A model for intraday volatility

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Volatility of stock returns

Let $y$ be the return on a stock. Then the instantaneous volatility is

$$\sigma_t^2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left\{ (y(t + \Delta t) - y(t))^2 \bigg| \mathcal{F}_t \right\}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_0^\infty P\{ (y(t + \Delta t) - y(t))^2 \geq \alpha \big| \mathcal{F}_t \} \, d\alpha$$

$$\approx \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{k=1}^K P\{ |y(t + \Delta t) - y(t)| \geq k\varepsilon \big| \mathcal{F}_t \} \, C(k, \varepsilon)$$

$$= \sum_{k=1}^K \lambda_t^{(k\varepsilon)} \, C(k, \varepsilon)$$

assuming the changes in returns are multiples of some small $\varepsilon > 0$. Here

$$\lambda_t^{(\Delta y)} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\{ N(t + \Delta t) - N(t) > 0 \big| \mathcal{F}_t \}$$

is the intensity of the counting process $N$, which counts the number of jumps of $y$ exceeding some fixed threshold $\Delta y$.

For convenience we call $\lambda_t^{(\Delta y)}$ the volatility.
Volatensity is a proxy for volatility
This can be seen, for example, by comparing it with the GARCH(1,1) fitted to the return series\(^1\), which is a very common representation of volatility.

\(x 10^{-3}\)

\[\text{Intraday volatility (conditional standard deviation) from GARCH(1,1) model}\]

\(x 10^{-3}\)

\[\text{Intraday intensity–based volatility (average # events per hour) for every day}\]

\[\text{Time [hours]}\]

\[\text{Time [hours]}\]

\(^1\) This and following based on TAQ data for IBM stock on 1/05/2007
Our model for volatensity

\[ \lambda_t = \lambda_\infty(N_t) + \int_0^t g(t-s) dN_s \]

where

\[ \lambda_\infty(n)^{-1} = C + \sum_{i=1}^{m} \alpha_i (T_{n-i} - T_{n-i-1}) + \sum_{j=1}^{q} \beta_j \lambda_\infty(n-j)^{-1}, \]

and \( T_i \) is the time of the \( i \)th jump.

We take \( g \) to be equal to

\[ g_{\text{general}}(x) = \frac{1}{\sum_{l=0}^{L} c_l x^l} \quad \text{or} \quad g_{\text{special}}(x) = \frac{a}{(b + x)^c} \]
Reproducing stylized features of volatensity

Slowly decaying autocorrelation:

![Graph of log of Absolute Sample Autocorrelation Function vs log(lag)](image)

- **GARCH(1,1)**
- **Volatensity model**
- **Exponential decay**

Regression line (slope = −1.07, R-squared = 0.83)
Reproducing stylized features of volatensity

Volatile bursts:
Why do we need the two terms?

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Calibrating the intraday volatensity model

1. Cleaning and converting the data:
   - deleting trades outside the standard trading hours,
   - averaging the trade price in single time points,
   - determining the counting process based on a fixed threshold.

2. Estimating model parameters using a two-step procedure:
   - least-squares fitting of the intensity curve to obtain a starting point,
   - MLE estimation using the starting point.

3. Statistical goodness-of-fit tests for time-changed interarrival times, which should be independent exponentials (Meyer 1971):
   - mean/variance,
   - Ljung-Box test,
   - Q-Q plot,
   - Kolmogorov-Smirnov Test.
Historical vs. fitted volatensity

![Graph showing historical and fitted volatensity over time.](image-url)
Statistical tests & forecasting

Q–Q Plot

Empirical CDF vs. Theoretical CDF +/- 5% K–S interval

Estimated / Forecasted Volatility

For time-changed interarrival times:

- sample mean = 1.00
- sample variance = 1.15
- Ljung–Box test (with lags 10–80) accepts the null hypothesis that the model fit is adequate
Conclusions and future work

- We have succeeded in incorporating in our model most of the stylized features of volatility: clustering, slow decay of autocorrelation, and power-law decay after a jump.

- This is an intraday model, but it can be extended to a multiday one.

- One way to achieve this is using the so-called ‘seasonality functions’ to take into account the fact that trading patterns exhibit similarity across different days.

- Another way is to regard the parameters of our model as a realization of a daily time-series and then fitting a model to it (e.g. ARMA).

- Both of those would allow us to make a forecast several days into the future and thus develop a trading strategy.