Volatility Term Structure in the Q-Alpha-Sigma Model

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Outline

Introduction
  - Implied volatility surface/Q-alpha.sigma model

Statistical Overview

GARCH analysis

PCA Analysis

Vega hedging?

Conclusion
Implied Volatility Surface

Black-Scholes assumes constant volatility

Observed: volatility surface

Surface fluctuations: How to model? Hedge?

vega = \frac{dC}{d\sigma}
Q-Alpha-Sigma Model
(Borland, PRL 2002)

New model for underlying: not GBM

Captures fat tails of stock return:

Successfully approximates the smile.
Term Structure

Volatility surface reduced to term structure:

(Use logarithms of volatilities)

PG: Two- (top) and five-month (bottom) calls

Correlation across maturities?
Statistical Properties of the Data

We can test the time series of fluctuations for

• Repeating patterns:

• Normal distribution:
  – Needed for PCA
  – Does log improve normality?

• ACF
  Box-Ljung Test

• Qq-plot
  Shapiro-Wilk Histogram
There is little autocorrelation.

The time series show normality near the center but the fat-tail shape of the histogram indicates some non-normality.
The GARCH Implied Volatility Model

- Assumes stationarity in the implied volatility time series
- Exhibits observed heteroskedasticity (vol of vol)
- Decomposes dynamics into those attributed to parallel shift and change of slope

\[ \sigma_k(\tau) = \sigma_k + (\tau - T/2) \Delta \sigma_k(\tau) \]

- Avellaneda, Marco and Zhu, Yingzi, "An E-ARCH Model for the Term Structure of Implied Volatility of FX Options", 1997
Variance of Mean Term-Structure

- \( v_k = \text{var} [x_k = \ln \left( \frac{\sigma_{k+1}}{\sigma_k} \right)] \),
- \( x_k = \mu + \varepsilon_k \), \( v_k = \alpha_0 + \alpha_1 \varepsilon_k^2 + \beta_1 v_{k-1} \)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>-3.599e-04</td>
<td>9.201e-04</td>
</tr>
<tr>
<td>( \alpha_0 )</td>
<td>8.505e-05</td>
<td>2.276e-05</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>4.984e-01</td>
<td>4.378e-02</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>8.069e-01</td>
<td>6.485e-03</td>
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### Variance of Slope of Term-Structure

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Std. Error</th>
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<tbody>
<tr>
<td>$\mu_\Delta$</td>
<td>-0.003558</td>
<td>0.011759</td>
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<tr>
<td>$\alpha_{\Delta,0}$</td>
<td>0.057078</td>
<td>0.008537</td>
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<tr>
<td>$\alpha_{\Delta,1}$</td>
<td>0.354185</td>
<td>0.045164</td>
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<tr>
<td>$\beta_{\Delta,1}$</td>
<td>0.651447</td>
<td>0.031985</td>
</tr>
</tbody>
</table>

- $w_k = \text{var} \ [y_k := \ln (\Delta \sigma_{k+1} / \Delta \sigma_k)]$
- $y_k = \mu_\Delta + \epsilon_{\Delta,k}$, $w_k = \alpha_{\Delta,0} + \alpha_{\Delta,1} \epsilon_{\Delta,k}^2 + \beta_{\Delta,1} w_{k-1}$
Principal Component Analysis

Finds uncorrelated axes of variation (eigenvectors of covariance matrix)

\[ \Sigma_{ij} = \mathbb{E}[(X_i - \overline{X_i})(X_j - \overline{X_j})] \]

For us: determines dominant deformations
Interpolate term structure curve from observed maturities and vols:

Sample curves at each month

Study daily displacement of sample points
Reducing Dimensionality

How much change is captured by the most dominant eigenvectors?

The first three capture 80% of the change.
Vega Hedging: Principles

How do you hedge against fluctuations of the volatility surface?

Q-alpha-sigma and PCA can help: they reduce the dimensionality of fluctuations.

Instead of hedging every strike and maturity (~30 options), you only hedge the dominant PCA components (in maturity space) (~3 such).
Vega Hedging: Practicalities

1. Compute the exposure of each option to the dominant PCA eigenvector.

2. Compute your portfolio exposure, using the options you hold.

3. Buy options to cancel this exposure (as cheaply as possible)
Large Vega Fluctuations

Portfolio exposure does not smoothly asymptote

Difficult to hedge: must include many eigenvectors

Potential Solution: Perform PCA on vega-convoluted surface?
Conclusions
Modeled volatility surface dynamics using GARCH
Performed PCA analysis of volatility surface fluctuations
Attempted simple vega-hedging strategy

Future Directions
Understand dynamics better
Study convolution of shifts and vega
ICA: an alternative way?
Acknowledgement

• Thanks to Kay Giesecke, Benjamin Armbruster and

• EvA: Christian Silva, Lisa Borland and Jeremy Evnine.
Appendix
ICA - Alternative Way?

- To find a transformation of the data in which the components are statistically as independent from each other as possible
- ICA proj condense to PCA proj.
- ICA vectors still preserve the sharp peaks.
Box-Ljung & Shapiro-Wilk Test

- **XRX**
  - BL: X-squared = 384.0319, df = 1, p-value < 2.2e-16
  - SW: W = 0.3153, p-value < 2.2e-16

- **UIS**
  - X-squared = 425.4982, df = 1, p-value < 2.2e-16
  - W = 0.7293, p-value < 2.2e-16

- **AMGN**
  - X-squared = 144.5561, df = 1, p-value < 2.2e-16
  - W = 0.816, p-value < 2.2e-16

- **DIS**
  - X-squared = 303.0574, df = 1, p-value < 2.2e-16
  - W = 0.8836, p-value < 2.2e-16

- **PG**
  - X-squared = 351.1064, df = 1, p-value < 2.2e-16
  - W = 0.7998, p-value < 2.2e-16