*, then a fastest path from* s *to* t *is characterized the same way as in Theorem 10, where* $x = s$*,* $y = t$ *and if* $V(\theta_{st}) = 0$ *we set* $k = s$*. (See Figure 8.)*

Proof. 1. Proof of the first statement follows directly from Theorem 13.

2. *[Sketch]* The proof of this statement is analogous to the proof of Theorem 12 part 2. The optimality of a path characterized in Theorem 10 has been only proven for the positive speed function, where s is an interior point of the corresponding LPAR. Thus, we first define the new speed function $V'(\theta)$ as given by equation (7), then we apply Theorem 10 to the new speed function, and finally, we show that the found path is also feasible and has the same travel time for the original speed function $V(\theta)$, making it an optimal path for our problem.

□

3.6 Fastest-Path Finding Algorithm

Sections 3.1-3.5 characterize an optimal path between any two points in $\mathbb{R}^2$ for an arbitrary speed function $V(\theta)$. In this section, we discuss the implementation of the presented results and provide an algorithm that can be implemented by a computer program (e.g., on-board autonomous navigation system) to find a fastest path from a given start point $s \in \mathbb{R}^2$ to a given target point $t \in \mathbb{R}^2$. The presented algorithm checks the feasibility of the problem as discussed in Section 3.5 and then implements the results of Theorem 10 in the case of a feasible problem.

Since, in practice, the direction-dependent speed is usually evaluated by a computer program, we assume that the values of a speed function $V(\theta)$ is given for a discrete set of equally spaced heading angles, θ, which we denote by the set of polar coordinates $S = \{(\theta_0, V(\theta_0)), ..., (\theta_n, V(\theta_n)\}$. (The speed values for the intermediate heading angles are assumed to be equal to the linear interpolation within a polar coordinate system, see Figure

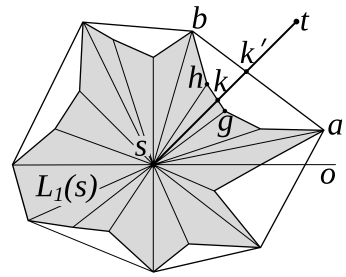

Figure 9: $L_1(s)$ and its convex hull: $\angle ost = \theta_{st}$, $\angle osg = \theta_L$, $\angle osh = \theta_U$, $\angle osa = \theta'_L$, and $\angle osb = \theta'_U$.

9 for example.) Note that in the case when an analytical function of $V(\theta)$ is available, we still have to discretize the speed function in order to be able to implement the fastest-path finding procedure by a computer.

Our first step is to check the feasibility of the problem. Theorem 13 states the necessary and sufficient condition for a problem to be infeasible. If those conditions are not satisfied, we know that an optimal path exist and we can proceed to finding such a path.

The first step in finding a fastest path is to construct a convex hull of the linear path attainable region. Construction of a convex hull of a finite number of points in $\mathbb{R}^2$ is a well studied problem, and its details are omitted. However, we recommend the use of Graham's Scan algorithm [11, 6] to accomplish this task. The advantage of this algorithm is that it uses a technique called "rotational sweep," processing vertices in the order of polar angles they form with a reference vertex. The polar nature of our LPAR makes Graham's Scan a favorable choice as it forgoes the sorting procedure required for other algorithms.

After the construction of a convex hull, we obtain a subset $S' \subseteq S$ corresponding to the extreme (corner) points of the resulting convex hull. Furthermore, the convex combination of two consecutive points in S' is part of the boundary of $conv(L_1(s))$. (Just like, all convex combinations of pairs of consecutive points in S is the boundary of $L_1(s)$.) Let l_{st} denote the straight line passing through points s and t, and θ_{st} the heading angle of the vector $t - s$. Then, to apply Theorem 10, we need to find the point of intersection of l_{st} with the boundary of $L_1(s)$, denoted by k, and the point of intersection of l_{st} with the boundary of a convex hull of $L_1(s)$, denoted by k'. To do so, we find between which two headings in sets S and S' our θ_{st} falls. We label such headings as θ_L and θ_U, and θ'_L and θ'_U, respectively (See Figure 9, L and U stand for the lower and upper headings).

We know that k lies on the line segment connecting points $(\theta_L, V(\theta_L))$ and $(\theta_U, V(\theta_U))$, and k' lies on the line segment connecting points $(\theta'_L, V(\theta'_L))$ and $(\theta'_U, V(\theta'_U))$. Based on that, we can use the found $\theta_L, \theta_U, \theta'_L$ and θ'_U to determine whether $k = k'$ without actually finding the points k and k'. Note that if points $(\theta_L, V(\theta_L)), (\theta_U, V(\theta_U)), (\theta'_L, V(\theta'_L))$ and $(\theta'_U, V(\theta'_U))$ (corresponding to points g, h, a and b on Figure 9) are all collinear, then k must equal k', and

$k \neq k'$ otherwise. If $k = k'$, we conclude that line segment st is the fastest path as proven in scenario 1 of Theorem 10. If $k \neq k'$, our problem reduces to scenario 2 of the theorem, and we need to compute the values of α and λ^*, as defined in Theorem 10. After some algebraic manipulations omitted here, we find that

$$\alpha\lambda^* = \frac{\|t - s\| \sin(\theta'_U - \theta_{st})}{V(\theta'_L) \sin(\theta'_U - \theta'_L)}. \tag{4}$$

Then, we know that a fastest path is piecewise-linear with a single waypoint $z = s + \alpha\lambda^*(\cos(\theta'_U), \sin(\theta'_U))$.

The following algorithm outlines a step-by-step procedure of finding the fastest path from s to t.

Algorithm 1. *Fastest Path from s to t in an Obstacle-Free Domain.*

Step 1. *Find $\underline{\theta}$ and $\bar{\theta}$ using equations (2) and (3).*
If $V(\theta_{st}) = 0$ and $\bar{\theta} - \underline{\theta} \geq \pi$, STOP. The problem is infeasible.
Else, go to step 2.

Step 2. *Find the convex hull of the linear path attainable region $L_1(s)$.*

Step 3. *Find the heading angle θ_{st} and compute the values of $\theta_L, \theta_U, \theta'_L$ and θ'_U.*

Step 4. *If points $(\theta_L, V(\theta_L)), (\theta_U, V(\theta_U)), (\theta'_L, V(\theta'_L))$ and $(\theta'_U, V(\theta'_U))$ are collinear (i.e., if $k = k'$), STOP. Straight line path st is an optimal path.*
Else (i.e., if $k \neq k'$), go to step 5.

Step 5. *Compute $\alpha\lambda^*$ using equation (4).*
Set $z = s + \alpha\lambda^*(\cos(\theta'_U), \sin(\theta'_U)) \in \mathbb{R}^2$. A fastest path from s to t is the two consecutive straight line segments sz and then zt.

4 Obstacle-Avoiding Fastest-Path Finding for an Anisotropic Speed Function

In this section we discuss obstacle-avoiding fastest-path finding by relaxing the assumption made in Section 3 that $\mathcal{P} = \emptyset$. Throughout this section, $\mathcal{P}$ is a nonempty set of polygonal obstacles that are not permitted to be intersected by any feasible path.

4.1 Fastest Path for a Convex Linear Path Attainable Region

Similarly to our analysis of path finding problems in an obstacle-free domain, we first restrict our attention to problems with speed functions corresponding to the convex linear path attainable regions (LPARs). The analysis of this special case demonstrates an interesting insight into the structure of the solution for the problems with arbitrary speed functions. In the following subsection, we relax the convexity assumption and show how the results presented here are extended to the unrestricted time and space homogenous anisotropic speed functions.

The visibility graph search method, used to solve Euclidean shortest-path finding problems, exploits the triangle inequality property of the distance function and restricts the optimal path search to the set of 'taut strings' connecting the points of origin and destination. Similar properties can be established for the fastest-path finding problem at hand. Our Theorem 10 states that in the case of a convex LPAR a fastest path between any two points in an obstacle-free anisotropic domain is the connecting straight line segment. Consequently, the triangle inequality, restated in Corollary 15 for completeness, also holds true for our anisotropic cost function (travel time). We use this property to develop a fastest-path finding algorithm analogous to the one used for Euclidean shortest path problems.

Corollary 15. *If a speed function $V(\theta)$ corresponds to a convex linear path attainable region, then the travel-time function $\tau(.)$ has the 'triangle inequality' property, that is, $\tau(a,b) \leq \tau(a,c)+\tau(c,b)$, $\forall a,b,c \in \mathcal{F} = \mathbb{R}^2 \backslash \mathcal{P}$, as long as neither one of the linear paths are obstructed by obstacles.*

Proof. Follows directly from Lemma 1. □

The triangle inequality stated in Corollary 15 provides the grounds for a direct extension of the earlier mentioned visibility graph search method to our anisotropic problem. In the case of minimizing Euclidean distance, "an easy geometric argument shows that in general the shortest path between two points must be a polygonal chain whose vertices are vertices of obstacles" [2]. A similar observation is true for our anisotropic medium, which validates the search of a modified visibility graph as an appropriate solution approach for our problem.

Theorem 16. *If a linear path attainable region $L_\delta(x)$ is convex, then there exists a fastest path from s to t in $\mathcal{F}$, which is piecewise-linear with all its waypoints (vertices) corresponding to the vertices of obstacles in $\mathcal{P}$.*

Proof. see A.10. □

Theorem 16 implies that when a given speed function corresponds to a convex linear path attainable region, a fastest-path search can be restricted to a directed visibility graph with the edge cost defined to be the travel time along the straight line connecting its nodes. Henceforth, we adapt the shortest path visibility graph approach to develop the algorithm below, which finds an obstacle-avoiding fastest path for an anisotropic speed function corresponding to a convex LPAR.

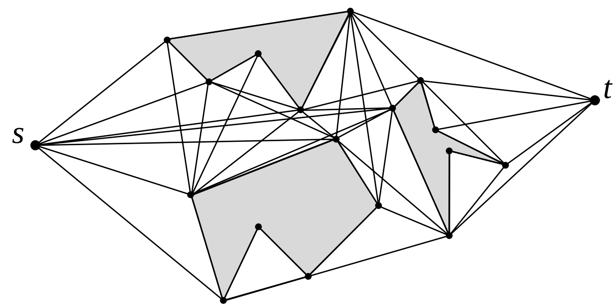

Figure 10: Construction of a visibility graph.

Algorithm 2. *Obstacle-Avoiding Fastest Path for a Speed Function $V(\theta)$ Corresponding to a Convex LPAR.*

Step 1. *Construct a visibility graph $\mathcal{VG}_V$ as follows.*

- *The set of $\mathcal{VG}_V$ vertices is composed of all the vertices of the obstacles in $\mathcal{P}$, as well as points s and t.*
- *The set of $\mathcal{VG}_V$ edges consists of all the straight line edges interconnecting these vertices such that they do not intersect any of the obstacles in $\mathcal{P}$.*
- *The cost associated with an edge (i, j) is equal to the travel time $\tau(i, j) = ||j - i||/V(\theta_{j-i})$. (Note that unlike the case of Euclidean metric, our visibility graph has to be directed since the cost of an arc (i, j) does not generally equal to the cost of an arc (j, i).)*

Figure 10 provides an example of constructing a visibility graph by illustrating all the visibility graph nodes and edges.

Step 2. *Apply Dijkstra's algorithm [7] to find an optimal path in the constructed network $\mathcal{VG}_V$ from node s to node t. The resulting path is an obstacle-avoiding fastest path.*

Published work discussing Euclidean shortest path problems notes that in some cases the visibility graph has quadratic size (i.e., the construction time of a graph with n vertices is $O(n^2)$), and is not the most efficient approach for such problems [16]. An alternative method, referred to as *continuous Dijkstra*, involves simulating the effect of a 'wavefront' propagating out from the source s and constructs the linear-size shortest path map directly [17, 18]. While this method was originally developed for the Euclidean shortest-path problems, applications to other scenarios have been also considered (e.g., L_1 shortest paths in the plane [15], and construction of Voronoi diagrams for convex distance functions [5]). We thus note that Algorithm 2 is not the only possible method to address our problem, and the extension of a continuous Dijkstra algorithm to an anisotropic medium with a convex LPAR is also a plausible approach.

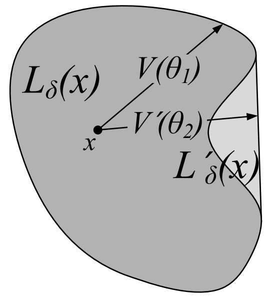

Figure 11: Example of $L_\delta(x)$ and $L'_\delta(x) := conv(L_\delta(x))$.

4.2 Fastest Path for an Arbitrary Anisotropic Speed Function

Subsection 4.1 discusses a direct extension of the shortest-path visibility graph approach to the obstacle-avoiding fastest-path finding problems with convex LPAR. However, the proposed algorithm does not apply to a general speed function $V(\theta)$ in the case when the correspond $L_\delta(x)$ is not convex. In Theorem 3 we have shown that a straight line path between a pair of points is not necessarily optimal for an arbitrary direction-dependent speed function. Therefore, in general, the triangle inequality does not hold true for the travel time function $\tau(.)$, and we cannot restrict our fastest-path search to the set of taut strings connecting s and t. In this subsection, we relax the convexity assumption of an LPAR, and analyze fastest-path finding problems for a general anisotropic speed function.

Consider an arbitrary speed function $V(\theta)$ and the corresponding linear path attainable region $L_\delta(x)$ which may or may not be convex. We introduce an augmented speed function $V'(\theta)$, such that, its corresponding LPAR, $L'_\delta(x)$, is the convex hull of $L_\delta(x)$, i.e., $L'_\delta(x) := conv(L_\delta(x))$ (see Figure 11). Note that the set $L'_\delta(x)$ and the speed function $V'(\theta)$ are unique, due to the uniqueness of a convex hull. By definition, a linear path attainable region $L'_\delta(x)$ is convex. Therefore, by constructing the visibility graph $\mathcal{VG}_{V'}$ as described in Algorithm 2, we can find an obstacle-avoiding fastest path from s to t corresponding to the new speed function $V'(\theta)$. We let $p_{V'}$ represent this optimal path and $t_{V'}(p_{V'})$ denote the travel time along the path $p_{V'}$ while traveling with speed $V'(\theta)$. Then, Proposition 17 below states that the minimum travel time from s to t corresponding to the original speed function $V(\theta)$ has to be greater than or equal to $t_{V'}(p_{V'})$.

Proposition 17. *The traveling time along a fastest path corresponding to an arbitrary speed function $V(\theta)$ has to be greater than or equal to the travel time along a fastest path corresponding to the speed function $V'(\theta)$, where $L'_\delta(x) = conv(L_\delta(x))$.*

Proof. see A.11. □

Proposition 17 concludes that if we let p_V denote an obstacle-avoiding fastest path for speed $V(\theta)$, then $t_V(p_V) \geq t_{V'}(p_{V'})$. This provides a lower bound on the minimum travel time for

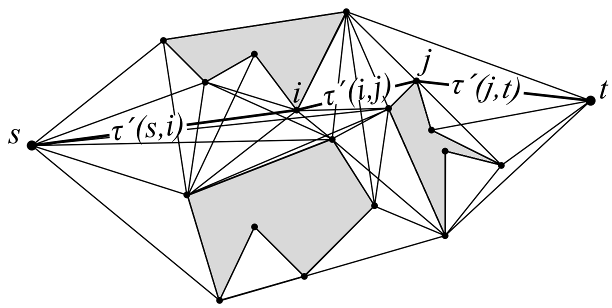

Figure 12: Fastest path corresponding to the speed function $V'(\theta)$ is shown in bold. Its travel time $t_{V'}(p_{V'}) = \tau'(s,i) + \tau'(i,j) + \tau'(j,t)$.

our original problem, which is an important component to characterizing an optimal path. We use this bound to demonstrate that the travel time for a path proposed below is equal to its lower bound, consequently proving the path's optimality.

Since path $p_{V'}$ lies in the visibility graph $\mathcal{VG}_{V'}$, it is piecewise-linear with the waypoints corresponding to the vertices of $\mathcal{P}$, and points s and t. Consequently, the total travel time of the path can be written as the sum of travel times along each individual link. Recall that the travel time for each linear link (i,j) of the path $p_{V'}$ is equal to $\tau'(i,j) = \frac{\|j-i\|}{V'(\theta_{j-i})}$ (see Figure 12). From Theorem 10, our obstacle-free analysis describes a fastest path from i to j for an arbitrary speed function $V(\theta)$ with the optimal travel time equal to $\tau'(i,j)$. Applying the theorem to each linear link of the path $p_{V'}$ and then combining them together results in a path corresponding to the original speed function $V(\theta)$ with the travel time equal to the lower bound $t_{V'}(p_{V'})$.

Recall that an optimal path described in the second scenario of Theorem 10 is not unique; it is just one of infinitely many paths with the same minimum travel time. In fact, as we attempt to implement the one-waypoint path along each linear link of the path $p_{V'}$ we might intersect the obstacle space $\mathcal{P}$. However, due to time and space homogeneity of the cost function, i.e., $V(\theta)$, we can construct a zigzag path with the same travel time by alternating the traveling directions between headings corresponding to vectors $x_1 - x$ and $x_2 - x$ as many times as needed. Our problem statement assumes that the distance between any two obstacles is always greater than zero. Therefore, we can always construct a zigzag path close enough to the line xy, such that it does not intersect with the neighboring obstacles. (See Figure 13.)

We now introduce an algorithm for finding an obstacle-avoiding fastest path for an arbitrary speed function.

Algorithm 3. *Obstacle-Avoiding Fastest Path for an Arbitrary Speed Function $V(\theta)$.*

Step 1. *Find $V'(\theta)$ for $\theta \in [0, 2\pi]$ such that $L'_\delta(x) = conv(L_\delta(x))$.*

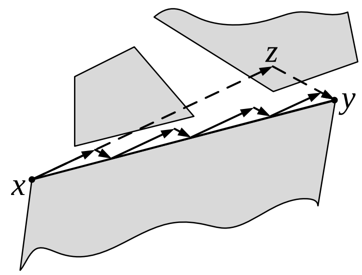

Figure 13: A feasible zigzag path from x to y with the total travel time equal to $\tau'(x, y)$.

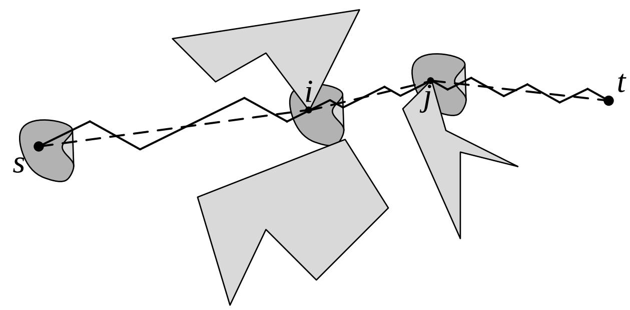

Figure 14: Example of a fastest path for speed function $V'(\theta)$ (dashed line), and an optimal path for speed $V(\theta)$ (solid line).

Step 2. *Use Algorithm 2 to find an optimal path corresponding to the speed function $V'(\theta)$. Let $p_{V'}$ denote the determined path, and let $(k_0, k_1, k_2, ..., k_n)$ be the sequence of vertices path $p_{V'}$ is traversing. Note that $k_0 = s$ and $k_n = t$. Then the corresponding travel time along the path $p_{V'}$, denoted by $t_{V'}(p_{V'})$, can we written as*

$$t_{V'}(p_{V'}) = \Sigma_{i=1}^{n} \tau'(k_{i-1}, k_i) \tag{5}$$

Step 3. *For each pair of consecutive points in $(k_0, k_1, ..., k_n)$, apply Algorithm 1 to find a fastest path between the two points corresponding to the speed function $V(\theta)$. If a given one waypoint path is infeasible due to the presence of obstacles, increase the number of waypoints in a zigzag path as discussed above.*

Step 4. *Combine together the optimal paths found in Step 3. The resulting path has a travel time equal to $t_{V'}(p_{V'})$ and is therefore a fastest obstacle-avoiding path for an arbitrary speed function $V(\theta)$. (See Figure 14.)*

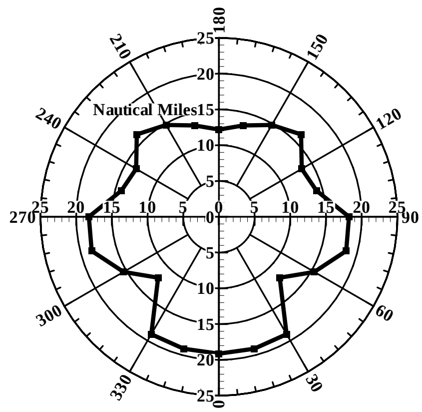

Figure 15: "An example of linear path attainable regions for the S-175 corresponding to voluntary speed loss at Sea State no.7" [8].

5 Applications and Conclusion

5.1 Application: Optimal Short-Range Routing of Vessels in a Seaway

A fastest-path finding problem for the direction-dependent speed functions arises in a wide range of applications. For example, the speed of a sail boat depends on the traveling heading angle it makes with wind, and a vehicle speed varies as the agent traverses up and down a hill. Airplanes have to deal with an anisotropic speed due to wind, while motor boats have similar effects caused by waves. To demonstrate an application in more details, we analyze optimal short-range routing of vessels in a seaway.

Any vessel traveling at a seaway encounters waves which add drag and affect the vessel's performance. In our collaboration with colleagues working on Optimal Vessel Performance in Evolving Nonlinear Wave-Fields project [8], we evaluate the added drag by computing the time average wave force acting on the vessel in the longitudinal direction. Then, by superimposing the added drag on the steady drag experienced by the moving ship in calm waters, we compute the maximum mean attainable speed for each given sea state (which describes the distribution of the waves) and the heading angles in the range from $0°$ to $180°$. Figure 15, borrowed from [8], illustrates an example of the linear path attainable region for the S-175 containership at Sea State no.7. Here, heading is measured as the angle a vessel makes with the dominant wave direction, which is assumed to be in the southerly direction.

For the given LPAR, we can use Theorem 10 to find a fastest path; Algorithm 1 describes the step-by-step procedure to construct such an optimal path. As an example, we consider two scenarios. In first case, let the target point t_1 lie directly east from the starting point s. This example corresponds to the scenario 1 of Theorem 10, since the straight line st_1

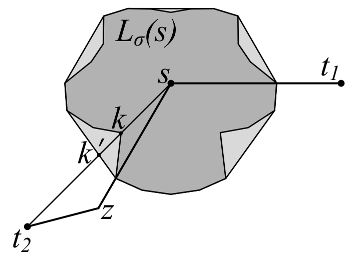

Figure 16: Illustration of the fastest paths from point s to points t_1 and t_2, paths st_1 and szt_2, respectively.

intersects the boundary of the linear path attainable region $L_\delta(s)$ and the boundary of its convex hull at the same point. Hence, we can conclude that the straight line path st_1 is a fastest path from s to t_1, illustrated in Figure 16.

In the second example, let the target point t_2 lie south-west from the starting point s. Then, the intersection points of the line st_2 with the boundary of $L_\delta(s)$ and the boundary of $L_\delta(s)$'s convex hull are not the same (i.e., $k \neq k'$), corresponding to the scenario 2 of Theorem 10. From the theorem we can conclude that a fastest path from s to t_2 is piecewise-linear with one waypoint. Thus, to reach the point t_2 as fast as possible, the vessel should first travel SSE, or 30° clockwise from the south direction, and then complete the travel heading 75° clockwise from south. This corresponds to the path szt_2, illustrated on the Figure 16.

In addition to finding a fastest path from s to t_2, we can use equations (1) to calculate how much improvement in travel time a vessel observes as it follows the optimal path szt_2 instead of following a straight line path st_2. By dividing the length of sk by the length of sk', we find that $\beta = 0.688$, which implies that by following an optimal path we can decrease our travel time at most by approximately 31.2%. This kind of information is particulary useful in evaluating the tradeoffs between following an optimal path as opposed to following a straight line.

In some applications seaway regions might be restricted for vessel's use due to severe weather, presence of land, other vessels, or imposed regulations. For such problems we approximate the restricted regions with polygonal obstacles and apply Algorithm 3 to find an optimal obstacle-avoiding path to the destination. In other applications the vessel's speed is often maintained constant by utilizing a greater amount of fuel and varying the engine thrust level. Our solution approach easily extends to such problems. We redefine the linear path attainable region to represent the set of points one can reach consuming a single unit of fuel. *Then, by using the algorithms presented in this paper we can find a path minimizing fuel consumption instead of traveling time.*

5.2 Conclusion

In this paper, we find the solution to a fastest-path finding problem for a direction-dependent time and space homogeneous speed function. We demonstrate that in an obstacle-free domain an optimal path is piecewise-linear with at most two line segments, regardless of the underline structure of the speed function. This analytical character of our results provides a computationally fast method for finding an optimal path, making it suitable for online applications. We also provide a tight bound on the improvement in travel time by following an optimal path as opposed to traversing a simpler straight line path. Algorithm 1 presented in the paper facilitates a simple implementation of these results.

We also use these results to address the obstacle-avoiding fastest-path problems in anisotropic media. We use the properties of speed functions with the convex polar plots to adapt the visibility graph search method, traditionally used for Euclidean shortest-path problems, to find a solution for these types of problems. Algorithm 2 outlines the fastest-path finding procedure for solving problems with the convex speed polar plots. We then address the case of an arbitrary speed function, which may not correspond to a convex liner path attainable region. For the general scenario, we introduce an augmented speed function such that its polar plot is the convex hull of the original speed plot. Then, to find a lower bound on the minimum travel time for our original problem we apply Algorithm 2 to the augmented speed function. By applying a fastest piecewise-linear path between the nodes of the visibility graph, we construct a path with the travel time equal to its lower bound, thus establishing its optimality. Algorithm 3 gives the detailed steps to finding an optimal obstacle-avoiding path for a general time and space homogeneous speed function.

We discuss the application of the results for optimal vessel routing in a seaway. The numerical example demonstrates over 30% decrease in vessel travel time when our path-finding algorithm is implemented instead of a straight line path.

Acknowledgment

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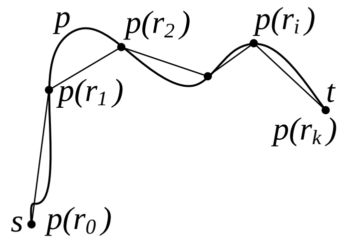

Figure 17: Polygonal path approximation.

A Proofs

A.1 Proof of Lemma 1

Proof.

$$\begin{aligned}\tau(x,y) &= m(y-x)\\ &= m(y-x+z-z)\\ &= m((z-x)+(y-z))\\ &\leq m(z-x)+m(y-z)=\tau(x,z)+\tau(z,y).\end{aligned}$$

□

A.2 Proof of Lemma 2

Proof. To compute $t(p)$ we apply polygonal approximation to the path p. Choose an arbitrary partition Π of interval $[0,1]$, i.e., let $\Pi = (r_0, r_1, r_2, ..., r_k)$ such that $0 = r_0 < r_1 < r_2 < ... < r_{k-1} < r_k = 1$. Let mesh $|\Pi|$ be the maximum length $r_i - r_{i-1}$ of a subinterval of Π, that is, $|\Pi| = \max_{1\leq i\leq k}\{r_i - r_{i-1}\}$. Then Π defines a polygonal approximation to p, i.e., the polygonal arc from $p(0)$ to $p(1)$ having successive vertices $p(r_0)$, $p(r_1)$, ..., $p(r_k)$ (see Figure 17).

Then, the traveling time along the polygonal approximation of the path can be written as $\eta(p,\Pi) = \sum_{i=1}^{k} \tau(p(r_{i-1}), p(r_i))$. However, as we let $|\Pi|$ approach zero, thus increasing the number of vertices, the polygonal approximation in the limit is equal to path p; then so are their travel times (this follows from the assumption that path p is rectifiable). Given this,

$$t(p) = \lim_{|\Pi|\to 0} \eta(p,\Pi). \tag{6}$$

Note that for any arbitrary partition Π, $\eta(p,\Pi) = \tau(p(r_0),p(r_1)) + \tau(p(r_1),p(r_2)) + ... + \tau(p(r_{k-1}),p(r_k))$. After iteratively applying inequality from Lemma 1 we obtain $\eta(p,\Pi) \geq \tau(s,t)$. Substituting this into equation (6) results in $\tau(s,t) \leq t(p)$. □

A.3 Proof of Proposition 4

Proof. First, we prove the **if** direction of the proposition by contradiction.

Assume $L_\delta(x) \neq A_\delta(x)$. From the definitions of $L_\delta(x)$ and $A_\delta(x)$, we know that $L_\delta(x) \subseteq A_\delta(x)$. Then, $L_\delta(x) \subset A_\delta(x)$, that is, $\exists y \in A_\delta(x)$, s.t. $y \notin L_\delta(x)$. Hence, there exists a non-linear path p from point x to point y, such that traveling time along this path is less than traveling time along a straight line path from x to y. However, Theorem 3 states that for a convex linear path attainable region, a straight line path is the fastest path between any two points in $\mathbb{R}^2$. Thus, we reach a contradiction and conclude that if $L_\delta(x)$ is convex, then $L_\delta(x) = A_\delta(x)$.

Next, we prove the **only if** direction of the proposition by contradiction.

Assume that $L_\delta(x) = A_\delta(x)$ but $L_\delta(x)$ is not convex. From Theorem 3 we know that if $L_\delta(x)$ is not convex, then $\exists x, y \in \mathbb{R}^2$ such that the straight line path from x to y is not a fastest path. Let δ_{xy} be the minimum travel time from x to y. Then $y \in A_{\delta_{xy}}(x)$ but $y \notin L_{\delta_{xy}}(x)$ since traveling time along the straight line segment from x to y will be greater than δ_{xy}. This contradicts our assumption that $L_\delta(x) = A_\delta(x)$. Hence, our assumption that $L_\delta(x)$ is not convex was incorrect. $\square$

A.4 Proof of Lemma 5

Proof. We first show that if $V^1(\theta) \leq V^2(\theta)\ \forall\theta$ then $L^1_\delta(x) \subseteq L^2_\delta(x)$. Select an arbitrary point $y \in L^1_\delta(x)$. Then from the definition of $L^1_\delta(x)$, we have $\|y - x\| \leq \delta V^1(\theta_{y-x})$. Since $V^1(\theta) \leq V^2(\theta)\ \forall\theta$, we know that $\|y - x\| \leq \delta V^2(\theta_{y-x})$ as well. Hence, $y \in L^2_\delta(x)$. Since point $y \in L^1_\delta(x)$ was chosen arbitrarily, we can conclude that $L^1_\delta(x) \subseteq L^2_\delta(x)$.

Next, we prove the other direction of the lemma by contradiction. We need to show that if $L^1_\delta(x) \subseteq L^2_\delta(x)$ then $V^1(\theta) \leq V^2(\theta)\ \forall\theta$. Assume that $V^1(\theta) \nleq V^2(\theta)$ for some θ. Then, $\exists\theta^* \in [0, 2\pi]$ such $V^2(\theta^*) < V^1(\theta^*)$. Select $y \in L^1_\delta(x)$ such that $\theta^* = \theta_{y-x}$ and $\|y - x\| = \delta V^1(\theta_{y-x})$. Then $\delta V^2(\theta^*) < \delta V^1(\theta^*) = \delta V^1(\theta_{y-x}) = \|y - x\|$ and $y \notin L^2_\delta(x)$. We reach a contradiction, since $L^1_\delta(x) \subseteq L^2_\delta(x)$. Hence, our assumption that $V^1(\theta) \nleq V^2(\theta)$ for some θ was incorrect. $\square$

A.5 Proof of Lemma 6

Proof. From Lemma 5, we know that if $L^1_\delta(x) \subseteq L^2_\delta(x) \Rightarrow V^1(\theta) \leq V^2(\theta)\ \forall\theta$. Thus, for any heading direction speed $V^2(\theta)$ is always at least as great as $V^1(\theta)$. Select an arbitrary $y \in A^1_\delta(x)$. From the definition of attainable region $A^1_\delta(x)$, there exists a continuous path

$p : [0, 1] \to \mathbb{R}^2$ from point x to point y, such that if a mobile agent's maximum speed function is $V^1(\theta)$, the travel time from x to y is no greater than δ, i.e., $t_{V^1}(p) \leq \delta$. Now, consider following this path p with the maximum speed given by function $V^2(\theta)$. Since $V^1(\theta) \leq V^2(\theta)\ \forall\theta$, we know that travel time along path p with speed corresponding to function $V^2(\theta)$, $t_{V^2}(p)$, will be at most $t_{V^1}(p)$. Hence, $t_{V^2}(p) \leq t_{V^1}(p) \leq \delta$ and thus, point y also belongs to set $A^2_\delta(x)$. Since $y \in A^1_\delta(x)$ was chosen arbitrarily, we can conclude that $A^1_\delta(x) \subseteq A^2_\delta(x)$. □

A.6 Proof of Lemma 7

Proof. Select arbitrary $x_1, x_2 \in L_\delta(x)$ and $\lambda \in [0, 1]$. Note that $\lambda x_1 + (1 - \lambda)x_2$ may not lie in $L_\delta(x)$, since set $L_\delta(x)$ might not be a convex set. Let $y = \lambda x_1 + (1 - \lambda)x_2$. Since $x_1, x_2 \in L_\delta(x)$, $\tau(x, x_1) \leq \delta$ and $\tau(x, x_2) \leq \delta$.

Now consider the following path p: from point x, we travel following the vector $\lambda(x_1 - x)$ and then, continue on following the vector $(1 - \lambda)(x_2 - x)$ (This path is the same path as the one constructed in the proof of Theorem 3 which can be seen in Figure 3.) Then, our path p starts at point x and ends at point $x + \lambda(x_1 - x) + (1 - \lambda)(x_2 - x) = \lambda x_1 + (1 - \lambda)x_2 = y$. Using time and space homogeneity, we can find the travel time for this path: $t(p) = \lambda\tau(x, x_1) + (1 - \lambda)\tau(x, x_2) \leq \lambda \cdot \delta + (1 - \lambda) \cdot \delta = \delta$. Consequently, $y \in A_\delta(x)$. Since x_1, x_2 and λ were chosen arbitrarily, we can conclude that the set of all convex combinations of points from $L_\delta(x)$ lies in $A_\delta(x)$. □

A.7 Proof of Lemma 11

Proof. If $V(\theta) = 0$ for some θ, then s has to be a boundary point of the convex set $L_\delta(s)$. Therefore, there exists a supporting line passing through s such that $L_\delta(s)$ lies on one side of this line. Consequently, there is no linear combination of feasible headings that would deliver us to any point belonging to the other half-space. □

A.8 Proof of Theorem 12

Proof.

1. Proof of this statement follows from Lemma 11. We can construct a supporting line to $L_\delta(s)$ at point s, that separates the LPAR and point t. Concluding, that no feasible path from s to t exists (see Figure 7).

2. To prove the second statement, we select an arbitrary $\epsilon > 0$, such that $\epsilon < \min_\theta\{V(\theta) :$

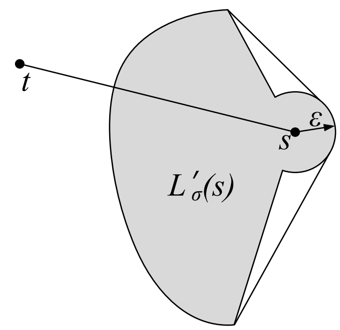

Figure 18: Linear path attainable region corresponding to the speed function $V'(\theta)$.

$V(\theta) > 0\}$. Then, define a new speed function $V'(\theta)$ as follows (see Figure 18).

$$V'(\theta) = \begin{cases} V(\theta), & \text{if } V(\theta) > 0 \\ \epsilon, & \text{if } V(\theta) = 0 \end{cases} \tag{7}$$

By construction, $V(\theta) \leq V'(\theta)$ for all $\theta \in [0, 2\pi]$. Then, from Lemma 5 and Lemma 6 we know that a fastest path corresponding to the speed function $V(\theta)$ cannot be faster than an optimal path corresponding to $V'(\theta)$. Since $V'(\theta) > 0$ for all θ, we can apply Theorem 10 to find a fastest path from s to t corresponding to that speed function. Note that from $\epsilon < \min_\theta\{V(\theta) : V(\theta) > 0\}$, we know that the intersection point of the line st with the boundaries of LPAR and the intersection point of line st with LPAR's convex hull are equal to each other, corresponding to scenario 1 of Theorem 10. Therefore, an optimal path for the speed function $V'(\theta)$ is a straight line path st. Since $V'(\theta_{st}) = V(\theta_{st})$, the straight line path is also feasible for the original speed function, and it has the same travel time. Hence, st is an optimal path for the original speed function $V(\theta)$.

□

A.9 Proof of Theorem 13

Proof. To prove the **if** statement of the theorem, we observe that if $V(\theta) = 0, \forall\theta \in (\underline{\theta}, \bar{\theta})$, $\bar{\theta} - \underline{\theta} \geq \pi$ and $\theta_{st} \in [\underline{\theta}, \bar{\theta}]$, then no linear combination of feasible headings would deliver you from point s to point t. Figure 7 provides a visual example.

Next, we prove the **only if** direction of the theorem by contradiction. Assume that there does not exist a feasible path from s to t, but either $V(\theta_{st}) \neq 0$ or $\bar{\theta} - \underline{\theta} < \pi$. Recall that $\underline{\theta}$ and $\bar{\theta}$ are not defined if $V(\theta_{st}) \neq 0$, thus it is not possible for both conditions to be violated simultaneously. Clearly, if $V(\theta_{st}) \neq 0 \Rightarrow V(\theta_{st}) > 0$, which would mean that the straight line path from s to t is feasible. On the other hand, if $\bar{\theta} - \underline{\theta} < \pi$, then $\exists \epsilon_1, \epsilon_2 \geq 0$ such that $V(\underline{\theta} - \epsilon_1) > 0, V(\bar{\theta} + \epsilon_2) > 0$ and $(\bar{\theta} + \epsilon_2) - (\underline{\theta} - \epsilon_1) < \pi$. Therefore, $\exists \alpha \in [0, 1]$ such that

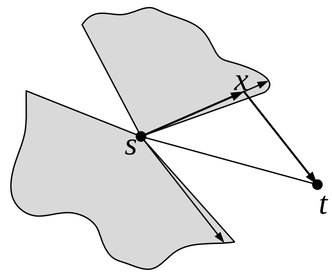

Figure 19: Existence of a feasible path from s to t: path sxt.

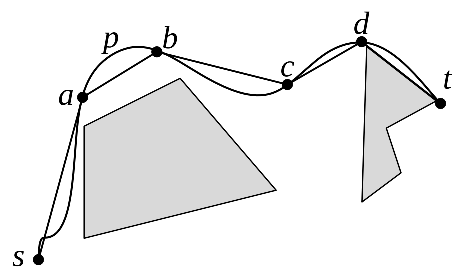

Figure 20: For a convex LPAR, the travel time along the piecewise-linear path *sabcdt* is not greater than along the curve p.

$\theta_{st} = \alpha(\underline{\theta} - \epsilon_1) + (1 - \alpha)(\bar{\theta} + \epsilon_2)$; and hence we can construct a feasible path from s to t by first traveling in the direction $\underline{\theta} - \epsilon_1$ and then turning to the direction $\bar{\theta} + \epsilon_2$. See Figure 19. We reached a contradiction which proves that the original assumption of nonexistence of a feasible path was incorrect. $\square$

A.10 Proof of Theorem 16

Proof. Corollary 15 and the polygonal structure of the obstacles imply that any continuous path $p \in \mathcal{F}$ from s to t can be replaced by a piecewise-linear path from s to t such that the travel time of the piecewise-linear path is not greater than that of the initial path p. Therefore, there exists a piecewise-linear path which is a fastest path from s to t. (See Figure 20 for a visual illustration.)

Next, we show that there exists an optimal piecewise-linear path such that all its vertices correspond to the obstacle vertices. Consider a piecewise-linear path with some vertex a not equal to a vertex of any obstacle in $\mathcal{P}$. Then, there exist two points b and c lying on each of the two line segments of the polygonal path joined by vertex a, such that the line segment bc does not intersect $\mathcal{P}$. We construct a new path by replacing the bac part of the path with a straight line segment bc. From the triangle inequality of Corollary 15 we know that the travel time for the resulting path is not greater than the travel time for the original path. It

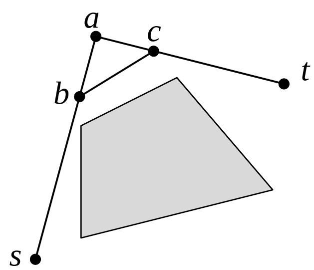

Figure 21: For a convex LPAR, the travel time along the piecewise-linear path *sbct* is not greater than along the path *sat*.

follows that there exists a fastest path which is piecewise-linear and its vertices correspond to the vertices of obstacles in $\mathcal{P}$. (See Figure 21.) □

A.11 Proof of Proposition 17

Proof. [by contradiction] Let p_V denote a fastest path from s to t corresponding to a speed function $V(\theta)$, and $t_V(p_V)$ be the path travel time at speed $V(\theta)$. Assume that $t_V(p_V) < t_{V'}(p_{V'})$. From Lemma 5 we know that since $L_\delta(x) \subseteq L'_\delta(x)$, then $V(\theta) \leq V'(\theta), \forall\theta$. Consequently, traveling along the path p_V with the speed described by function $V'(\theta)$ (denoted by $t_{V'}(p_V)$), constrains the travel time to be less than or equal to $t_V(p_V)$. Hence, we find a feasible path corresponding to the speed function $V'(\theta)$ with the travel time less than or equal to $t_V(p_V)$. Then, the travel time along an optimal path corresponding to the speed function $V'(\theta)$ will be less than or equal to the travel time along this feasible path. That is $t_{V'}(p_{V'}) \leq t_{V'}(p_V) \leq t_V(p_V)$, implying that assumption $t_V(p_V) < t_{V'}(p_{V'})$ is contradictory, and thus proving the proposition. □

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