Finding Equitable Convex Partitions of Points in a Polygon

Efficiently

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The problem:

Find a partition of $C$ into convex pieces of equal area, each containing one point.
$C$: convex polygon with $m$ vertices

$P$: set of $n$ points in general position

Claim: In $O(Nn \log N)$ we can find a partition, where $N = m + n$. 
Equitable 2-partition

\[
\frac{\text{Area}(R)}{\text{points in } R} = \frac{\text{Area}(B)}{\text{points in } B}
\]

Equitable 3-partition

\[
\frac{\text{Area}(R)}{\text{points in } R} = \frac{\text{Area}(B)}{\text{points in } B} = \frac{\text{Area}(G)}{\text{points in } G}
\]
Helper Lemma 1: If $R$ contains $q$ points, $q \leq \lceil n/2 \rceil$, and area($R$) < area($C$) $\cdot q/n$, then we can find an equitable 2-partition in $O(N \log N)$ time.
**Helper Lemma 2:** If two half-spaces through $x$ cut-off areas $A_1$ and $A_2$, then for any $A \in [A_1, A_2]$, we can find a half-space in between cutting off area $A$ in $O(m)$ time.
Helper Lemma 3: We can calculate the area of a convex polygon with $k$ sides in $O(k)$ time.
Main Claim:

\( n = 2q + 1 \) points. In \( O(N \log N) \) time, we find an equitable 2- or 3-partition:
Region Partition Algorithm:

Find a 2- or 3-partition using Lemma 1. Solve smaller sub-problems recursively.
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Proof outline of Main Claim
area(L) ∈ area(C)[\frac{q}{n}, \frac{q+1}{n}] and area(R) ∈ area(C)[\frac{q}{n}, \frac{q+1}{n}]
Case 1

Step 1

too small
correct size

Step 2

correct size
correct size
Case 2

Step 1
- too big
- correct size
- too small

Step 2
Case 3

Step 1

Step 2

Too big

Too small

Correct size

Too big

Too small

Correct size
Case 3, continued

Correct size

Step 3

Correct size

Correct size
Overview of Cases

<table>
<thead>
<tr>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>(−, −)</td>
<td>1</td>
</tr>
<tr>
<td>(−, +)</td>
<td>2</td>
</tr>
<tr>
<td>(+, +)</td>
<td>3</td>
</tr>
</tbody>
</table>
More detailed proof of Main Claim

$v$: Vertex of $\text{conv}(P)$; $\ell$: line through $v$ such that $q$ points lie to the left and right of $\ell$
\(\ell_L, \ell_R\): half-spaces left and right of \(\ell\). Check that

\[
\frac{q}{n} \cdot A \leq \text{area}(\ell_L \cap C), \text{area}(\ell_R \cap C) \leq \frac{q + 1}{n} \cdot A
\]

Otherwise, Helper Lemma 1 gives equitable 2-partition.
$v_L, v_R$: vertices of $\text{conv}(P)$ left and right of $v$.

$H_L, H_R$: upward half-spaces cut off by $vv_L$ and $vv_R$. Check that

$$\text{area}(H_L \cap C), \text{area}(H_L \cap C) \geq A/n$$

Otherwise, Helper Lemma 1 gives equitable 2-partition.
3 Cases: the red and blue regions will each contain $q$ points.
Case 1:

Step 1

- too big
- still too small
- too small

Step 2

- correct size
Case 1, cont.:
Case 2:

Step 1

Step 2

too small

too big

just right

too big

too small
Case 2 cont:

Step 1

too big

⇒ too small

just right

⇒ too small

Step 2

too big

just right

too small
Case 2 result:
Case 3:

Step 1

too big

⇒
too small

Step 2

 Too big

⇒ too small
Case 3 cont:

Step 3

Step 4

⇒ too small

just right

too big

⇒ too small

too big

just right
Case 3 cont:

Step 4

Step 5