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INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL (II): OPTIMAL DESIGNS FOR ONE-SIDED COMPARISONS WHEN $p = 2(1)6$, $k = 2$ AND $p = 3$, $k = 3$

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SUMMARY. In this article we continue the study of balanced treatment incomplete block (BTIB) designs initiated in Bechhofer and Tamhane (1981). These designs are appropriate for comparing simultaneously $p \geqslant 2$ test treatments with a control treatment in blocks of common size $k < p+1$. The general class of BTIB designs was characterized in that first article. In the present article we study in detail the particular cases $p \geqslant 2$, $k = 2$ and $p = 3$. $k = 3$. These cases share a special property, namely that there are only two so-called generator designs in the minimal complete set. This fact enables us to give for these cases a simple characterization of admissible designs which are the only contenders for optimal designs.

We have computed tables of discrete optimal designs for joint one-sided comparisons for the cases $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$. The special property possessed by these cases also enables us to develop a simple continuous approximation to the discrete optimal designs. Using this approximation we have computed analogous tables of continuous optimal designs; these tables can be used when large b-values are required. The theory underlying the approximation is developed, and its goodness is assessed.

1. Introduction

In Bechhofer and Tamhane (1981) (referred to hereinafter as B-T) we initiated the study of balanced treatment incomplete block (BTIB) designs which are appropriate for comparing simultaneously $p \geqslant 2$ test treatments with a control treatment in blocks of common size $k < p+1$. This general class of designs was characterized in B-T. In the present article we obtain optimal designs within this class for $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$.

The cases $p \geqslant 2$, $k = 2$ and $p = 3$, $k = 3$ share a special property, namely that there are exactly two so-called generator designs in the minimal complete set for each of the (p, k)-values. (See Section 2.2 for definitions of the various technical terms used in this section.) This fact enables us to give a simple characterization of admissible designs for these cases.

When the minimal complete set consists of only two generator designs, it is possible to develop a simple continuous approximation to the discrete optimal designs. The problem of obtaining the continuous optimal designs is easy to solve on a computer. Also, use of the approximation substantially reduces the number of designs to be tabulated. In the present article we give discrete as well as continuous optimal designs for *one-sided* comparisons with a control. More detailed tables of discrete optimal designs for the (p, k)-values considered in the present article as well as for many additional ones of practical interest are given in Bechhofer and Tamhane (1983) both for one-sided and for two-sided comparisons.

In order to make the present article self-contained we state below the key definitions and results from B-T. We shall use the following notation (also used in B-T) : Let the treatments be indexed by $0, 1, \ldots, p$ with 0 denoting the control treatment and $1, 2, \ldots, p$ denoting the test treatments. The $N = kb$ experimental units can be arranged in b blocks each of common size k. If treatment i is assigned to the h-th plot of the j-th block $(0 \leqslant i \leqslant p,\ 1 \leqslant h \leqslant k,\ 1 \leqslant j \leqslant b)$, let Y_{ijh} denote the corresponding random variable; we assume the usual additive linear model (no treatment$\times$block interaction)

$$Y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} \qquad \ldots \quad (1.1)$$

with $\sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$; the e_{ijh} are assumed to be i.i.d. $N(0, \sigma^2)$ random variables, and σ^2 is assumed to be known. It is desired to make joint interval estimates (employing one-sided or two-sided intervals) of the p differences $\alpha_0 - \alpha_i$ based on their best linear unbiased estimators (BLUE's) $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leqslant i \leqslant p)$.

In Section 3.1 of B-T we proposed a class of incomplete block designs which are *balanced* with respect to the *test* treatments in the following sense : $\text{var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \tau^2\sigma^2$ $(1 \leqslant i \leqslant p)$ and $\text{corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \rho$ $(i_1 \neq i_2;\ 1 \leqslant i_1, i_2 \leqslant p)$; the parameters τ and ρ depend on the design employed. We refer to designs with this property as BTIB designs. (We have recently learned that Pearce (1960) had proposed designs with this same property; he called them designs with "supplemented balance.") Conditions that a design must satisfy in order that it be BTIB were given in Theorem 3.1 of B-T. This theorem states that if $\{r_{ij}\}$ is the incidence matrix of the design, r_{ij} being the total number of times the i-th treatment appears in the j-th block, and if $\lambda_{i_1 i_2} = \sum_{j=1}^{b} r_{i_1 j} r_{i_2 j}$ which is the total number of times that the i_1-th treatment appears with the i_2-th treatment in the same block over the whole design

$(i_1 \neq i_2;\ 0 \leqslant i_1, i_2 \leqslant p)$, then the necessary and sufficient conditions for a design to be BTIB are that

$$\lambda_{01} = \lambda_{02} = \ldots = \lambda_{0p} = \lambda_0 \text{ (say)}$$
$$\lambda_{12} = \lambda_{13} = \ldots = \lambda_{p-1,\,p} = \lambda_1 \text{ (say)}$$

for some $\lambda_0, \lambda_1 \geqslant 0$. In Section 4 of B-T we restricted consideration to BTIB designs, and showed how to use such designs for experiments leading to joint one-sided (or two-sided) confidence interval estimates of the $\alpha_0-\alpha_i$ $(1 \leqslant i \leqslant p)$ when σ^2 is known (or unknown).

The specific multiple comparisons with a control (MCC) problem with which we are concerned in the present article is that of obtaining joint *one-sided* confidence intervals of the form

$$\{\alpha_0-\alpha_i \geqslant \hat{\alpha}_0-\hat{\alpha}_i-a \ (1 \leqslant i \leqslant p)\} \qquad \ldots \quad (1.2)$$

for given values of (p, k) when σ^2 is *known*, and $a > 0$ is a *specified* "allowance" associated with the common "width" of the confidence intervals. For this problem we seek an *optimal* design in the class of all admissible BTIB designs, an optimal design being one which minimizes b, the total number of blocks required to achieve a *specified* confidence coefficient $1-\alpha$ associated with (1.2).

2. Preliminaries

2.1. *Expressions for joint confidence interval estimates.* For ease of reference we record here the expressions derived in B-T for the estimators $\hat{\alpha}_0-\hat{\alpha}_i$ $(1 \leqslant i \leqslant p)$, and their variances and correlations. Let T_i denote the sum of all observations obtained with the i-th treatment $(0 \leqslant i \leqslant p)$, and let B_j denote the sum of all observations in the j-th block $(1 \leqslant j \leqslant b)$. Define $B_i^* = \sum_{j=1}^{b} r_{ij}B_j$ and let $Q_i = kT_i-B_i^*$ $(0 \leqslant i \leqslant p)$. Then

$$\hat{\alpha}_0-\hat{\alpha}_i = \frac{\lambda_1 Q_0-\lambda_0 Q_i}{\lambda_0(\lambda_0+p\lambda_1)} \quad (1 \leqslant i \leqslant p). \qquad \ldots \quad (2.1)$$

Also,

$$\text{var}\{\hat{\alpha}_0-\hat{\alpha}_i\} = \tau^2\sigma^2 \quad (1 \leqslant i \leqslant p) \qquad \ldots \quad (2.2)$$

where

$$\tau^2 = \frac{k(\lambda_0+\lambda_1)}{\lambda_0(\lambda_0+p\lambda_1)}, \qquad \ldots \quad (2.3)$$

and

$$\rho = \text{corr}\{\hat{\alpha}_0-\hat{\alpha}_{i_1}, \hat{\alpha}_0-\hat{\alpha}_{i_2}\} = \frac{\lambda_1}{\lambda_0+\lambda_1} \quad (i_1 \neq i_2;\ 1 \leqslant i_1, i_2 \leqslant p). \qquad \ldots \quad (2.4)$$

The probability associated with (1.2) is given by

$$P\{\alpha_0-\alpha_i \geqslant \hat{\alpha}_0-\hat{\alpha}_i-a \quad (1 \leqslant i \leqslant p)\}$$
$$= \int_{-\infty}^{\infty} \Phi^p \left[\frac{x\sqrt{\rho}+\xi/\eta}{\sqrt{1-\rho}}\right] d\Phi(x); \quad \ldots \quad (2.5)$$

here $\Phi(\cdot)$ denotes the standard normal c.d.f. and for notational simplicity we have let

$$\eta^2 = kb\tau^2 = \frac{k^2b(\lambda_0+\lambda_1)}{\lambda_0(\lambda_0+p\lambda_1)} \quad \ldots \quad (2.6)$$

and

$$\xi = a\sqrt{kb}/\sigma, \quad \ldots \quad (2.7)$$

both of which are pure numbers.

2.2. *Generator designs, admissible designs, and minimal complete set of generator designs.* We begin with the concept of a *generator* design. For given (p, k) a generator design is a BTIB design such that no proper subset of its blocks forms a BTIB design and none of its blocks contains only one of the $p+1$ treatments.

Next we define an admissible design. If for given (p, k) we have two BTIB designs D and D', with parameters $(b, \lambda_0', \lambda_1', \tau, \rho)$ and $(b', \lambda_0', \lambda_1', \tau', \rho')$, respectively, with $b \leqslant b'$, and if for every a and σ, D yields a confidence coefficient no smaller than (resp., larger than) that yielded by D' when $b < b'$ (resp., $b = b'$) then we say that D' is *inadmissible* with respect to (w.r.t.) D. In Theorem 5.1 of B-T the following condition was shown to be necessary and sufficient for D' to be inadmissible w.r.t. $D : b \leqslant b'$, $\tau \leqslant \tau'$, and $\rho \geqslant \rho'$ with at least one inequality strict. If a design is not inadmissible then it is said to be *admissible.* If $b = b'$, $\tau = \tau'$, $\rho = \rho'$ then we say that D and D' are *equivalent* since they yield the same confidence coefficient (2.5) for all values of a and σ. A *minimal complete set* of generator designs is the smallest set of generator designs $\boldsymbol{D} = \{D_1, D_2, \ldots, D_n\}$ from which all admissible designs can be constructed for given (p, k) (except possibly any equivalent designs). A method for obtaining the minimal complete set for any given (p, k) is described in Section 5 of B-T.

2.3. *Optimal designs.* For given (p, k), let $\boldsymbol{D} = \{D_1, D_2, \ldots, D_n\}$ denote the minimal complete set of generator designs. Let $\lambda_0^{(i)}, \lambda_1^{(i)}$ be the design parameters associated with D_i, and let b_i be the number of blocks required by D_i $(1 \leqslant i \leqslant n)$. Then a BTIB design D obtained by forming unions of $f_i \geqslant 0$ replications of D_i, (represented as $D = \bigcup_{i=1}^{n} f_iD_i$) has the design parameters $\lambda_0 = \sum_{i=1}^{n} f_i\lambda_0^{(i)}$ and $\lambda_1 = \sum_{i=1}^{n} f_i\lambda_1^{(i)}$ and requires $b = \sum_{i=1}^{n} f_i b_i$ blocks. We shall consider only *implementable* D, i.e., those for which $\lambda_0 > 0$. It should be noted that for given $\boldsymbol{D}$ the design D is completely determined by its *frequency vector* $\boldsymbol{f} = (f_1, ..., f_n)$.

As mentioned at the end of Section 1, an optimal design minimizes b in the class of all admissible designs which for (1.2) achieve at least a *specified* confidence coefficient $1-\alpha$. The problem of finding an optimal design is solved numerically in two steps which are described below.

In the first step b is fixed and for given (p, k), $\boldsymbol{D}$, and specified a/σ, $\boldsymbol{f}$ is chosen to maximize (2.5) among all admissible $\boldsymbol{f}$ satisfying $\sum_{i=1}^{n} f_i b_i = b$, $\sum_{i=1}^{n} f_i\lambda_0^{(i)} > 0$ and $f_i \geqslant 0$ $(1 \leqslant i \leqslant n)$. In this setup the integral expression (2.5) for the confidence coefficient can be regarded as a function of $\boldsymbol{f}$ for given (p, k), $\boldsymbol{D}$, b and for specified $\xi = a\sqrt{kb}/\sigma$. We therefore denote (2.5) by $g(\boldsymbol{f}\,|\,\boldsymbol{D};\ p, k, b; \xi) = g$ (say). Let $\hat{g}$ denote the maximum value of g for that b and let $\hat{\boldsymbol{f}}$ denote the BTIB design that yields $\hat{g}$. This procedure of finding $\hat{\boldsymbol{f}}$ and its associated $\hat{g}$ is repeated for all values of b for which admissible designs exist. Thus this first step generates a table of $\hat{\boldsymbol{f}}$ and $\hat{g}$ for different b and the specified a/σ.

In the second step, the specified $1-\alpha$ and a/σ are fixed and b is varied. Then by referring to the table of $(\hat{\boldsymbol{f}}, \hat{g})$, the $\hat{\boldsymbol{f}}$ with the smallest b (say, $\hat{b}$) for which $\hat{g} \geqslant 1-\alpha$ is determined. This procedure of finding an optimal design is illustrated in Section 4 for the special case $p = 2$, $k = 2$.

3. Minimal complete sets of generator designs and admissible designs for $p \geqslant 2$, $k = 2$, and $p = 3$, $k = 3$

3.1. *Minimal complete sets of generator designs for* $p \geqslant 2$, $k = 2$ *and* $p = 3$, $k = 3$. For $p \geqslant 2$, $k = 2$ and $p = 3$, $k = 3$ the minimal complete sets of generator designs always have cardinality two. For $p \geqslant 2$, $k = 2$ this is clear since the only two generator designs possible are

$$D_0 = \begin{Bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & p \end{Bmatrix}, \qquad D_1 = \begin{Bmatrix} 1 & 1 & \cdots & p-1 \\ 2 & 3 & \cdots & p \end{Bmatrix}. \qquad \ldots (3.1)$$

For D_0 of (3.1) we have

$$b_0 = p, \ \lambda_0^{(0)} = 1, \ \ \lambda_1^{(0)} = 0, \qquad \ldots (3.2)$$

and for D_1 of (3.1) we have

$$b_1 = \frac{p(p-1)}{2}, \quad \lambda_0^{(1)} = 0, \quad \lambda_1^{(1)} = 1. \qquad \dots \quad (3.3)$$

Thus for any BTIB design $D = f_0 D_0 \bigcup f_1 D_1$ for $p \geqslant 2$, $k = 2$ we have

$$b = p\left\{f_0 + \frac{(p-1)}{2} f_1\right\}, \quad \lambda_0 = f_0, \quad \lambda_1 = f_1. \qquad \dots \quad (3.4)$$

For $p = 3$, $k = 3$ it is shown in Notz and Tamhane (1983) that

$$D_0 = \left\{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} \qquad \dots \quad (3.5)$$

constitutes the minimal complete set. (The problem of constructing the minimal complete sets of generator designs for $p > 3$, $k = 3$ is nontrivial; this problem is addressed in the Notz-Tamhane article where the minimal complete sets are given for $p = 3(1)10$, $k = 3$.) For D_0 of (3.5) we have

$$b_0 = 3, \quad \lambda_0^{(0)} = 2, \quad \lambda_1^{(0)} = 1, \qquad \dots \quad (3.6)$$

and for D_1 of (3.5) we have

$$b_1 = 1, \quad \lambda_0^{(1)} = 0, \quad \lambda_1^{(1)} = 1. \qquad \dots \quad (3.7)$$

Thus for any BTIB design $f_0 D_0 \bigcup f_1 D_1$ for $p = 3$, $k = 3$ we have

$$b = 3f_0 + f_1, \quad \lambda_0 = 2f_0, \quad \lambda_1 = f_0 + f_1. \qquad \dots \quad (3.8)$$

In the sequel we will only consider the BTIB designs obtained for given (p, k) from the generator designs in the minimal complete set for that (p, k).

3.2. *Characterization of admissible designs for* $p \geqslant 2$, $k = 2$ *and* $p = 3$, $k = 3$. To characterize the admissible designs we first introduce the concept of a *b-admissible* design : For given (p, k), a BTIB design D requiring b blocks is said to be *b-inadmissible* if it is inadmissible w.r.t. another BTIB design also requiring b blocks. If a design is not b-inadmissible then it is said to be b-admissible. (See Table 4.1A for examples of b-inadmissible designs.) A b-admissible design sometimes can be inadmissible w.r.t. a design with a smaller b, but a b-inadmissible design is always inadmissible.

The importance of the b-admissibility concept lies in the fact that for $p \geqslant 2$, $k = 2$ *almost all* b-admissible designs are admissible with only a very small number of exceptions while for $p = 3$, $k = 3$ *all* b-admissible designs are admissible. This fact suggests that it usually is sufficient to restrict consideration to b-admissible designs for the purpose of obtaining optimal designs.

We now recall the necessary and sufficient condition, given in Section 2.2, for a design to be inadmissible w.r.t. another design. For b-inadmissibility that condition can be stated simply as follows : For given (p, k) let D and D' be two BTIB designs with $b = b'$, and parameters (η, ρ) and (η', ρ'), respectively. The design D' is *b-inadmissible* w.r.t. the design D iff $\eta \leqslant \eta'$ and $\rho \geqslant \rho'$ with at least one inequality strict. If a design is not b-inadmissible then it is *b-admissible.*

We note that this simpler condition compares designs in terms of their η-values rather than their τ-values. This is permissible because from (2.6) we see that $\eta/\eta' = \tau/\tau'$ when $b = b'$.

Using this condition, b-admissible designs are characterized in the following theorem.

Theorem 3.1 : *Let (p, k) and the associated minimal complete set of generator designs $\{D_0, D_1\}$ be given where D_0 contains both the control and the test treatments while D_1 contains only the test treatments. Let $(b_i, \lambda_0^{(i)}, \lambda_1^{(i)})$ be the parameters associated with D_i $(i = 0, 1)$ where $\lambda_0^{(1)} = 0$. For fixed b consider all designs $D = f_0D_0 \bigcup f_1D_1$ with $f_0b_0 + f_1b_1 = b$. Let f_0^U denote the upper bound on f_0. Then one of the following two cases obtain depending on the value of*

$$\beta = (p-1)\lambda_0^{(0)}\lambda_1^{(1)} - b_1(\lambda_0^{(0)} + p\lambda_1^{(0)})(\lambda_0^{(0)} + \lambda_1^{(0)}). \qquad \ldots \quad (3.9)$$

Case 1 : If $\beta > 0$ then there exists an integer $f_0^* \geqslant 2$ which is the smallest value of f_0 satisfying $\eta^2(f_0) \geqslant \eta^2(f_0 - d)$; here d is the smallest positive integer such that b_1 divides (b_0d). If $f_0^* \leqslant f_0^U$ then all designs D with $f_0 \geqslant f_0^*$ are *b-admissible.*

Case 2 : If $\beta \leqslant 0$ then all designs D are b-admissible.

Corollary : *For $p \geqslant 2$, $k = 2$, Case 1 holds while for $p = 3$, $k = 3$, Case 2 holds.*

The proof of the theorem is given in Appendix 1. If Case 1 holds, then b must be sufficiently large in order that $f_0^* \leqslant f_0^U$. As b increases the number of designs eliminated as being b-inadmissible increases.

The corollary follows directly for $p \geqslant 2$, $k = 2$ by substituting (3.2) and (3.3) in (3.9) and verifying that $\beta = p(p-1)/2 > 0$ and for $p = 3$, $k = 3$ by substituting (3.6) and (3.7) in (3.9) and verifying that $\beta = -3 < 0$.

The proof of Theorem 3.1 uses a technique (discussed in detail in Section 5) which regards $\gamma = b_0 f_0/b$ as a continuous variable taking values in (0, 1]. If γ^* denotes the minimizing value of η^2 (regarded as a function of γ) then for Case 1, γ^* is given by solving the equation $\partial\eta^2/\partial\gamma = 0$ (see (A.4) in Appendix 1 for an expression for $\partial\eta^2/\partial\gamma$); for Case 2, $\gamma^* = 1$. The quantity γ^* is the maximum permissible proportion of blocks that can be allocated to D_0 in a design D with the latter being b-admissible (assuming that a BTIB design exists for every $\gamma \in (0, 1]$). Therefore a characterization of b-inadmissibility in the continuous case can be given as follows : For given (p, k), $\{D_0, D_1\}$ and b, a design $f_0D_0 \bigcup f_1D_1$ is b-inadmissible iff

$$f_0 > \frac{b\gamma^*}{b_0}. \qquad \dots \quad (3.10)$$

This characterization of b-inadmissibility in the *continuous* case can be expected to approximate closely the exact characterization in the discrete case (cf. Theorem 3.1) for sufficiently large b. However, for small or moderate b the former characterization may classify a design as b-inadmissible when, in fact, it is not. This can happen because of one of two reasons : (i) For given b there may exist only one BTIB design in which case it is automatically b-admissible although it may satisfy (3.10). (ii) The critical number f_0^* (defined in Theorem 3.1) which provides an exact characterization of b-admissibility for the discrete case is always greater than $b\gamma^*/b_0$. Therefore, a design $f_0D_0 \bigcup f_1D_1$ with $b\gamma^*/b_0 < f_0 < f_0^*$ is b-admissible although it satisfies (3.10).

For $p \geqslant 2$, $k = 2$, the value of γ^* is given by

$$\gamma^* = \begin{cases} \dfrac{2}{(p-3)}\left[\dfrac{p-1}{\sqrt{p+1}}-1\right] & \text{for } p \geqslant 2,\ p \neq 3,\ k = 2 \\[2ex] 3/4 & \text{for } p = 3,\ k = 2. \end{cases} \qquad \dots \quad (3.11)$$

The corresponding exact values of f_0^* are given by (4.2).

We now state

Theorem 3.2 : *For given (p, k) and $\{D_0, D_1\}$ let $D = f_0D_0 \bigcup f_1D_1$ and $D' = f_0'D_0 \bigcup f_1'D_1$ be two BTIB designs where $\{D_0, D_1\}$ is given by* (3.1) *for $p \geqslant 2$, $k = 2$ and by* (3.5) *for $p = 3$, $k = 3$, respectively. Let (b, τ, ρ) and (b', τ', ρ') with $b < b'$ denote the parameters associated with D and D', respectively.*

(i) *For $p \geqslant 2$, $k = 2$, if $f_0 \leqslant b\gamma^*/b_0$ and $f_0' \leqslant b'\gamma^*/b_0'$ where γ^* is given by* (3.11), *then D' cannot be inadmissible w.r.t. D.*

(ii) *For $p = 3$, $k = 3$, D' cannot be inadmissible w.r.t. D.*

Proof: See Appendix 1.

For $p = 3$, $k = 3$, the corollary of Theorem 3.1 states that all BTIB designs are b-admissible; this together with part (ii) of Theorem 3.2 implies that all BTIB designs for $p = 3$, $k = 3$ are admissible. Also note that part (i) of Theorem 3.2 does *not* assert that if a design is b-admissible then it is admissible; this statement, unfortunately, is not always true. By an exhaustive enumerative computer search for the cases $p = 2(1)6$, $k = 2$ with $b \leqslant 200$ a total of only four b-admissible designs that are inadmissible were found. These designs are listed below along with the corresponding dominating design with smaller b:

(i) $p = 4$, $k = 2$: The design $4D_0$ which is b-admissible for $b = 16$ and has ($\tau^2 = 0{\cdot}5000$, $\rho = 0{\cdot}0000$) is inadmissible w.r.t. the design $2D_0 \bigcup D_1$ with $b = 14$ and ($\tau^2 = 0{\cdot}5000$, $\rho = 0{\cdot}3333$).

(ii) $p = 6$, $k = 2$: (a) The design $5D_0$ which is b-admissible for $b = 30$ and has ($\tau^2 = 0{\cdot}4000$, $\rho = 0{\cdot}0000$) is inadmissible w.r.t. the design $2D_0 \bigcup D_1$ with $b = 27$ and ($\tau^2 = 0{\cdot}3750$, $\rho = 0{\cdot}3333$).

(b) The design $9D_0 \bigcup D_1$ which is b-admissible for $b = 69$ and has ($\tau^2 = 0{\cdot}1481$, $\rho = 0{\cdot}1000$) is inadmissible w.r.t. the design $6D_0 \bigcup 2D_1$ with $b = 66$ and ($\tau^2 = 0{\cdot}1481$, $\rho = 0{\cdot}2500$).

(c) The design $10D_0 \bigcup D_1$ which is b-admissible for $b = 75$ and has ($\tau^2 = 0{\cdot}1375$, $\rho = 0{\cdot}0909$) is inadmissible w.r.t. the design $7D_0 \bigcup 2D_2$ with $b = 72$ and ($\tau^2 = 0{\cdot}1353$, $\rho = 0{\cdot}2222$).

It can be expected that for large b such exceptions will not arise. Therefore, for convenience, we restrict consideration to b-admissible designs, and we search for optimal designs among them (recognizing the fact that a very small number of b-admissible designs will yield confidence coefficients lower than those yielded by some designs with smaller b-values, and hence the former cannot be optimal).

4. Discrete optimal designs

4.1. *Results for* $p \geqslant 2$, $k = 2$. For $p \geqslant 2$, $k = 2$, any BTIB design D can be written as $f_0D_0 \bigcup f_1D_1$ where $\{D_0, D_1\}$ is given by (3.1) with $f_0 \geqslant 1$, $f_1 \geqslant 0$. The values of b, λ_0 and λ_1 for D are given by (3.4) which when substituted in (2.6) and (2.4) yield

$$\eta^2(f_0) = \frac{4b\{2b+p(p-3)f_0\}}{pf_0\{2b-(p+1)f_0\}}, \qquad \ldots \quad (4.1a)$$

and

$$\rho(f_0) = \frac{2(b-pf_0)}{2b+p(p-3)f_0}. \qquad \ldots \quad (4.1b)$$

The corollary to Theorem 3.1 states that Case 1 holds for all $p \geqslant 2$. In what follows, we fix b in order to determine the b-admissible designs. The critical number f_0^* referred to in Theorem 3.1 is the smallest f_0 satisfying equation (3.4) for given b, and for which $\eta^2(f_0) \geqslant \eta^2(f_0-d)$; here $d = p-1$ if p is even, and $d = (p-1)/2$ if p is odd. We thus obtain

$$f_0^* = \begin{cases} \text{int}\left[\dfrac{2b+1}{2} - \sqrt{\dfrac{b^2}{3}+\dfrac{1}{4}}\right] & \text{for } p = 2 \quad \ldots \quad (4.2a) \\[2ex] \text{int}\left[\dfrac{b+2}{4}\right] & \text{for } p = 3 \quad \ldots \quad (4.2b) \\[2ex] \text{int}\left[\dfrac{p(p-3)d-4b}{2p(p-3)} + \sqrt{\dfrac{4(p-1)^2d^2}{(p+1)p^2(p-3)^2}+\dfrac{d^2}{4}}\right] & \text{for } p \geqslant 4 \quad \ldots \quad (4.2c) \end{cases}$$

where int$[z]$ denotes the smallest integer $\geqslant z$.

We now consider in detail the special case $p = 2$, $k = 2$ in order to show how we obtain optimal designs using the two-step method described at the end of Section 2.

For $p = 2$, $k = 2$, all designs $D = f_0D_0 \bigcup f_1D_1$ are generated from $\{D_0, D_1\}$ of (3.1) for arbitrary $b = 2f_0+f_1$ $(b = 2, 3, \ldots)$ where $1 \leqslant f_0 \leqslant b/2$,

TABLE 4.1A. ENUMERATION OF DESIGNS[1] FOR $p = 2$, $k = 2$ AND $b = 2(1)10$

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \qquad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

b	f_0	f_1	η^2	ρ	b	f_0	f_1	η^2	ρ
2	1	0	8.00	0.000	8	1	6	17.23	0.857
					8	2	4	9.60	0.667
					8	3	2	7.62	0.400
3	1	1	8.00	0.500	8	*4	0	8.00	0.000
4	1	2	9.60	0.667	9	1	7	19.20	0.875
4	2	0	8.00	0.000	9	2	5	10.50	0.714
					9	3	3	8.00	0.500
					9	4	1	7.50	0.200
5	1	3	11.43	0.750					
5	2	1	7.50	0.333					
					10	1	8	21.18	0.889
					10	2	6	11.43	0.750
6	1	4	3.33	0.800	10	3	4	8.48	0.571
6	2	2	8.00	0.500	10	4	2	7.50	0.333
6	*3	0	8.00	0.000	10	*5	0	8.00	0.000
7	1	5	15.27	0.833					
7	2	3	8.75	0.600					
7	3	1	7.47	0.250					

[1] The designs marked with an asterisk (*) are b-inadmissible.

$0 \leqslant f_1 \leqslant b-2$ for b even, and $1 \leqslant f_0 \leqslant (b-1)/2$, $1 \leqslant f_1 \leqslant b-2$ for b odd. Equation (4.2a) gives f_0^* while $f_0^U = b/2$ or $(b-1)/2$ according as b is even or odd; all b-inadmissible f_0 then satisfy $f_0^* \leqslant f_0 \leqslant f_0^U$. Thus for $b \leqslant 5$, all (f_0, f_1) are b-admissible; for $b = 6$, $(f_0, f_1) = (3, 0)$ is b-inadmissible; for $b = 20$, $(f_0, f_1) = (9, 2)$ and $(10, 0)$ are b-inadmissible, etc. In Table 4.1A we have enumerated all designs for $b = 2(1)10$, and have given the associated η^2 and ρ; the b-inadmissible designs are noted.

Conceptually, one then proceeds as follows: We are given (p, k), $\{D_0, D_1\}$, and a/σ is specified. We fix b and list all b-admissible designs for that b. Thus from Table 4.1A we see that for $p = 2$, $k = 2$, there are four b-admissible designs for $b = 10$. For each b-admissible design we then calculate g as a function of a/σ and find the design which is associated with $\hat{g}$, the maximum value of g for that b and a/σ. Table 4.1B shows for $b = 10$ the designs maximizing g and their associated $\hat{g}$-values for $a/\sigma = 0{\cdot}5(0{\cdot}1)1{\cdot}0$. (For a/σ sufficiently small the design $(\hat{f}_0, \hat{f}_1) = (1, 8)$ is optimal while for a/σ sufficiently large the design $(\hat{f}_0, \hat{f}_1) = (4, 2)$ is optimal; this an example of the standard phenomenon referred to in the proof of Theorem 5.1 of B-T.) Finally, such tables can be prepared for arbitrary $b \geqslant 2$.

TABLE 4.1B. OPTIMAL DESIGN AND ASSOCIATED CONFIDENCE COEFFICIENT AS A FUNCTION OF a/σ FOR $p = 2$, $k = 2$ WHEN $b = 10$

a/σ	0.5	0.6	0.7	0.8	0.9	1.0
$\hat{f}_0$	3	3	4	4	4	4
$\hat{f}_1$	4	4	2	2	2	2
$\hat{g}$	0.6673	0.7248	0.7806	0.8303	0.8719	0.9057

If $\hat{g}$ is strictly increasing in b for all values of a/σ then such tables provide optimal designs with each listed design being optimal for the corresponding values of a/σ. However, this is not the case for all (p, k); e.g., this is not the case for $p = 4$, $k = 2$. For fixed a/σ, if $\hat{g}$ decreases or stays constant as b increases then one must delete the designs having the larger b-values when the associated $\hat{g}$-values are no larger than that yielded by a design with a smaller b-value. Using this elimination procedure, detailed tables have been prepared for $p = 2(1)6$, $k = 2$ and $p = 3(1)6$, $k = 3$ for selected values of b and a/σ; these are given in Bechhofer and Tamhane (1983). In the present paper we have obtained the optimal designs from such tables for selected $1-\alpha$ and a/σ. Tables 4.2–4.6 list for $p = 2(1)6$, respectively, with $k = 2$ the optimal designs for $1-\alpha = 0{\cdot}80,\ 0{\cdot}90,\ 0{\cdot}95,\ 0{\cdot}99$ and $a/\sigma = 0{\cdot}2(0{\cdot}2)2{\cdot}0$.

4.2. *Results for* $p = 3,\ k = 3$. For $p = 3,\ k = 3$ any BTIB design D can be written as $f_0D_0 \bigcup f_1D_1$ where $\{D_0, D_1\}$ is given by (3.5) with $f_0 \geqslant 1$, $f_1 \geqslant 0$. The values of b, λ_0, λ_1 for D are given by (3.8) which when substituted in (2.6) and (2.4) yield

$$\eta^2(f_0) = \frac{9b^2}{2f_0(3b-4f_0)} \qquad \dots \quad (4.3a)$$

and

$$\rho(f_0) = \frac{b-2f_0}{b}. \qquad \dots \quad (4.3b)$$

The corollary to Theorem 3.1 states that Case 2 holds for $p = 3,\ k = 3$. This together with Theorem 3.2 shows that all designs $D = f_0D_0 \bigcup f_1D_1$ are *admissible* for $p = 3,\ k = 3$. Table 4.7 lists for $p = 3,\ k = 3$ the optimal designs for $1-\alpha = 0{\cdot}80,\ 0{\cdot}90,\ 0{\cdot}95,\ 0{\cdot}99$ and $a/\sigma = 0{\cdot}2(0{\cdot}2)2{\cdot}0$.

TABLE 4.2 DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1-α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$.)

$$p = 2,\ k = 2,\ D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\hat{b} = 2\hat{f}_0 + \hat{f}$$

confidence coefficient (1-α)	a/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
	258	64	29	16	10	8	5	4	3	3
0.99	100	26	11	7	5	2	3	2	2	1
	144	36	16	9	6	4	3	2	2	2
0.95	63	16	7	4	3	2	2	2	1	0
	96	24	11	6	4	3	2	2	2	1
0.90	49	13	5	4	2	1	1	1	0	1
	51	13	6	3	2	2	1	1	1	1
0.80	34	8	4	3	2	1	1	1	0	0

TABLE 4.3 DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1-α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$)

$$p = 3,\ k = 2,\ D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}$$

$$\hat{b} = 3\hat{f}_0 + 3\hat{f}_1$$

confidence coefficient (1–α)	a/σ 0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
	241	61	27	16	10	7	5	4	4	3
0.99	84	21	10	5	4	3	2	2	1	1
	141	36	16	9	6	4	4	3	2	2
0.95	56	14	6	4	2	2	1	1	1	1
	98	25	11	6	4	3	2	2	1	1
0.90	44	11	5	3	2	1	1	1	1	1
	56	14	6	4	3	2	1	1	1	1
0.80	32	8	4	2	1	1	1	1	1	1

TABLE 4.4. DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1–α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$)

$$p = 4,\ k = 2,\ D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{Bmatrix}$$

$$\hat{b} = 4\hat{f}_0 + 6\hat{f}_1$$

confidence coefficient (1–α)	a/σ 0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
	226	57	24	14	9	7	4	4	3	2
0.99	73	18	9	5	3	2	2	1	1	1
	136	33	16	9	6	5	3	2	2	1
0.95	49	13	5	3	2	1	1	1	1	1
	97	24	10	7	5	3	2	3	1	2
0.90	39	10	5	2	1	1	1	0	1	0
	56	15	7	4	3	2	1	1	2	2
0.80	30	7	3	2	1	1	1	1	0	0

TABLE 4.5. DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1–α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$)

$$p = 5, k = 2, D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 & 5 & 3 & 4 & 5 & 4 & 5 & 5 \end{Bmatrix}$$

$$\hat{b} = 5\hat{f}_0 + 10\hat{f}_1$$

confidence coefficient (1–α)	a/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
	213	54	25	14	8	6	6	4	3	2
0.99	65	16	7	4	3	2	1	1	1	1
	132	33	15	8	5	5	3	2	2	1
0.95	44	11	5	3	2	1	1	1	1	1
	94	24	11	7	5	3	2	1	1	1
0.90	36	9	4	2	1	1	1	1	1	1
	58	14	7	4	3	2	1	1	2	2
0.80	27	7	3	2	1	1	1	1	0	0

TABLE 4.6. DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1–α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$)

$$p = 6, k = 2, D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 5 & 6 & 6 \end{Bmatrix}$$

$$\hat{b} = 6\hat{f}_0 + 15\hat{f}_1$$

confidence coefficient (1–α)	a/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
	205	52	24	12	9	5	5	3	2	2
0.99	57	14	6	4	2	2	1	1	1	1
	129	32	13	7	7	4	3	2	1	3
0.95	39	10	5	3	1	1	1	1	1	0
	94	24	12	6	5	3	2	1	3	3
0.90	32	8	3	2	1	1	1	1	0	0
	55	15	6	3	3	2	1	3	2	2
0.80	26	6	3	2	1	1	1	0	0	0

TABLE 4.7. DISCRETE OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1–α) AS A FUNCTION OF a/σ FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{f}_0$, AND THE LOWER ENTRY IS $\hat{f}_1$)

$$p = 3, k = 3, D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \qquad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$\hat{b} = 3\hat{f}_0 + \hat{f}_1$$

confidence coefficient (1–α)	a/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	164 0	41 0	18 1	10 1	7 0	5 0	3 2	3 0	2 1	2 0
0.95	98 2	25 0	11 0	6 1	4 0	3 0	2 1	2 0	1 1	1 0
0.90	71 1	18 0	8 0	4 2	3 0	2 0	2 0	1 1	1 0	1 0
0.80	41 8	10 3	5 0	3 0	2 0	1 1	1 0	1 0	1 0	1 0

5. Continuous optimal designs

5.1. *Preliminaries.* As in Section 3 we continue to deal with situations in which the minimal complete set consists of two generator designs D_0 and D_1 where D_0 contains both the control and the test treatments while D_1 contains only the test treatments. In Section 4 we noted that, in general (but not always), the number of competing admissible designs increases with b for fixed (p, k). We have seen that for each (p, k) the optimal design depends on a/σ; also, the determination of the optimal design requires that the design maximizing g be found for each (p, k, b) and a/σ combination. This represents a formidable computing and tabulation task. The solutions for many of the useful combinations are given in Tables 4.2–4.7.

In order to extend the results given in Section 4, and to do so in a compact form we introduce a method for finding an approximation to the discrete optimal designs. We shall refer to such designs as *continuous* optimal designs. The problem of obtaining the continuous optimal designs is analytically more tractable and computationally easier to solve. Moreover, since its solution does not depend individually on b and a/σ but only on these quantities through $\xi = a\sqrt{kb}/\sigma$, the number of solutions that must be tabulated is drastically reduced. In Section 5.7 we assess how closely the approximate discrete optimal designs (obtained from the continuous optimal designs) agree with the exact discrete optimal designs.

5.2. *Definition of* γ. For given (p, k) and the associated minimal complete set of generator designs $\{D_0, D_1\}$ we define for an arbitrary BTIB design $D = f_0D_0 \bigcup f_1D_1$, the quantity

$$\gamma = \frac{f_0b_0}{b} = \frac{f_0b_0}{f_0b_0+f_1b_1} \qquad \ldots \quad (5.1)$$

which is the proportion of the total number of blocks allocated to D_0. For large values of b we shall treat $\gamma \in (0, 1]$ as a nonnegative continuous variable. Then regarding η^2 and ρ of (2.6) and (2.4) (see also (A.8) and (A.9)), respectively, as continuous functions of γ, we consider the integral (2.5) as a function of γ for each (p, k, ξ)-combination, and denote its value by $g(\gamma \mid D_0, D_1; p, k; \xi) = g$ (say). Note that in Section 2.3 we regarded g as a *discrete* function of f (which for the special case of two generator designs can be regarded as a function only of f_0 for fixed b); here we regard g as a *continuous* function of γ. Thus we are considering a continuous extension of the discrete function g. For convenience, we denote this continuous extension by the *same* symbol g.

5.3. *Optimization problem for continuous designs.* Analogous to the optimization problem of obtaining discrete optimal designs stated in Section 2.3, the problem of obtaining continuous optimal designs can be stated as follows : For given (p, k) and $\{D_0, D_1\}$, find the smallest ξ, say $\hat{\xi}$, and the associated optimal value of γ, say $\hat{\gamma}$, to guarantee a specified confidence coefficient $1-\alpha$. Note that here the confidence coefficient $1-\alpha$ is achieved exactly. The method of obtaining the approximate discrete optimal design $\hat{f} = (\hat{f}_0, \hat{f}_1)$ from the continuous optimal design $(\hat{\xi}, \hat{\gamma})$ is explained in Section 5.6.

As in the case of discrete optimal designs, it is helpful to conceptualize the optimal solution $(\hat{\gamma}, \hat{\xi})$ for given (p, k), $\{D_0, D_1\}$, and specified $1-\alpha$ as being obtained in two steps. However, in contrast to the discrete case, the solution in the continuous case is obtained in one step by solving a pair of simultaneous equations (5.8) and (5.9) given below. To this end we regard ξ as specified and fixed, and consider the maximization of g w.r.t. γ alone. To maximize g w.r.t. γ a study of the behavior of g as a function of γ is required; this study is carried out in the following section.

5.4. *Maximization of g with respect to γ :*

5.4.1. *Derivative $\partial g/\partial\gamma$.* We seek to obtain the maximizing value $\hat{\gamma}$ as the solution in γ of the equation $\partial g/\partial \gamma = 0$. In doing so we must be assured that a feasible solution in γ exists, that it is unique, and that it is in fact associated with the maximum of g. Actually it turns out that either a unique solution in γ of $\partial g/\partial\gamma = 0$ exists, lies in the interval (0, 1), and corresponds to the maximizing value, or that the maximum value of g for $\gamma \in [0, 1]$ occurs at the boundary $\gamma = 0$ for ξ sufficiently small or at the boundary $\gamma = 1$ for ξ sufficiently large; the solution $\hat{\gamma} = 1$ occurs only for Case 2 (cf. Theorem 3.2).

We show in Appendix 2 that

$$\frac{\partial g}{\partial \gamma} = \frac{p\phi(\xi/\eta)}{2\eta^2} h(\gamma \,|\, D_0, D_1; p, k; \xi) \qquad \ldots \quad (5.2)$$

where

$$h(\gamma \,|\, D_0, D_1; p, k; \xi)$$

$$= \frac{(p-1)\eta^2}{(1-\rho^2)^{1/2}} \frac{\partial \rho}{\partial \gamma} \phi\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\right] \Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}} \,\middle|\, \frac{\rho}{1+2\rho}\right]$$

$$-\frac{\xi}{\eta}\frac{\partial \eta^2}{\partial \gamma} \Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}} \,\middle|\, \frac{\rho}{1+\rho}\right]. \qquad \ldots \quad (5.3)$$

In (5.3), $\phi(\cdot)$ denotes the standard normal p.d.f. and $\Phi_r(x \,|\, \rho)$ denotes the c.d.f. at the equicoordinate point x of an r-variate equicorrelated standard normal distribution with common correlation coefficient ρ. The quantities η^2, ρ, $\partial\rho/\partial\gamma$ and $\partial\eta^2/\partial\gamma$ are given as functions of γ by (A.1)–(A.4), respectively, in Appendix 1. Note that the sign of $\partial g/\partial\gamma$ depends only on the sign of $h(\gamma, \,|\, D_0, D_1; p, k; \xi)$.

5.4.2. *Study of g and its maximum.* In this section we study the behavior of g as a function of γ and ξ for fixed (p, k) and $\{D_0, D_1\}$. It is straightforward to check that in the limiting case $\gamma = 0$ we have $(\eta^2 = \infty, \rho = 1)$ and hence $g = 1/2$. For fixed $\gamma > 0$, we note that as ξ increases from 0 to ∞, g increases from $\Phi_p(0 \,|\, \rho)$ to 1.

All of our calculations and studies lead us to conclude that g regarded as a function of γ attains a *unique* maximum at $\hat{\gamma}$, the value of which depends on ξ and $\{D_0, D_1\}$. For all ξ $(0 \leqslant \xi \leqslant \xi_0 = \xi_0(D_1; p, k))$ where ξ_0 is given by (5.6) below, we see that g is strictly decreasing in γ, and hence $\hat{\gamma} = 0$ maximizes g, the maximum being equal to 1/2. This result parallels the one obtained in Bechhofer (1969).

To study the behavior of g as a function of γ for $\xi > \xi_0$, we note that for large ξ the second term in (5.3) dominates and hence for large ξ we have $\mathrm{sgn}(\partial g/\partial\gamma) = \mathrm{sgn}(h) = -\mathrm{sgn}(\partial\eta^2/\partial\gamma)$. In Appendix 1 we show that η^2 is a quasiconvex function of γ for $0 < \gamma \leqslant 1$. We now obtain the following two cases depending on whether η^2 achieves a minimum in the interior of $[0, 1]$ (when $\beta > 0$) or at the boundary $\gamma = 1$ (when $\beta \leqslant 0$) :

Case 1 ($\beta > 0$) : In this case g has a unique maximum at $\hat{\gamma}$ $(0 < \hat{\gamma} < 1)$ for all $\xi > \xi_0$; here $\hat{\gamma}$ is the unique solution in γ of the equation

$$h(\gamma \mid D_0, D_1; p, k; \xi) = 0. \qquad \ldots \quad (5.4)$$

The maximizing solution $\hat{\gamma}$ is a strictly increasing function of ξ for $\xi > \xi_0$. In the limit ($\xi \to \infty$) the maximizing solution approaches γ^* where γ^* is the largest limiting proportion of blocks that can be allocated to D_0 in order that the design be b-admissible. Thus

$$\gamma^* = \lim_{b \to \infty} \frac{b_0 f_0^*}{b}.$$

This limiting value of γ can also be found by minimizing the common variance of the $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leqslant i \leqslant p)$, i.e., by solving the equation $\partial\eta^2/\partial\gamma = 0$.

Case 2 ($\beta \leqslant 0$) : In this case there exists a positive constant $\xi_1 = \xi_1(D_0, D_1; p, k)$, $(0 < \xi_0 < \xi_1 < \infty)$ such that for every $\xi \in (\xi_0, \xi_1)$, g has a unique maximum at $\hat{\gamma} < 1$; here $\hat{\gamma}$ is the unique solution in γ to (5.4). The maximizing solution $\hat{\gamma}$ is a strictly increasing function of ξ for $\xi \in (\xi_0, \xi_1)$ with $\hat{\gamma} \to 1$ as $\xi \to \xi_1$. For all $\xi \geqslant \xi_1$ the maximizing solution is also $\hat{\gamma} = 1$ (which implies *no* replications of D_1). As with Case 1, in the limit ($\xi \to \infty$), the maximizing solution is the value of γ which minimizes the common variance of the $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leqslant i \leqslant p)$.

5.4.3. *Definition of* ξ_0. As the first step in finding a closed expression for ξ_0 we consider $\lim_{\gamma \to 0} h(\gamma \mid D_0, D_1; p, k; \xi)$. From (A.3), (A.4) and (A.5) we see that

$$\lim_{\gamma \to 0} \frac{\partial \rho}{\partial \gamma} < 0 \quad \text{and} \quad \lim_{\gamma \to 0} \frac{\partial \eta^2}{\partial \gamma} = -\infty < 0.$$

Therefore

$$\lim_{\gamma \to 0} h(\gamma \mid D_0, D_1; p, k; \xi) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \Longleftrightarrow \xi \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \xi_0$$

where $\xi_0 = \xi_0(D_1; p, k)$ is defined by

$$\xi_0 = \lim_{\gamma\to 0} \frac{(p-1)\eta^3 \frac{\partial\rho}{\partial\gamma}}{(1-\rho^2)^{\frac{1}{2}} \frac{\partial\eta^2}{\partial\gamma}} \left\{ \frac{\phi\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\right]\Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}}\,\Big|\,\frac{\rho}{1+2\rho}\right]}{\Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\,\Big|\,\frac{\rho}{1+\rho}\right]} \right\} \quad \ldots \quad (5.5)$$

We evaluate this limit in Appendix A2 and show it to be

$$\xi_0 = \frac{1}{2}\, p(p-1)k\Phi_{p-2}(0\,|\,1/3)\sqrt{\frac{b_1}{\pi p\lambda_1^{(1)}}}. \quad \ldots \quad (5.6)$$

Note that ξ_0 depends only on D_1 but does not depend on D_0. Since D_1 is either a **BIB** or a **RB** design between the p test treatments, we can substitute $\lambda_1^{(1)} = b_1k(k-1)/p(p-1)$ in (5.6) to obtain

$$\xi_0 = \frac{1}{2}\, p\Phi_{p-2}(0\,|\,1/3)\sqrt{\frac{k(p-1)^3}{(k-1)\pi}}. \quad \ldots \quad (5.7)$$

Remark 5.1 : It is known (see, e.g., Gupta, 1963) that $\Phi_0(0\,|\,1/3) = 1$, $\Phi_1(0\,|\,1/3) = 1/2$, $\Phi_2(0\,|\,1/3) = 1/4+(1/2\pi)\text{arc sin}(1/3)$, and $\Phi_3(0\,|\,1/3) = 1/8+(3/4\pi)\text{arc sin}(1/3)$; $\Phi_t(0\,|\,1/3)$ has been computed to five decimal places for $t = 1(1)12$ by Gupta (1963, Table II, p. 817).

The values of ξ_0 for $p = 2(1)6$, $k = 2$ are 0·7979, 1·6926, 2·5214, 3·2894 and 4·0073, respectively, while $\xi_0 = 1{\cdot}4658$ for $p = 3$, $k = 3$.

5.4.4. *Definition of* ξ_1. For Case 2, we define ξ_1 as the smallest value of ξ for which $\hat{\gamma}$ equals unity; hence for $\xi \geqslant \xi_1$ we have $\hat{\gamma} = 1$. Thus ξ_1 is the solution in ξ of the equation

$$h(\gamma\,|\,D_0, D_1; p, k; \xi)\,|_{\gamma=1} = 0.$$

In general ξ_1 depends on *both* D_0 and D_1. The value of ξ_1 for $p = 3$, $k = 3$ is 4·5081.

5.4.5. *Uniqueness of maximum of* g *as a function of* γ. As mentioned in Section 5.4.2, we have not yet proved analytically the existence of a unique maximum for g as a function of γ when $\xi_0 \leqslant \xi < \infty$ (nor was the corresponding result proved in Bechhofer (1969) or Bechhofer and Nocturne (1972)); however, all of our numerical calculations and certain analytical considerations point to this conclusion.

We have computed g as a function of γ for selected values of ξ, and given the results in Tables 5.1A and 5.1B to illustrate Cases 1 and 2, respectively. Table 5.1A is for $p = k = 2$ (with generator designs $\{D_0, D_1\}$ of (3.1)), and

Table 5.1B is for $p = k = 3$ (with generator designs $\{D_0, D_1\}$ of (3.5)); these computations give representative pictures of the behaviour of g as a function of γ. The behavior of g in Table 5.1A is analogous to that of g in Figure 1 of Bechhofer (1969) in that g has a unique maximum at $\hat{\gamma} < \gamma^*$ for $\xi > \xi_0$. However, unlike the situation in Figure 1 where $\lim_{\gamma \to 1} g = 1/2^p$ we now have $\lim_{\gamma \to 1} g$ depending on ξ and other parameters of the design.

5.5. *Solution to the problem of obtaining continuous optimal designs.* We now describe the method of solution to the problem of obtaining continuous optimal designs.

(a) If Case 1 holds or if Case 2 holds but $\hat{\xi} < \xi_1$ (see comment below), then solve simultaneously the two equations

$$h(\gamma \mid D_0, D_1; p, k; \xi) = 0 \qquad \text{... (5.8)}$$

$$\int_{-\infty}^{\infty} \Phi^p \left[\frac{x\sqrt{\rho}+\xi/\eta}{\sqrt{1-\rho}} \right] d\Phi(x) = 1-\alpha \qquad \text{... (5.9)}$$

for γ and ξ, the solutions being $\hat{\gamma}$ and $\hat{\xi}$; here h, η^2 and ρ are given by (5.3), (A.1) and (A.2), respectively.

TABLE 5.1A. VALUES OF g AS FUNCTION OF γ FOR SELECTED ξ FOR $p = k = 2$ (CASE 1 : $\xi_0 = 0.7979$)

ξ	γ										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.5	.5000	.4794	.4680	.4569	.4451	.4321	.4174	.4004	.3802	.3558	.3251
2.0	.5000	.5731	.5993	.6161	.6272	.6334	.6352	.6321	.6231	.6063	.5780

TABLE 5.1B. VALUES OF g AS A FUNCTION OF γ FOR SELECTED ξ for p = k = 3 (Case 2 : $\xi_0 = 1.4658$, $\xi_1 = 4.5081$)

ξ	γ										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	.5000	.4707	.4561	.4431	.4305	.4179	.4049	.3914	.3774	.3625	.3468
3.0	.5000	.5879	.6196	.6407	.6556	.6661	.6730	.6769	.6779	.6762	.6716
5.0	.5000	.6978	.7639	.8059	.8350	.8560	.8712	.8822	.8897	.8944	.8965

(b) If Case 2 holds and $\hat{\xi} \geqslant \xi_1$, then solve only (5.9) for ξ with $\gamma = 1$, the solutions being $\hat{\gamma} = 1$, and $\hat{\xi}$.

To see whether or not $\hat{\xi} < \xi_1 = 4{\cdot}5081$ for $p = k = 3$ for any specified $1-\alpha$, it is useful to note that $\hat{g} = 0{\cdot}8561$ for $\xi = \xi_1 = 4{\cdot}5081$ which is calculated by evaluating (2.5). Thus $\hat{\xi} \lesseqgtr \xi_1$ iff $1-\alpha \lesseqgtr 0{\cdot}8561$.

5.6. *Use of tables of continuous optimal designs for $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$.* For given (p, k) and $\{D_0, D_1\}$, tables of continuous optimal designs can be computed using the method described in Section 5.5. This has been done for $p = 2(1)6$, $k = 2$ for $\{D_0, D_1\}$ given by (3.1) and for $p = k = 3$ for $\{D_0, D_1\}$ given by (3.5). The results are summarized in Table 5.2 which for $1-\alpha = 0{\cdot}80$, $0{\cdot}90$, $0{\cdot}95$, $0{\cdot}99$ gives $\hat{\gamma}$ and $\hat{\xi}$.

Table 5.2 is intended for large values of b (which occur when a/σ is small and/or $1-\alpha$ is close to unity). The table is to be used as follows : (p, k), $\{D_0, D_1\}$ and σ^2 are given and the experimenter specifies a and $1-\alpha$. Entering the table with $(p, k, 1-\alpha)$ the experimenter obtains $\hat{\gamma}$ and the associated $\hat{\xi}$. Then $\hat{b} = \text{int}[(\hat{\xi}\sigma/a)^2/k]$. Finally, $\hat{f}_0$ is chosen so that $b_0\hat{f}_0/\hat{b} \sim \hat{\gamma}$ *and* $b_0\hat{f}_0 + b_1\hat{f}_1$ is as close as possible $(\geqslant)\hat{b}$. This process yields an approximately optimal discrete design with associated confidence coefficient of approximately $1-\alpha$.

The approximations referred to in the paragraphs above arise because we use a discrete design which is as "as close as possible" to the optimal continuous design. These approximations become increasingly more accurate as b increases. The goodness of the approximation is assessed in the next section.

5.7. *Comparison of exact and approximate optimal designs.* To indicate the accuracy of the approximation provided by the continuous optimal designs we computed the exact discrete optimal design and the corresponding continuous optimal design for $p = 2$, $k = 2$, $a/\sigma = 0{\cdot}2$ and selected values of b (and thus ξ). The results are displayed in Table 5.3. We compared the approximate discrete optimal design obtained from the continuous optimal design (by the procedure described in the preceding paragraph) and found that it is the *same* as the corresponding exact discrete optimal design in every case listed in Table 5.3. We would expect that the continuous optimal designs will provide excellent approximations to the corresponding discrete optimal designs even for relatively small values of ξ (associated with low values of confidence coefficients). Our computations have shown that the g-function is quite flat in the neighborhood of its maximum. As a reuslt, $\hat{g}$ for a discrete optimal design is only slightly smaller than $\hat{g}$ for the corresponding continuous optimal design.

TABLE 5.2. CONTINUOUS OPTIMAL DESIGNS TO ACHIEVE A SPECIFIED CONFIDENCE COEFFICIENT (1–α) FOR ONE-SIDED COMPARISONS (THE UPPER ENTRY IN EACH CELL IS $\hat{\xi}$, AND THE LOWER ENTRY IS $\hat{\gamma}$.)

$p = 2(1)6,\ k = 2$ WITH $D_0 = \begin{Bmatrix} 0 & 0 & & 0 \\ 1 & 2 & \cdots & p \end{Bmatrix}$, $D_1 = \begin{Bmatrix} 1 & 1 & & p-1 \\ 2 & 3 & \cdots & p \end{Bmatrix}$

AND $p = 3,\ k = 3$ WITH $D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}$, $D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

confidence coefficient (1–α)	$k = 2$					$k = 3$
	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 3$
0.99	7.0218 0.8373	8.8362 0.7401	10.3624 0.6729	11.7052 0.6226	12.9186 0.5831	7.6870 1.0000
0.95	5.2989 0.8188	6.8621 0.7174	8.1885 0.6491	9.3623 0.5987	10.4276 0.5595	5.9551 1.0000
0.90	4.3894 0.7968	5.8265 0.6912	7.0531 0.6220	8.1429 0.5718	9.1349 0.5331	5.0482 1.0000
0.80	3.2870 0.7440	4.5741 0.6312	5.6833 0.5616	6.6750 0.5126	7.5817 0.4756	3.9613 0.9468

TABLE 5.3. COMPARISON OF DISCRETE AND CONTINUOUS OPTIMAL DESIGNS FOR $p = 2,\ k = 2$ AND $a/\sigma = 0.2$

b	ξ	discrete optimal design			continuous optimal design	
		$\hat{f}_0$	$\hat{\gamma}$	$\hat{g}$	$\hat{\gamma}$	$\hat{g}$
10	0.8944	1	0.2000	0.5028	0.1001	0.5041
15	1.0955	2	0.2667	0.5210	0.2567	0.5210
20	1.2649	4	0.4000	0.5390	0.3528	0.5393
25	1.4142	5	0.4000	0.5572	0.4195	0.5572
50	2.0000	15	0.6000	0.6352	0.5881	0.6352
75	2.4495	25	0.6667	0.6965	0.6627	0.6965
100	2.8284	35	0.7000	0.7457	0.7062	0.7457

5.8. *Asymptotically optimal allocation on the control treatment.* For a completely randomized design, Dunnett (1955, 1106–1107) recommended for $1-\alpha \geqslant 0{\cdot}95$ (approximately) that for every observation made on each one of the test treatments, $\sqrt{p}$ observations should be made on the control treatment, i.e., the proportion of the total observations N that should be allocated to the control treatment is $(1+\sqrt{p})^{-1}$. Bechhofer (1969) showed that this allocation is asymptotically ($N \to \infty$) optimal (i.e., maximizes the confidence coefficient for given N and specified a) for one-sided comparisons;

Bechhofer and Nocturne (1972) extended this result to two-sided comparisons. It is of some interest to find an analog of this result for BTIB designs. We do this in the present section by employing the continuous approximation.

Let θ denote the proportion of observations allocated to the control treatment in a BTIB design; note that $\theta = \gamma/2$ for $p \geqslant 2$, $k = 2$ and $\theta = \gamma/3$ for $p = 3$, $k = 3$. From Section 5.4.2 we know that for Case 1 (which holds when $p \geqslant 2$, $k = 2$), the optimal limiting value of γ, namely γ^*, is obtained by solving $\partial\eta^2/\partial\gamma = 0$ where $\partial\eta^2/\partial\gamma$ is given by (A.4); for Case 2 (which holds when $p = 3$, $k = 3$), $\gamma^* = 1$. The solution γ^* was given in (3.11) for $p \geqslant 2$, $k = 2$. If θ^* denotes the asymptotically optimal proportion of observations allocated to the control treatment then we have $\theta^* = \gamma^*/2$ for $p \geqslant 2$, $k = 2$ and $\theta^* = \gamma^*/3 = 1/3$ for $p = 3$, $k = 3$. The values of θ^* are given in Table 5.4.

TABLE 5.4. ASYMPTOTICALLY OPTIMAL PROPORTION OF OBSERVATIONS (θ^*) ON THE CONTROL TREATMENT USING A BTIB DESIGN

p	k	θ^*	$(1+\sqrt{p})^{-1}$
2	2	0.4227	0.4142
3	2	0.3750	0.3660
4	2	0.3417	0.3333
5	2	0.3165	0.3090
6	2	0.2966	0.2899
3	3	0.3333	0.3660

This table shows that although the asymptotically optimal proportion θ^* for BTIB designs is different from $(1+\sqrt{p})^{-1}$ which holds for a completely randomized design, the θ^*-values are quite close to $(1+\sqrt{p})^{-1}$. Of course, this statement applies only to the (p, k)-combinations studied here all of which involve two generator designs in the minimal complete set.

6. Comparison of an optimal BTIB design with a BIB design between all treatments

In the preceding sections we have considered the problem of choosing an optimal design from the class of BTIB designs. A BIB design between all $p+1$ treatments (including the control treatment) is also a member of this class. It is of some interest to ascertain the potential gains achievable in terms of the decreased total size of the experiment if an optimal BTIB design is used instead of the corresponding BIB design; it is assumed here that (p, k), $\{D_0, D_1\}$ are given and a/σ and $1-\alpha$ are specified. In this section we make such a comparison for large b; thus continuous approximations can be used, and the problem of existence of designs for a given b can be ignored.

We first note that in the case of BIB designs, (2.2), (2.4), and (2.7) simplify to

$$\text{var}\{\hat{\alpha}_0-\hat{\alpha}_i\} = \frac{2p\sigma^2}{b(k-1)} \ (1 \leqslant i \leqslant p) \qquad \ldots \quad (6.1)$$

$$\rho = \text{corr}\{\hat{\alpha}_0-\hat{\alpha}_{i_1}, \hat{\alpha}_0-\hat{\alpha}_{i_2}\} = 1/2, \ (i_1 \neq i_2;\ 1 \leqslant i_1, i_2 \leqslant p) \qquad \ldots \quad (6.2)$$

and

$$\int_{-\infty}^{\infty} \Phi^p\left[x+\xi\sqrt{\frac{k-1}{pk}}\right]d\Phi(x), \qquad \ldots \quad (6.3)$$

respectively. Denote by $\hat{b}_{BIB}$ the minimum number of blocks required, to guarantee a specified confidence coefficient $1-\alpha$ using a BIB design. Then $\hat{b}_{BIB}$ is given by solving the equation

$$\xi\sqrt{(k-1)pk} = (a\sqrt{kb}/\sigma)\sqrt{(k-1)/pk} = c,$$

i.e.,

$$\hat{b}_{BIB} = \frac{pc^2\sigma^2}{(k-1)d^2}, \qquad \ldots \quad (6.4)$$

where $c = c_{p,1-\alpha}$ is the solution of the equation

$$\int_{-\infty}^{\infty} \Phi^p(x+c)d\Phi(x) = 1-\alpha. \qquad \ldots \quad (6.5)$$

The values of c have been tabulated by Bechhofer (1954), Gupta (1963) and Milton (1963) for selected values of p and $1-\alpha$; Bechhofer's λ equals c while Gupta's and Milton's H equals $c/\sqrt{2}$.

Denote the corresponding minimum number of blocks required, using the optimal BTIB design by $\hat{b}_{BTIB}$. Note that $\hat{b}_{BTIB}$ is given by

$$\hat{b}_{BTIB} = \frac{\sigma^2\hat{\xi}^2}{kd^2} \qquad \ldots \quad (6.6)$$

where $\hat{\xi}$ is given in Table 5.2.

For given (p, k), $\{D_0, D_1\}$, σ^2 and specified a and $1-\alpha$ we define the efficiency of a BIB design relative to that of an optimal BTIB design by

$$RE = \frac{\hat{b}_{BTIB}}{\hat{b}_{BIB}} = \left(\frac{\hat{\xi}}{c}\right)^2 \frac{(k-1)}{kp} \qquad \ldots \quad (6.7)$$

Since a BIB design is a special case of BTIB designs we see that the relative efficiency (RE) is $\leqslant 1$. The values of RE for selected (p, k) and $1-\alpha$ are listed in Table 6.1.

TABLE 6.1. EFFICIENCY OF A BIB DESIGN RELATIVE TO AN OPTIMAL BTIB DESIGN

p	k	confidence coefficient $(1-\alpha)$			
		0.80	0.90	0.95	0.99
2	2	0.9892	0.9684	0.9557	0.9420
3	2	0.9729	0.9414	0.9228	0.9027
4	2	0.9581	0.9201	0.8979	0.8737
5	2	0.9454	0.9029	0.8783	0.8512
6	2	0.9346	0.8887	0.8623	0.8330
3	3	0.9729	0.9423	0.9267	0.9109

From Table 6.1 we note that for fixed k and $1-\alpha$, RE decreases as p increases; also for fixed p and k, RE decreases as $1-\alpha$ increases. Thus it is seen that substantial improvements in efficiency (i.e., savings in the total number of blocks) can be achieved by using an optimal BTIB design instead of the corresponding BIB design.

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Appendix 1

Proof of Theorem 3.1 : For mathematical convenience and without loss of generality we shall regard $\gamma = f_0b_0/b$ as a continuous variable taking values in the interval (0, 1]. (We use the same continuous approximation in Section 5.) For $p = 2$, $k = 2$ and $p = 3$, $k = 3$ we substitute $\lambda_0 = f_0\lambda_0^{(0)}$, $\lambda_1 = f_0\lambda_1^{(0)} + f_1\lambda_1^{(1)}$, $b = f_0b_0 + f_1b_1$ and $f_0 = b\gamma_0/b$ in (2.6) and (2.4) to obtain

$$\eta^2(\gamma) = \frac{k^2b_0\{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}}{\lambda_0^{(0)}\gamma\{\gamma[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}} \quad \ldots \text{(A.1)}$$

and

$$\rho(\gamma) = \frac{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_0\lambda_1^{(1)}}{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}}, \quad \ldots \quad (A.2)$$

respectively. It follows that

$$\frac{\partial\rho}{\partial\gamma} = \frac{-b_0 b_1 \lambda_0^{(0)}\lambda_1^{(1)}}{\{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}^2} < 0 \quad \ldots \quad (A.3)$$

for b_1, $\lambda_1^{(1)} > 0$, and therefore ρ is strictly decreasing in γ (and hence in f_0). Next we have

$$\frac{\partial\eta^2}{\partial\gamma} = \frac{k^2 b_0 \psi(\gamma)}{\lambda_0^{(0)}\gamma^2\{\gamma[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}^2} \quad \ldots \quad (A.4)$$

where

$$\psi(\gamma) = \gamma^2(b_0\lambda_1^{(1)} - b_1\lambda_1^{(0)} - b_1\lambda_0^{(0)})[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}]$$
$$-2\gamma b_0\lambda_1^{(1)}[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] - p(b_0\lambda_1^{(1)})^2. \quad \ldots \quad (A.5)$$

Since $\lim_{\gamma \to 0} \eta^2 = \infty$, it follows that η^2 must be decreasing, at least in a small neighborhood of $\gamma = 0+$; thus it suffices to show that η^2 has at most one stationary point in (0, 1], i.e., that the equation $\psi(\gamma) = 0$ has at most one root in (0, 1].

Since the constant term $-p(b_0\lambda^{(1)})^2$ in (A.5) is negative, a necessary (but clearly not sufficient) condition for both roots of $\psi(\gamma)$ to be real, positive and distinct is that the coefficients of γ^2 and γ in $\psi(\gamma)$ be negative and positive, respectively, i.e., we require that

$$p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)} < 0.$$

Therefore,

$$\left.\frac{\partial\psi(\gamma)}{\partial\gamma}\right|_{\gamma=1} = -2b_1(\lambda_0^{(0)} + \lambda_1^{(0)})[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] > 0.$$

The latter inequality shows that $\psi(\gamma)$ is increasing at $\gamma = 1$. This together with the fact that $\psi(\gamma) \to -\infty$ as $\gamma \to \pm\infty$ implies that at most one root of $\psi(\gamma)$ must be in (0, 1].

The proof of the theorem now follows easily since Case 1 or Case 2 holds depending on whether or not

$$\text{sgn}\left\{ \left.\frac{\partial\eta^2}{\partial\gamma}\right|_{\gamma=1} \right\} = \text{sgn}\{\psi(\gamma)\,|_{\gamma=1}\} = \text{sgn}\{b_1\beta\}$$

is > 0 or $\leqslant 0$, respectively, where β is given by (3.9). If Case 1 holds then η^2 first decreases and then increases with f_0 (i.e., γ) while ρ always decreases with f_0. Hence there exists a critical number f_0^* which is the smallest value of f_0 at which η^2 starts increasing (for fixed b). Thus f_0^* is the smallest value of f_0 satisfying $\eta^2(f_0) \geqslant \eta^2(f_0-d)$ where d is the smallest positive integer such that db_0/b_1 is a positive integer. If $f_0^* \leqslant f_0^U$, then clearly designs with $f_0 \geqslant f_0^*$ are inadmissible. If Case 2 holds then since both η^2 and ρ are strictly decreasing in f_0 (i.e., γ), all designs $D = f_0D_0 \bigcup f_1D_1$ with $f_0 > 0$ are admissible.

Proof of Theorem 3.2 : We can express τ^2 of (2.3) and ρ of (2.4) as

$$\tau^2 = \frac{k\{f_0(b_1\lambda_0^{(0)}+b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b\lambda_1^{(1)}\}}{f_0\lambda_0^{(0)}\{f_0(b_1\lambda_0^{(0)}+p[b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}])+pb\lambda_1^{(1)}\}} \quad \ldots \quad (A.6)$$

$$\rho = \frac{f_0(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b\lambda_1^{(1)}}{f_0(b_1\lambda_0^{(0)}+b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b\lambda_1^{(1)}}; \quad \ldots \quad (A.7)$$

τ'^2 and ρ' have analogous expressions with f_0 replaced by f_0' and b replaced by b'.

We shall show that $b < b'$ and $\rho \geqslant \rho'$ implies that $\tau^2 > \tau'^2$. From $\rho \geqslant \rho'$ and (A.7) we get that $f_0'b-f_0b' \geqslant 0$, i.e., $f_0/f_0' \leqslant b/b' < 1$. Therefore we have

$$f_0'-f_0 > 0. \quad \ldots \quad (A.8)$$

Now using (A.7) and (A.8) we shall show that $\tau^2 > \tau'^2$, i.e., we shall show that

$$(f_0'-f_0)(f_0f_0'AB+bb'p\lambda_1^{(1)2}) > \lambda_1^{(1)}[f_0f_0'(b-b')pA+(f_0^2b'-f_0'^2b)B] \quad \ldots \quad (A.9)$$

where $A = b_1\lambda_0^{(0)}+b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}$ and $B = b_1\lambda_0^{(0)}+p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})$. We consider the cases $p \geqslant 2$, $k = 2$ and $p = 3$, $k = 3$ separately.

Case 1 ($p \geqslant 2$, $k = 2$) : Using (3.2) and (3.3) we obtain $A = p(p-3)/2$, $B = -p(p+1)/2$. Substituting in (A.9) for A, B and for b, b' from (3.4) we find after a lengthy algebraic manipulation that (A.9) will follow if we show that

$$(f_0'-f_0)(f_0f_0'+pf_1f_1'+f_0'f_1+f_0f_1')+(p-1)f_0f_0'(f_1'-f_1) > 0. \quad \ldots \quad (A.10)$$

Now note that $b'-b > 0$ yields

$$f_1'-f_1 > -\frac{2}{(p-1)}(f_0'-f_0),$$

and since from (A.8) we have $f_0'-f_0 > 0$, (A.10) will follow if we show that

$$pf_1f_1'+f_0f_1+f_0f_1'-f_0f_0' \geqslant 0. \quad \ldots \quad (A.11)$$

We use the bounds $f_0 \leqslant b\gamma^*/b_0, f_0' \leqslant b'\gamma^*/b$ where γ^* is given by (3.11) to obtain the following bound on f_1 (and an analogous bound on f_1') :

$$f_1 \geqslant \begin{cases} \left(\dfrac{\sqrt{p+1}-1}{p}\right) f_0 & p \geqslant 2, p \neq 3, k = 2 \\ \dfrac{1}{3} f_0 & p = 3,\ k = 2. \end{cases} \qquad \ldots \text{(A.12)}$$

Thus a lower bound on the l.h.s. of (A.11) will be obtained by substituting in it (A.12) (and an analogous bound on f_1'). It is easy to verify that this substitution yields a lower bound on the l.h.s. of (A.12) of exactly zero which completes the proof of this case.

Case 2 ($p = 3,\ k = 3$) : Using (3.6) and (3.7) we obtain $A = 0,\ B = -4$. Substituting these in (A.9) we find that (A.9) will follow if we show that

$$\frac{4(f_0'^2 b - f_0^2 b')}{3b(f_0' - f_0)} < b'. \qquad \ldots \text{(A.13)}$$

But the l.h.s. of (A.13) is less than

$$4(f_0'^2 b - f_0^2 b)/3b(f_0' - f_0) < 4(f_0' + f_0)/3 < 8f_0'/3 < 3f_0' \leqslant b'$$

which completes the proof of this case and the theorem. In the preceding line of the proof we have used the inequalities $f_0 < f_0'$ and $b < b'$.

Appendix 2

Derivation of Results in Section 5

Evaluation and simplification of $\partial g/\partial\gamma$. From (2.7) and the definition of g given in Section 5.2 we obtain by direct calculation

$$\frac{\partial g}{\partial \gamma} = \int_{-\infty}^{\infty} p\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)$$

$$\times \left\{ \frac{\eta\sqrt{1-\rho}\left(x\eta'\sqrt{\rho}+\dfrac{x\eta\rho'}{2\sqrt{\rho}}\right) - \left(x\eta\sqrt{\rho}+\xi\right)\left(\eta'\sqrt{1-\rho}-\dfrac{\eta\rho'}{2\sqrt{1-\rho}}\right)}{\eta^2(1-\rho)} \right\} dx. \qquad \ldots \text{(A.14)}$$

In (A.14), $\phi(\cdot)$ denotes the standard normal p.d.f., $\Phi(\cdot)$ denotes the standard normal c.d.f., $\eta' = \partial\eta/\partial\gamma = (1/2\eta)\partial\eta^2/\partial\gamma$ where $\partial\eta^2/\partial\gamma$ is given by (A.4), and $\rho' = \partial\rho/\partial\gamma$ is given by (A.3). After some simplification (A.14) can be written as

$$\frac{\partial g}{\partial\gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}}\left\{\frac{\eta^2\rho'}{\sqrt{\rho}}\int_{-\infty}^{\infty} x\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)dx\right.$$

$$\left.-\xi[2\eta'(1-\rho)-\eta\rho']\int_{-\infty}^{\infty}\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)dx\right\}. \quad \ldots \text{(A.15)}$$

Making the change of variables

$$y = (x\eta\sqrt{\rho}+\xi)/\eta\sqrt{1-\rho}$$

(A.15) can be expressed as

$$\frac{\partial g}{\partial\gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}}\left\{\frac{\eta^2\rho'}{R^2\sqrt{\rho}}E_1-\left[\frac{S\eta^2\rho'}{R^2\sqrt{\rho}}+\frac{\xi}{R}\{2\eta'(1-\rho)-\eta\rho'\}E_2\right]\right\} \quad \ldots \text{(A.16)}$$

where

$$R = \left(\frac{\rho}{1-\rho}\right)^{1/2}, \quad S = \frac{\xi}{\eta\sqrt{1-\rho}}, \quad \ldots \text{(A.17)}$$

$$E_1 = \int_{-\infty}^{\infty} y\Phi^{p-1}(y)\phi(y)\phi^*(y)dy, \quad \ldots \text{(A.18)}$$

$$E_2 = \int_{-\infty}^{\infty} \Phi^{p-1}(y)\phi(y)\phi^*(y)dy, \quad \ldots \text{(A.19)}$$

and $\phi^*(y)$ denotes $\phi((y-S)/R)$. We now evaluate E_1 and E_2. Integrating by parts in E_1 with $U = \Phi^{p-1}(y)\phi^*(y)$ and $dV = y\phi(y)dy$ we obtain

$$E_1 = \frac{-1}{R^2}E_1+\frac{S}{R^2}E_2+(p-1)E_3 \quad \ldots \text{(A.20)}$$

where

$$E_3 = \int_{-\infty}^{\infty} \Phi^{p-2}(y)\phi^2(y)\phi^*(y)dy. \quad \ldots \text{(A.21)}$$

From (A.20) we have

$$E_1 = \frac{S}{R^2+1} E_2 + \frac{(p-1)R^2}{R^2+1} E_3. \qquad \ldots \quad (A.22)$$

Substituting (A.22) in (A.16) we obtain

$$\frac{\partial g}{\partial \gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}} \left\{ -\left[\frac{S\eta^2\rho'}{\sqrt{\rho}(R^2+1)} + \frac{\xi}{R}\{2\eta'(1-\rho)-\eta\rho'\} \right] E_2 \right.$$

$$\left. + \frac{\eta^2\rho'(p-1)}{\sqrt{\rho}(R^2+1)} E_3 \right\}. \qquad \ldots \quad (A.23)$$

By developments similar to those in Bechhofer (1969) we can write

$$E_2 = \frac{R}{\sqrt{R^2+1}} \phi\left(\frac{S}{\sqrt{R^2+1}} \right) \Phi_{p-1}\left[\frac{S}{\sqrt{(2R^2+1)(3R^2+1)}} \,\middle|\, \frac{R^2}{R^2+1} \right], \qquad \ldots \quad (A.24)$$

$$E_3 = \frac{R}{\sqrt{(2R^2+1)2\pi}} \phi\left(\sqrt{\frac{2S^2}{2R^2+1}} \right) \Phi_{p-2}\left[\frac{S}{\sqrt{(2R^2+1)(3R^2+1)}} \,\middle|\, \frac{R^2}{3R^2+1} \right]. \qquad \ldots \quad (A.25)$$

Substituting E_2 and E_3 from (A.24) and (A.25) in (A.23), and replacing R and S by their definitions in (A.17) we obtain

$$\frac{\partial g}{\partial \gamma} = \frac{p}{2\eta^2\sqrt{1-\rho}} \left\{ -2\xi\eta'\sqrt{1-\rho}\,\phi\left(\frac{\xi}{\eta}\right) \Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}} \,\middle|\, \frac{\rho}{1+\rho} \right] \right.$$

$$\left. + \frac{\eta^2\rho'(p-1)}{\sqrt{2\pi(1+\rho)}} \phi\left(\frac{\xi}{\eta}\sqrt{\frac{2}{1+\rho}} \right) \Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}} \,\middle|\, \frac{\rho}{1+2\rho} \right] \right\}$$

$$= \frac{p\phi(\xi/\eta)}{2\eta^2} h(\gamma \mid D_0, D_1; p, k; \xi)$$

where $h(\gamma \mid D_0, D_1; p, k; \xi)$ is given by (5.3).

Evaluation of the limit in expression (5.5) *for* ξ_0 : We note from (A.1) and (A.2) that $\lim_{\gamma \to 0} \eta^2 = \infty$ and $\lim_{\gamma \to 0} \rho = 1$. Since $\Phi_{p-1}(0 \mid 1/2) = 1/p$ we have from (5.5) that

$$\xi_0 = \frac{p(p-1)\Phi_{p-2}(0 \mid 1/3)}{\sqrt{2\pi}} \lim_{\gamma \to 0} \left\{ \frac{\eta^3 \dfrac{\partial \rho}{\partial \gamma}}{\sqrt{1-\rho^2}\dfrac{\partial \eta^2}{\partial \gamma}} \right\}. \qquad \ldots \quad (A.26)$$

Using (A.3) and (A.4) we write

$$\frac{\eta^3}{\dfrac{\partial\eta^2}{\partial\gamma}} = \frac{k}{\psi(\gamma)}\sqrt{\frac{b_0\gamma}{\lambda_0^{(0)}}}\;\frac{\{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)})+b_0\lambda_1^{(1)}\}^{3/2}}{\{\gamma[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b_1\lambda_0^{(0)}]+pb_0\lambda_1^{(1)}\}^{-1/2}} \quad \ldots \quad \text{(A.27)}$$

where $\psi(\gamma)$ is given by (A.5). Using (A.2) and (A.3) we write

$$\frac{\dfrac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}} = -b_0\lambda_1^{(1)}\sqrt{\frac{b_1\lambda_0^{(0)}}{\gamma}}\;\frac{\{\gamma[2(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b_1\lambda_0^{(0)}]+2b_0\lambda_1^{(1)}\}^{-1/2}}{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)})+b_0\lambda_1^{(1)}}. \quad \ldots \quad \text{(A.28)}$$

Combining (A.27) and (A.28) we obtain

$$\frac{\eta^3\dfrac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}\,\dfrac{\partial\eta^2}{\partial\gamma}} = -\frac{k\lambda_1^{(1)}}{\psi(\gamma)}\,b_0^{3/2}b_1^{1/2}$$

$$\times\left[\frac{\{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)})+b_0\lambda_1^{(1)}\}\{\gamma[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b_1\lambda_0^{(0)}]+pb_0\lambda_1^{(1)}\}}{\gamma[2(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)})+b_1\lambda_0^{(0)}]+2b_0\lambda_1^{(1)}}\right]^{1/2}. \quad \ldots \quad \text{(A.29)}$$

Since $\lim\limits_{\gamma\to 0}\psi(\gamma) = -p(b_0\lambda_1^{(1)})^2$ we have

$$\lim_{\gamma\to 0}\frac{\eta^3\dfrac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}\,\dfrac{\partial\eta^2}{\partial\gamma}} = k\sqrt{\frac{b_1}{2\lambda_1^{(1)}p}}. \quad \ldots \quad \text{(A.30)}$$

Substituting (A.30) in (A.26) we obtain the desired result (5.6).

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