Information Matrices of
Irregular Factorial Designs

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A Combinatorial Classification of arrays in 2 symbols:

Extended Partially Balanced

PB1

Partially Balanced

Orthogonal
Orthogonal array: (R47)

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No. factors $m = 3$

Only $q=1$ group of factors

All pairs of factors have the same constant Pairwise Index set \{2, 2, 2\}: (number of (0,0)'s, number of (1,0) or (0,1)'s and number of (1,1)'s)

Runs $N = 2\binom{2}{0} + 2\binom{2}{1} + 2\binom{2}{2} = 8$

Information matrix is *invariant* under any permutation of the factors.

Partially Balanced array (Ch56)

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No. of factors $m = 3$, No. of groups $q=1$

All pairs of factors have same Pairwise Index set: \{1, 1, 0\} (KT01)

Runs $N = 1\binom{2}{0} + 1\binom{2}{1} + 0\binom{2}{2} = 3$

Information matrix is *invariant* under any permutation of the factors.
**PB1 array** (KK86)

No. factors $m = 4 = 2 + 2$, No. of groups $q = 2$

| 0 1 | 1 1 |
| 1 0 | 1 1 |
| 0 1 | 0 0 |
| 1 0 | 0 0 |

All pairs of factors in the same group have the same Pairwise Index set:

- $\{0, 2, 0\}$
- $\{2, 0, 2\}$

(Joint) Index set:

- $\text{strength} (t_k)$ of each group  
  \begin{align*}
  \{ \mu_{2,2}^{1,2} = 1, \mu_{2,2}^{1,0} = 1, \mu_{2,2}^{i,j} = 0 \text{ for all other possible } i, j \} \\
  (**) \text{ weight } = \text{ strength}
  \end{align*}

Runs $N = 1 \left( \binom{2}{1} \binom{2}{2} + 1 \binom{2}{1} \binom{2}{0} \right) = 4$

Information matrix is invariant under any permutation within each group, but not between groups.

(*) **Strength** indicates for how many factors in each group there is a common index set.

(**) **Weight** is the no. of 1’s in a vector of 0’s and 1’s.
**Extended Partially Balanced array** (KT02)

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No. of factors $m=2+2+2$, No. of groups $q=3$

All pairs of factors in the **same group** have the same Pairwise Index set: \{0, 2, 0\}; \{2,0, 2\}; \{2,0,2\}

(Joint) Index set:

\[
\{ \mu^{2,2,2}_{1,2,0} = 1, \mu^{2,2,2}_{1,0,2} = 1, \mu^{2,2,2}_{i,j,k} = 0 \text{ for all other possible } i, j, k \}
\]

Runs $N=1 \begin{pmatrix} \binom{2}{1} \binom{2}{2} \binom{2}{0} \end{pmatrix} + 1 \begin{pmatrix} \binom{2}{1} \binom{2}{0} \binom{2}{2} \end{pmatrix} = 4$

Information matrix is *invariant* under any permutation *within* each group but *not between* groups.

In general $q = m$. 

Key concepts

- The “building blocks” (generating arrays) of Orthogonal and Partially Balanced arrays are the *Unit Simple arrays*, consisting of all possible row vectors of weight \( i \), given \( m \).

**Example**: \( m=3 \) factors, weight \( i=2 \):

\[
\begin{array}{ccc}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array}
\]

Index set \( \mu^3_0 = 0, \mu^3_1 = 0, \mu^3_2 = 1 \), Runs \( N = 1 \binom{3}{2} = 3 \)

- The “building blocks” (generating arrays) of PB1 and Extended Partially Balanced arrays are the *Unit Compound arrays*: the Cartesian product of 2 or more Unit Simple arrays.
Example:

\[
\begin{align*}
m &= m_1 + m_2 = 3 + 3 \\
\{ & \mu_{3,3}^{2,1} = 1, \mu_{i,j}^{3,3} = 0 \} \\
\text{for all other possible } i, j \}\end{align*}
\]

for all other possible \( i, j \}

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

Runs \( N = 1 \left( \binom{3}{2} \binom{3}{1} \right) = 9 \)

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
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\begin{pmatrix}
0 & 1 & 0 \\
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0 & 1 & 1
\end{pmatrix}
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\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

- The (joint) **Index set** records how many and which Unit Simple or Compound arrays are present among \( q \) groups of factors \((1 = q = m)\).

- **Marginal Index set** (of order \( p \)) is the Index set of a subarray with \( p (= q) \) groups.
Result 1:

Elements of an information matrix can be written as linear combinations of an Index set (marginal or joint).

Example

Can be represented as PB1 array (q=2) with m= 2+ 2 factors

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strength t= 2+ 2 and Index set:

\{ \mu_{2,2}^1 = 1, \mu_{2,2}^1,0 = 2, \mu_{2,2}^0,1 = 2, \mu_{2,2}^0,0 = 1, \mu_{2,2}^1,1 = 1, \text{ all other 0} \} \}

\[ N = 1 \binom{2}{2} \binom{2}{2} + 2 \binom{2}{2} \binom{2}{1} + 1 \binom{2}{2} \binom{2}{0} + 2 \binom{2}{0} \binom{2}{1} + 1 \binom{2}{2} \binom{2}{0} = 11 \]
Calculation of Information matrix elements:

\[
\sum_{i \in \{1, 2, 3\}} x_i = \mu^{2.2}_{2.2} + \mu^{2.2}_{1.0} + \mu^{2.2}_{2.0} = 4
\]

\[
\sum_{i \in \{4, 5\}} x_i = \mu^{2.2}_{2.2} + \mu^{2.2}_{0.1} = 3
\]

\[
\sum_{i, j \in \{1, 2, 3\}} x_i x_j = \mu^{2.2}_{2.2} + \mu^{2.2}_{2.0} = 2
\]

\[
\sum_{i, j \in \{4, 5\}} x_i x_j = \mu^{2.2}_{2.2} = 1
\]

\[
\sum_{i \in \{1, 2, 3\}, j \in \{4, 5\}} x_i x_j = \mu^{2.2}_{2.2} + \mu^{2.2}_{1.1} + \mu^{2.2}_{1.2} + \mu^{2.2}_{2.1} = 1
\]

\[
\sum_{i \in \{1, 2, 3\}, j, k \in \{4, 5\}} x_i (x_j x_k) = \mu^{2.2}_{1.2} + \mu^{2.2}_{2.2} = 1
\]

\[
\sum_{i, j \in \{1, 2, 3\}, k \in \{4, 5\}} (x_i x_j) x_k = \mu^{2.2}_{2.1} + \mu^{2.2}_{2.2} = 1
\]

\[
\sum_{i, j \in \{1, 2, 3\}, k, \ell \in \{4, 5\}} x_i x_j x_k = \mu^{2.1}_{2.1} = \mu^{3}_{3} = 1
\]

\[
\sum_{i, j \in \{1, 2, 3\}, k, \ell \in \{4, 5\}} (x_i x_j) (x_k x_\ell) = \mu^{2.2}_{2.2} = 1
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**Result 2:**

Any array in 2 symbols can always be represented as an Extended Partially Balanced array with $q^*$ groups ($q^*>q$), with each group being of strength 2 (or 1). Such a representation provides a computationally efficient way to calculate marginal indices.
Result 3

Any *optimal design criterion* that can be written as a function of the elements of the information matrix can be expressed as a linear combination of a (marginal or joint) Index set.

Example:

The *Minimum Contamination criterion*, (XCW02) a measure of aliasing between main effects and two factor interactions, is equivalent to minimizing $| (\mu_{0,0,0}^{1,1,1} + \beta_{1,1,0}^{1,1,1} ) - (\mu_{1,1,1}^{1,1,1} + \beta_{1,0,0}^{1,1,1} ) |$, for all three factor elementwise products.

Note that, if the given design is Orthogonal array, this difference is equal to zero. (KT03)
If such terms are sequentially minimized by considering element wise products of j columns \((j= 3, 4,\ldots)\) the Generalized Minimum Aberration criterion \((XW01)\) is obtained.

**FUTURE RESEARCH**

- Extend Results to arrays for more than 2 symbols.
- Explore new optimality criteria for irregular factorial designs.
- Explore other uses of marginal index.

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REFERENCES

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