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AN OPTIMAL PROCEDURE FOR PARTITIONING A SET OF NORMAL POPULATIONS WITH RESPECT TO A CONTROL

By AJIT C. TAMHANE

Northwestern University, Evanston, USA

SUMMARY. **An optimal solution is derived for the procedure proposed by Tong (1969) for the problem of simultaneously partioning the test population means μ_i $(1 \leqslant i \leqslant k)$ as being less than $\mu_0 + \delta_1^*$ or being greater than $\mu_0 + \delta_2^*$ where μ_0 denotes the control population mean and $\delta_1^*, \delta_2^* (\delta_1^* < \delta_2^*)$ are pressigned constants. The tables of the optimal design constants are provided for implementing the procedure which guarantees a specified probability of making a correct decision regardless of the true values of the μ_i. Savings associated with this optimal solution are illustrated numerically.**

1. Introduction

The purpose of the present paper is to provide an optimal version of the procedure proposed by Tong (1969) for the following problem : Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be $(k+1)$ normal populations with unknown means $\mu_0, \mu_1, \ldots, \mu_k$ and a common variance σ^2 which is assumed to be known. Here Π_0 denotes the control population and Π_i $(1 \leqslant i \leqslant k)$ denotes a test population. For arbitrary but fixed constants δ_1^* and δ_2^* $(\delta_1^* \leqslant \delta_2^*)$, the set of populations $\Omega = (\Pi_1, \Pi_2, \ldots, \Pi_k)$ is divided into three mutually exclusive and collectively exhaustive subsets : $\Omega_B = (\Pi_i : \mu_i \leqslant \mu_0 + \delta_1^*)$, $\Omega_I = (\Pi_i : \mu_0 + \delta_1^* < \mu_i < \mu_0 + \delta_2^*)$ and $\Omega_G = (\Pi_i : \mu_i \geqslant \mu_0 + \delta_2^*)$. The *goal* of the experimenter is to partition Ω into two disjoint subsets S_B and S_G such that $S_B \supset \Omega_B$ and $S_G \supset \Omega_G$. Any such partition is termed as a *correct decision* (CD). (This means that any decision regarding $\Pi_i \in \Omega_I$ is correct.) The experimenter specifies a third constant P^* $(2^{-k} < P^* < 1)$ and requires a procedure which guarantees the *probability requirement* that

$$P(\text{CD} \mid \boldsymbol{\mu}; \sigma^2) \geqslant P^* \;\forall\; \boldsymbol{\mu}, \qquad \ldots \quad (1.1)$$

where $\boldsymbol{\mu}$ denotes the mean vector $(\mu_0, \mu_1, \ldots, \mu_k)$. See Gibbons, Olkin and Sobel (1977, Ch. 10) and Gupta and Panchapakesan (1979, Ch. 20) for some practical applications of this problem and for alternative formulations.

Tong proposed the following procedure to meet the above goal : Take N_0 observations X_{ij} $(1 \leqslant j \leqslant N_0)$ from each Π_i $(0 \leqslant i \leqslant k)$; all the observa-

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tions being independent. Compute $\bar{X}_i = N_0^{-1} \sum_{j=1}^{N_0} X_{ij}$ $(0 \leqslant i \leqslant k)$, and use the decision rule $S_B = (\Pi_i : \bar{X}_i - \bar{X}_0 < \delta^*)$ and $S_G = (\Pi_i : \bar{X}_i - \bar{X}_0 \geqslant \delta^*)$ where $\delta^* = (\delta_1^* + \delta_2^*)/2$. Tong provided tables of N_0 (actually a reparametrized version of N_0 namely $\lambda = \{(\delta_2^* - \delta_1^*)/2\sigma\}\sqrt{N_0/2})$ which guarantee (1.1) for selected values of k and P^*. An extract of this table also appears in Gibbons, Olkin and Sobel (1977).

A natural question that arises after studying this procedure is the following : "How much can you save on the total sample size and still guarantee the same probability requirement by (1) allocating unequal sample sizes to Π_0 and Π_i $(1 \leqslant i \leqslant k)$; and (2) choosing a critical constant possibly different from δ^* in the decision rule described above ?" Sobel and Tong (1971) have considered the optimal choice of the sample sizes although they have not given any tables for their procedure. The second option does not appear to have received any attention.

The purpose of the present paper is to study the simultaneous optimal (in a sense to be made more precise later) choice of both the options and produce the necessary tables. In turns out that savings of up to 20% in the total sample size are possible with our procedure over that of Tong's procedure for the range of k and P^* values studied. For this reason, the results of this paper should be of importance to practitioners. On the theoretical side, the method used in obtaining the optimal value of the critical constant and proving the least favourable (LF) configuration of the decision procedure (note that Tong's proof does not hold here since it depends on the fact that the value of the critical constant $= \delta^*$) also appears to be interesting and of potential use in some other problems.

We consider the following generalization of Tong's procedure : Take N_0 observations X_{0j} $(1 \leqslant j \leqslant N_0)$ from Π_0 and $N_1 = c^2 N_0$ $(c > 0)$ observations X_{ij} $(1 \leqslant j \leqslant N_1)$ from each Π_i $(1 \leqslant i \leqslant k)$, all the observations being independent. Compute the sample means $\bar{X}_i$ $(0 \leqslant i \leqslant k)$ and use the decision rule $S_B = (\Pi_i : \bar{X}_i - \bar{X}_0 < d\sigma/\sqrt{N})$, $S_G = (\Pi_i : \bar{X}_i - \bar{X}_0 \geqslant d\sigma/\sqrt{N})$ where $N = N_0 + kN_1 = (1+kc^2)N_0$ denotes the total sample size and c, d and N are to be determined so that (1.1) is guaranteed. Denote this generalized procedure by $R(c, d) = R(c, d \mid \delta_1^*, \delta_2^*, P^*, k)$. In this paper we determine an optimal choice of c and d, say $\hat{c}$ and $\hat{d}$, and then give tables for implementing $R(\hat{c}, \hat{d})$. Note that in the present notation Tong's procedure is denoted by $R(1, \Delta)$ and Sobel and Tong's procedure by $R(\hat{c}, \Delta)$ where $\Delta = (\Delta_1 + \Delta_2)/2$ and $\Delta_i = \delta_i^* \sqrt{N}/\sigma$ for $i = 1, 2$.

The following is a brief outline of the paper. In Section 2 we first formulate the optimization problem. Then we show that $\hat{d} = \Delta$ for $k = 1$ and k even, but $\hat{d} \neq \Delta$ for k odd ($k \geqslant 3$). The LF configuration for $R(\hat{c}, \hat{d})$ is the same as that obtained by Tong for $R(1, \Delta)$; see Theorem 2.1 for these results. In Sections 2.1 and 2.2 we derive the equations necessary for computing the optimal solution $\hat{c}$, $\hat{d}$ and the associated $\hat{N} = N(\hat{c}, \hat{d})$ (actually a reparametrized version of $\hat{N}$ namely $\hat{b} = (\delta_2^* - \delta_1^*)\sqrt{\hat{N}^{1/2}/2\sigma)}$. In Remark 2.2 we observe that for k odd ($k \geqslant 3$) $\hat{b}$ and $\hat{c}$ do not depend on δ_1^* and δ_2^* and $\hat{d}$ depends on them only through their ratio $\gamma = \delta_2^*/\delta_1^*$. More importantly, for fixed (k, P^*), if given $\hat{b}$ and $\hat{d}$ for some γ, the d-value for any other γ' can be obtained by a very simple relation (2.21). This last result greatly simplifies the extent of the tabulation required. The necessary tables are presented in Section 3 where an example of their use and numerical comparisons with Tong's procedure are also given. Some suggestions are given in Section 4 regarding the case where σ^2 is unknown.

2. Analytical results

2.1. *The infimum of $P(CD)$ and the optimal choice of d.* Since any decision regarding $\Pi_i \in \Omega_I$ is correct, to find the infimum of P(CD) of $R(c, d)$ w.r.t. $\boldsymbol{\mu}$, we may restrict attention to $\boldsymbol{\mu}$ such that Ω_I is empty. Furthermore, since P(CD) is increasing in $\mu_i - \mu_0$ for $\Pi_i \in \Omega_G$ and decreasing in $\mu_i - \mu_0$ for $\Pi_i \in \Omega_B$, it is minimized when $\mu_i - \mu_0 = \delta_2^*$ for $\Pi_i \in \Omega_G$ and $\mu_i - \mu_0 = \delta_1$ for $\Pi_i \in \Omega_B$. Thus in finding the infimum of P(CD) w.r.t. $\boldsymbol{\mu}$, without loss of generality, we can restrict attention to parameter configurations $\boldsymbol{\mu}^0(r)$ where $\mu_1^0 - \mu_0^0 = \ldots = \mu_r^0 - \mu_0^0 = \delta_1^*$ and $\mu_{r+1}^0 - \mu_0^0 = \ldots = \mu_k^0 - \mu_0^0 = \delta_2^*$ for some r, $0 \leqslant r \leqslant k$. Therefore our optimization problem can be formulated as follows. (For mathematical convenience we shall relax the integer restrictions on N_0, N_1 and N and henceforth regard them as nonnegative variables.)

For specified values of δ_1^*, δ_2^*, P^*, k and σ^2 find the minimum N such that

$$\max_{c>0} \max_{d} \min_{0 \leqslant r \leqslant k} P\{\text{CD} \mid \boldsymbol{\mu}^0(r), \sigma^2\} \geqslant P^*. \qquad \ldots (2.1)$$

In (2.1),

$$
\begin{aligned}
&P\{\text{CD} \mid \boldsymbol{\mu}^0(r), \sigma^2\} \\
&= P\{\bar{X}_i - \bar{X}_0 < d\sigma/\sqrt{N}\ (i = 1, \ldots, r),\ \bar{X}_i - \bar{X}_0 \geqslant d\sigma/\sqrt{N} (i = r+1, \ldots, k) \mid \boldsymbol{\mu}^0(r), \sigma^2\} \\
&= P\{Y_i \leqslant c(d - \Delta_1)(1+c^2)^{-1/2}(1+kc^2)^{-1/2}\ (i = 1, \ldots, r), \\
&\qquad Y_i \leqslant c(\Delta_2 - d)(1+c^2)^{-1/2}(1+kc^2)^{-1/2}\ (i = r+1, \ldots, k)\} \qquad \ldots (2.2)
\end{aligned}
$$

where Y_i are standard normal random variables (r.v.'s) with $\text{corr}(Y_i, Y_j) = \rho = c^2/(1+c^2)$ for $1 \leqslant i, j \leqslant r$ or $r+1 \leqslant i, j \leqslant k$ and $\text{corr}(Y_i, Y_j) = -\rho$ for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant k$. An alternative expression for (2.2) is

$$P\{\text{CD} \mid \boldsymbol{\mu}^0(r), \sigma^2\}$$

$$= \int_{-\infty}^{\infty} \Phi^r[c\{z+(d-\Delta_1)(1+kc^2)^{-1/2}\}]\Phi^{k-r}[c\{-z+(\Delta_2-d)(1+kc^2)^{-1/2}\}]\phi(z)dz \quad \ldots (2.3)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ denote, respectively, the c.d.f. and the p.d.f. of a standard normal r.v. Denote the r.h.s. of (2.3) by $\psi(c, d, r) = \psi(c, d, r \mid \Delta_1, \Delta_2, k)$. We consider two cases :

Case 1. $k = 1$: Here it is straightforward to see that

$$\max_{c>0} \max_{d} \min_{r=0,1} \psi(c, d, r)$$

$$= \max_{c>0} \max_{d} \min\{\Phi[c(d-\Delta_1)/(1+c^2)], \Phi[c(\Delta_2-d)/(1+c^2)]\}$$

$$= \max_{c>0} \Phi[c(\Delta_2-\Delta_1)/2(1+c^2)]$$

$$= \Phi[(\Delta_2-\Delta_1)/4] \quad \ldots (2.4)$$

where the optimal (saddle-point) value $\hat{d} = (\Delta_1+\Delta_2)/2 = \Delta$ and $\hat{c} = 1$. Thus $R(1, \Delta)$ is optimal in this case and the corresponding minimum total sample size is given by

$$\hat{N} = 16\sigma^2\{\Phi^{-1}(P^*)\}^2/(\delta_2^*-\delta_1^*)^2. \quad \ldots (2.5)$$

Next consider

Case 2. $k > 1$: We note that for fixed c,

$$\max_{d} \min_{r} \psi(c, d, r) = \min_{r} \max_{d} \psi(c, d, r),$$

and therefore we may solve this part of the problem by fixing r $(0 \leqslant r \leqslant k)$ and obtaining the maximizing value of d, say, $\hat{d}_r = \hat{d}_r(c)$ first. From (2.2) it is obvious that $\hat{d}_0 = -\infty$ and $\hat{d}_k = +\infty$ and $\psi(c, \hat{d}_0, 0) = \psi(c, \hat{d}_k, k) = 1$ for any $c > 0$. Next for $1 \leqslant r \leqslant k-1$, $\hat{d}_r$ satisfies (2.6) below obtained by setting the derivative of the r.h.s. of (2.3) w.r.t. d (the differentiation under the integral sign is permissible here) equal to zero :

$$r \int_{-\infty}^{\infty} \Phi^{r-1}[c\{z+A(\hat{d}_r-\Delta_1)\}]\Phi^{k-r}[c\{-z+A(\Delta_2-\hat{d}_r)\}]\phi[c\{z+A(\hat{d}_r-\Delta_1)\}]d\Phi(z)$$

$$= (k-r) \int_{-\infty}^{\infty} \Phi^r[c\{z+A(\hat{d}_r-\Delta_1)\}]\Phi^{k-r-1}[c\{-z+A(\Delta_2-\hat{d}_r)\}]$$

$$\times\phi[c\{-z+A(\Delta_2-d_r)\}]d\Phi(z), \quad \ldots (2.6)$$

where we have put $A = A(k, c) = (1+kc^2)^{-1/2}$. A similar equation can be obtained for $\hat{d}_{k-r}$ which by a change of variables $z' = -z$ in both the integrals and dropping the primes can be written as

$$r \int_{-\infty}^{\infty} \Phi^{r-1}[c\{z+A(\Delta_2-\hat{d}_{k-r})\}]\Phi^{k-r}[c\{-z+A(\hat{d}_{k-r}-\Delta_1)\}]\phi[c\{z+A(\Delta_2-\hat{d}_{k-r})\}]d\Phi(z)$$

$$= (k-r) \int_{-\infty}^{\infty} \Phi^{r}[c\{z+A(\Delta_2-\hat{d}_{k-r})\}]\Phi^{k-r-1}[c\{-z+A(\hat{d}_{k-r}-\Delta_1)\}]$$

$$\times \phi[c\{-z+A(\hat{d}_{k-r}-\Delta_1)\}]d\Phi(z). \quad \ldots \quad (2.7)$$

Remark 2.1. Here we are assuming that solutions exist to (2.6) and (2.7), that they are unique and that they are in fact associated with the maximums. It is difficult to analytically verify these assumptions. But we made a numerical study of the integral (2.3) for $2 \leqslant k \leqslant 5$, $1 \leqslant r \leqslant k-1$, and for selected values of $c \in [0{\cdot}5, 1]$ and of (Δ_1, Δ_2). We found that the integral behaves as a unimodal function of d with a unique maximum. This provides an empirical justification of the stated assumptions. In fact it is observed that $\hat{d}_r$ is increasing in r which is intuitive upon inspecting expression (2.2).

From (2.6) and (2.7) it follows that

$$\hat{d}_r+\hat{d}_{k-r} = \Delta_1+\Delta_2 \quad \ldots \quad (2.8)$$

for $1 \leqslant r \leqslant k-1$, $\hat{d}_r < \Delta$ for $r < k/2$ and vice-versa and $\hat{d}_{k/2} = \Delta$ for k even.

It is also easy to see that

$$\psi(c, \hat{d}_r, r) = \psi(c, \hat{d}_{k-r}, k-r). \quad \ldots \quad (2.9)$$

We are now in a position to prove the following theorem.

Theorem 2.1 : *Let $m = [k/2]$ where $[x]$ denotes the largest integer $\leqslant x$ and let $c > 0$ be fixed. Then for k even we have*

$$\min_{0 \leqslant r \leqslant k} \psi(c, \hat{d}_r, r) = \psi(c, \Delta, m). \quad \ldots \quad (2.10)$$

For k odd ($k \geqslant 3$) we have

$$\min_{0 \leqslant r \leqslant k} \psi(c, \hat{d}_r, r) = \psi(c, \hat{d}_m, m)$$

$$= \psi(c, \hat{d}_{m+1}, m+1) \quad \ldots \quad (2.11)$$

where $\hat{d}_m < \Delta < \hat{d}_{m+1}$.

Proof: To prove (2.10) for k even we only have to show that $\psi(c, \Delta, r) \geqslant \psi(c, \Delta, m)$ for $r \neq m$. This follows by the theorem given in the appendix of Tong (1969) or using Slepian's (1962) inequality in (2.2).

To prove (2.11) for k odd ($k \geqslant 3$) again we only have to show that $\psi(c, \hat{d}_m, r) \geqslant \psi(c, \hat{d}_m, m)$ for $r \neq m$, $m+1$. Because of (2.9) we can take $r < m$. Then from (2.2) we obtain

$$\psi(c, \hat{d}_m, r) = P\{Y_i \leqslant (\hat{d}_m - \Delta_1)B\,(i = 1, \ldots, r),\ Y_i \leqslant (\Delta_2 - \hat{d}_m)B\,(i = r+1, \ldots, k)\} \quad \ldots \quad (2.12)$$

where we have put $B = B(k, c) = (1+c^2)^{-1/2}A$ and the Y_i are as given below (2.2). Now since $m < k/2$, we have that $\hat{d}_m - \Delta_1 \leqslant \Delta_2 - \hat{d}_m$ and therefore if we increase r to m then we decrease (do not increase) the upper limits on the Y_i. Next if we increase r to m then we maximize the number of negative correlations between the Y_i's (from $r(k-r)$ to $m(k-m)$). This latter fact together with Slepian's inequality and the former fact that the upper limits on the Y_i decrease, allow us to conclude that the probability in (2.12) decreases, i.e., $\psi(c, \hat{d}_m, r) \geqslant \psi(c, \hat{d}_m, m)$ for $r < m$ and hence (2.11) follows. The fact that $\hat{d}_m < \Delta < \hat{d}_{m+1}$ follows from the earlier observation that $\hat{d}_r < \Delta$ for $r < k/2$. This proves the theorem.

Henceforth it would be convenient to use the reparametrization

$$b = (\delta_2^* - \delta_1^*)\sqrt{N}/2\sigma = (\Delta_2 - \Delta_1)/2,$$

$$b_1 = d - \Delta_1 = d - 2b/(\gamma - 1),$$

and

$$b_2 = \Delta_2 - d = \{2b\gamma/(\gamma-1)\} - d. \quad \ldots \quad (2.13)$$

In the new notation we have $\hat{d}_m = b(\gamma+1)/(\gamma-1)$ for k even and

$$\psi(c, \hat{d}_m, m) = \begin{cases} \int_{-\infty}^{\infty} \Phi^m[c(z+bA)]\Phi^m[c(-z+bA)]d\Phi(z) \text{ (for } k \text{ even)} & \ldots \quad (2.14a) \\ \int_{-\infty}^{\infty} \Phi^m[c(z+b_1A)]\Phi^{k-m}[c(-z+b_2A)]d\Phi(z) \text{ (for } k \text{ odd)} & \ldots \quad (2.14b) \end{cases}$$

where in (2.14b), d satisfies the equation (for fixed c)

$$m \int_{-\infty}^{\infty} \Phi^{m-1}[c(z+Ab_1)]\Phi^{k-m}[c(-z+Ab_2)\phi[c(z+Ab_1)]d\Phi(z)$$

$$= (k-m) \int_{-\infty}^{\infty} \Phi^m[c(z+Ab_1)]\Phi^{k-m-1}[c(-z+Ab_2)]\phi[c(-z+Ab_2)]d\Phi(z). \quad \ldots \quad (2.15)$$

2.2. *The optimal choice of c.* In this section we consider the problem of maximizing $\psi(c) = \psi(c, \hat{d}_m, m)$ w.r.t. c. As in Sobel and Tong (1971) (also see Bechhofer, 1969) we set $\psi'(c) = \partial\psi(c)/\partial c = 0$ and obtain c as the solution to the resulting equation. We give in (2.16) below the equation obtained by setting the derivative of the r.h.s. of (2.14b) (note that (2.14b) reduces to (2.14a) if we put for k even, $b_1 = b_2 = b$ and $m = k-m$) w.r.t. c equal to zero; we omit the details since similar details may be found in Sobel and Tong (1971). In taking this derivative, we regard b_1 and b_2 as fixed (and not dependent on c through (2.15)); eventually the equations $\psi'(c) = 0$ and (2.15) will be solved simultaneously thus accounting for the interdependence of various quantities. We have

$$m \int_{-\infty}^{\infty} (y-b_1D)\Phi^{m-1}(y)\Phi^{k-m}[-y+c(b_1+b_2)A]\phi(y/c-b_1A)d\Phi(y)$$
$$+(k-m) \int_{-\infty}^{\infty} (y-b_2D)\Phi^{k-m-1}(y)\Phi^{m}[-y+c(b_1+b_2)A]\phi(y/c-b_2A)d\Phi(y) = 0, \qquad \ldots (2.16)$$

where we have put $D = D(k, c) = kc^3A^3$. For k even (2.16) simplifies to the equation

$$\int_{-\infty}^{\infty} (y-bD)\Phi^{m-1}(y)\Phi^{m}(-y+2bcA)\phi(y/c-bA)d\Phi(y) = 0. \qquad \ldots (2.17)$$

For $k = 2$, as in Sobel and Tong (1971), (2.17) simplifies even further to the equation

$$\{b(1-2c^4)/(1+2c^2)(1+c^2)^{1/2}\}\Phi[bc(1+c^2)^{-1/2}] = c\phi[bc(1+c^2)^{-1/2}]. \qquad \ldots (2.18)$$

The two integrals on the l.h.s. of (2.16) can be integrated by parts and from the resulting expression it can be shown that as $P^* \to 1$ (i.e., as $b_1, b_2 \to \infty$), the solution $\hat{c}^2 \to k^{-1/2}$; a result also obtained by Sobel and Tong (1971) although for the less general case of $d = \Delta$ and, in a different context, by Bechhofer (1969). Thus for P^* close to one, an approximation to optimal allocation is obtained by letting $\hat{N}_0 \cong \hat{N}_1\sqrt{k}$.

2.3. *Equations for computing the optimum solutions.* The constrained optimization problem (2.1) involves a single inequality which can therefore be replaced by the corresponding equality. This fact together with the results obtained in the previous two sections enable us to reduce the optimization problem (2.1) to that of solving a system of simultaneous equations as follows.

Case 1. *k even* : For given values of P^* and k find the solution in b and c ($\hat{b}$ and $\hat{c}$) to the following equations :

(i) The r.h.s. of (2.14a) $= P^*$.

(ii) Eq. (2.17) (for $k > 2$).

Eq. (2.18) (for $k = 2$). ... (2.19)

Case 2. *k odd* ($k \geqslant 3$) : For given values of $\gamma = \delta_2^*/\delta_1^*$, P^* and k find the solution in b, c and d ($\hat{b}$, $\hat{c}$ and $\hat{d}$) to the following equations :

(i) The r.h.s. of (2.14b) $= P^*$.

(ii) Eq. (2.15).

(iii) Eq. (2.16). ... (2.20)

The quantities b_1 and b_2 appearing in the above equations are given by (2.13).

Remark 2.2 : Since $\hat{d} = \Delta$ for $k = 1$ and k even, $\hat{b}$ and $\hat{c}$ do not depend on δ_1^* and δ_2^* for fixed (k, P^*) although $\hat{N}$ does. Now consider the case k odd, $k \geqslant 3$. From (2.13), (2.14b), (2.15) and (2.16) it is clear that $\hat{b}$, $\hat{c}$, $\hat{d}$ depend on δ_1^*, δ_2^* only through their ratio $\gamma = \delta_2^*/\delta_1^*$. For fixed (k, P^*), let $(\hat{b}, \hat{c}, \hat{d})$ denote the optimal solution corresponding to γ and $(\hat{b}', \hat{c}', \hat{d}')$ corresponding to γ'. Now note that b and d appear in the relevant equations only through b_1 and b_2 and that if $\hat{b}' = \hat{b}$ and

$$\hat{d}' = \hat{d} + \hat{b}(\gamma - \gamma')/\{(\gamma - 1)(\gamma' - 1)\} \qquad \ldots \quad (2.21)$$

then $\hat{b}_i' = \hat{b}_i$ $(i = 1, 2)$. Therefore, $\hat{b}' = \hat{b}$, $\hat{c}' = \hat{c}$ and $\hat{d}'$ given by (2.21) is the optimal solution corresponding to γ'. Thus $\hat{b}$ and $\hat{c}$ do not depend on γ although $\hat{d}$ does. Because of (2.21), however, it suffices to compute $(\hat{b}, \hat{c}, \hat{d})$ just for one value of γ.

3. Numerical results

3.1. *Tables.* The sets of simultaneous equations (2.19) and (2.20) were solved on Northwestern's CDC 6600 computer using the IMSL subroutine ZSYSTM for $k = 2(1)10$, $P^* = 0{\cdot}75$, $0{\cdot}90$, $0{\cdot}95$, $0{\cdot}99$ and $\gamma = 2$ (for k odd, $k \geqslant 3$). The values of $(\hat{b}, \hat{c})$ computed via (2.19) for even values of k are given in Table 1. The values of $(\hat{b}, \hat{c}, \hat{d})$ computed via (2.20) for odd values of k are given in Table 2. For $k = 1$ the standard normal tables can be used in conjunction with (2.5).

TABLE 1. VALUES OF $\hat{b}$ AND $\hat{c}$ FOR EVEN k

k \ P^*	0.75	0.90	0.95	0.99
2	2.7586	3.9691	4.7314	6.2186
	0.8232	0.8387	0.8404	0.8409
4	4.4251	5.8026	6.6787	8.4059
	0.7329	0.7228	0.7174	0.7110
6	5.7241	7.2315	8.1947	10.1048
	0.6733	0.6578	0.6509	0.6432
8	6.8373	8.4532	9.4891	11.5523
	0.6310	0.6139	0.6066	0.5988
10	7.8319	9.5424	10.6417	12.8389
	0.5990	0.5813	0.5741	0.5663

Note : The upper entry in each cell is $\hat{b}$ and the lower entry is $\hat{c}$.

TABLE 2. VALUES OF $\hat{b}$, $\hat{c}$ AND $\hat{d}$ FOR ODD $k(k > 3)$

k \ P^*	0.75	0.90	0.95	0.99
3	3.5671	4.8826	5.7144	7.3443
	0.7679	0.7684	0.7660	0.7623
	10.1544	14.1926	16.7337	21.6950
5	5.0697	6.5206	7.4453	9.2732
	0.6991	0.6861	0.6799	0.6729
	14.8672	19.2702	22.0705	27.5971
7	6.2798	7.8461	8.8484	10.8404
	0.6501	0.6338	0.6268	0.6190
	18.5810	23.3153	26.3412	32.3485
9	7.3350	9.0012	10.0706	12.2042
	0.6138	0.5964	0.5892	0.5814
	21.7932	26.8194	30.0427	36.4687

Note : The upper entry in each cell is $\hat{b}$, the middle entry $\hat{c}$ and the lower entry $\hat{d}$. The $\hat{d}$ values are computed for $\gamma = 2$. For any other γ use the relation $\hat{d}$ = tabulated $\hat{d}+\hat{b}(2-\gamma)/(\gamma-1)$.

It might be noted that the approach of $\hat{c}$ to $k^{-1/4}$ as P^* increases is fairly rapid and appears to be monotonic (for $P^* \geqslant 0{\cdot}75$) for $k \geqslant 4$. For k odd ($k \geqslant 3$), the values of $\hat{d}$ are not too much off from Tong's choice $d = \Delta = b(\gamma+1)/(\gamma-1) = 3b$ for $b = \hat{b}$. Not surprisingly, the agreement between the two d-values gets closer as k increases (and/or P^* increases).

A numerical comparison with Tong's procedure shows that the savings because of using $R(\hat{c}, \hat{d})$ instead of $R(1, \Delta)$ are in the range of 3% to 20% for the (k, P^*) values studied in this paper and they in general increase with both k and P^*. The ratio of the total sample size required by $R(\hat{c}, \hat{d})$ to that required by $R(1, \Delta)$ is given in Table 3 for $k = 1(1)10$ and $P^* = 0{\cdot}75$, $0{\cdot}90$, $0{\cdot}95$ and $0{\cdot}99$.

TABLE 3. RATIO $N(\hat{c}, \hat{d})/N(1, \Delta)$ OF THE TOTAL SAMPLE SIZES REQUIRED BY $R(\hat{c}, \hat{d})$ AND $R(1, \Delta)$ TO GUARANTEE GIVEN P^*

k \ P^*	0.75	0.90	0.95	0.99
1	1.0000	1.0000	1.0000	1.0000
2	0.9652	0.9708	0.9713	0.9714
3	0.9138	0.9193	0.9205	0.9226
4	0.9278	0.9170	0.9115	0.9050
5	0.9043	0.8902	0.8835	0.8758
6	0.8984	0.8785	0.8693	0.8583
7	0.8835	0.8611	0.8509	0.8388
8	0.8764	0.8506	0.8390	0.8250
9	0.8657	0.8381	0.8257	0.8108
10	0.8593	0.8295	0.8101	0.8000

We now give an example of the use of the tables.

Example : Suppose that we have $k = 9$ test populations with means μ_i $(1 \leqslant i \leqslant 9)$ which are to be compared with a given control population with mean μ_0, all the populations being normal with a common known variance $\sigma^2 = 0{\cdot}5$. Let $\delta_1^* = 0{\cdot}2$, $\delta_2^* = 1{\cdot}0$ and $\gamma = \delta_2^*/\delta_1^* = 5{\cdot}0$. Thus populations with means $\mu_i \leqslant \mu_0 + 0{\cdot}2$ are regarded as "bad" (belong to the set Ω_B)

while those with means $\mu_i \geqslant \mu_0+1\cdot 0$ are regarded as "good" (belong to the set Ω_G). We wish to select a subset S_G of test populations (S_B being the complement of S_G) which contains all "good" populations but no "bad" populations with probability at least $P^* = 0\cdot 90$. For this purpose the sample sizes $\hat{N}_0$, $\hat{N}_1$ and the critical constant $\hat{d}$ to be used in the procedure $R(\hat{c}, \hat{d})$ can be obtained as follows.

From Table 2 we get $\hat{b} = 9\cdot 0012$, $\hat{c} = 0\cdot 5964$ and $\hat{d} = 26\cdot 8914$ (for $\gamma = 2\cdot 0$). Therefore,

$$\hat{N} = \left[\left(\frac{9\cdot 0012 \times 2 \times 0\cdot 5}{1-0\cdot 2} \right)^2 \right] = 127.$$

Out of these 127 observations we can allocate $\hat{N}_0 = 28$ to the control population and $\hat{N}_1 = 11$ to each of the nine test populations giving the ratio $c = \sqrt{11/28} = 0\cdot 6268$ which is fairly close to the optimal $\hat{c}$. To determine $\hat{d}$ for $\gamma = 5\cdot 0$ we use the formula in the footnote of Table 2 to obtain

$$\hat{d} = 26\cdot 8914 + 9\cdot 0012 \times (-3)/4 = 20\cdot 1405.$$

Thus the desired set S_G is given by

$$S_G = \left\{ \Pi_i : \bar{X}_i - \bar{X}_0 \geqslant \frac{20\cdot 1405 \times 0\cdot 5}{\sqrt{127}} = 0\cdot 8936 \right\}. \quad \ldots \quad (3.1)$$

Let us now compare this optimal procedure with Tong's procedure $R(1, \Delta)$ which uses a common sample size N_0 on each population including the control and critical constant $\delta^* = (\delta_1^* + \delta_2^*)/2 = 0\cdot 6$ in (3.1) in place of $0\cdot 8936$. Table 1 of Tong (1969) gives $\lambda = \{(\delta_2^* - \delta_1^*)/2\sigma\} \sqrt{N_0/2} = 2\cdot 1986$ for $k = 9$ and $P^* = 0\cdot 90$ from which we obtain $N_0 = 16$. Thus the total sample size required by Tong's procedure is $N = 10 \times 16 = 160$. Notice that the optimal procedure gives a substantial saving of 33 observations.

4. Unknown σ

This paper solves a design problem which assumes the knowledge of σ. If σ is not known then, strictly speaking, the probability requirement (1.1) cannot be guaranteed by a single-stage procedure; one must use a two-stage or a sequential procedure as described in Sections 2 and 3, respectively, of Tong (1969). If practical considerations dictate the use of a single-stage procedure then one can use the results of the present paper to provide approximate solutions as follows : Either an upper bound on σ can be specified for design purposes or one can specify δ_1^* and δ_2^* as multiplesof unknown σ rather than as fixed constants. In both cases $\hat{N}_0$ and $\hat{N}_1$ can be calculated using the methods given in the paper. Once the data are collected, the decision rule $R(\hat{c}, \hat{d})$ should be applied by substituting the usual pooled sample standard deviation in place of σ.

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