Incomplete Block Designs for Comparing Treatments With a Control: General Theory

Robert E. Bechhofer
School of Operations Research and Industrial Engineering
Cornell University
Ithaca, NY 14853

Ajit C. Tamhane
Department of Industrial Engineering and Management
Technological Institute
Northwestern University
Evanston, IL 60201

In this paper we develop a theory of optimal incomplete block designs for comparing several treatments with a control. This class of designs is appropriate for comparing simultaneously \( p \geq 2 \) test treatments with a control treatment (the so-called multiple comparisons with a control (MCC) problem) when the observations are taken in incomplete blocks of common size \( k < p + 1 \). For this problem we propose a new general class of incomplete block designs that are \textit{balanced} with respect to \textit{(wrt) test} treatments. We shall use the abbreviation BTIB to refer to such designs. We study their structure and give some methods of construction. A procedure for making exact joint confidence statements for this multiple comparisons problem is described. By using a new concept of admissibility of designs, it is shown how "inferior" designs can be eliminated from consideration, and attention limited to a small class of BTIB designs that can be constructed from so-called generator designs in the minimal complete class of such designs. Some open problems concerning construction of BTIB designs are posed.

KEY WORDS: Multiple comparisons with a control; Balanced treatment incomplete block (BTIB) designs; Generator designs; Admissible designs; Optimal designs.

1. INTRODUCTION

In many industrial, agricultural, and biological experiments, it is often desired to compare simultaneously several test treatments with a control treatment. The earliest correct work on this problem was carried out by Dunnett (1955, 1964). Dunnett (1955) also posed (but did not solve) the problem of optimally allocating experimental units to control and test treatments so as to maximize the probability associated with the joint confidence statement concerning the many-to-one comparisons between the mean of the control treatment and the means of the test treatments. This optimal allocation problem was solved by Bechhofer and his coworkers (1969, 1970, 1971).

In all of the aforementioned papers it was tacitly assumed that a completely randomized (CR) design was used. However, many practical situations may require the blocking of experimental units in order to cut down on bias and improve the precision of the experiment. If the block size is large enough to accommodate one replication of all of the test treatments and additional control treatments as well, then the design and analysis of replications of the experiment can be carried out using the optimal allocations described in Bechhofer (1969) and Bechhofer and Nocturne (1970) with only the usual modifications. We shall study the multiple comparisons problem in the situation that commonly occurs in practice, that is, when all of the blocks have a common size but the block size is less than the total number of treatments. Robson (1961) pointed out that Dunnett's procedure can be extended to the case in which a balanced incomplete block (BIB) design between all of the treatments (including the control treatment) is used. Cox (1958, p. 238) noted that BIB designs are perhaps not appropriate for the multiple comparisons with the control (MCC) problem because of the special role played by the control treatment. He suggested a design that employs the control treatment an equal number of times (once, twice, etc.) in each block, the test treatments forming a BIB design in the remaining plots of the blocks; no analytical details were given for this proposed design. Pešek (1974) has given analytical details for a special case of Cox's design (i.e., the control treatment is employed once in each block); he shows that this design is more
efficient than a BIB design for comparisons with a control, but it is less efficient for pairwise comparisons between the test treatments. It should be noted that even Cox's more general design is quite restrictive.

In this paper we propose a new general class of incomplete block designs that is appropriate for the MCC problem. We give the basic theory underlying these designs, and include as well some methods of construction. The method of analysis is described. Admissibility and optimality considerations are discussed in some detail.

2. PRELIMINARIES

Let the treatments be indexed by 0, 1, ..., p with 0 denoting the control treatment and 1, 2, ..., p denoting the p ≥ 2 test treatments. Let k < p + 1 denote the common size of each block, and let b denote the number of blocks available for experimentation. Thus N = kb is the total number of experimental units. If treatment i is assigned to the hth plot of the jth block (0 ≤ i ≤ p, 1 ≤ h ≤ k, 1 ≤ j ≤ b), let Y_{ihj} denote the corresponding random variable; we assume the usual additive linear model (no treatment × block interaction)

\[ Y_{ihj} = \mu + \alpha_i + \beta_j + e_{ijh} \]  

(2.1)

with \( \sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{1} \beta_j = 0 \); the \( e_{ijh} \) are assumed to be iid \( N(0, \sigma^2) \) random variables. It is desired to make an exact joint confidence statement (employing one-sided or two-sided intervals) concerning the differences \( \alpha_0 - \alpha_i \) based on their BLUE's \( \hat{\alpha}_0 - \hat{\alpha}_i \) (1 ≤ i ≤ p).

3. CHOICE OF THE CLASS OF DESIGNS

3.1 BTIB Designs

Since it is desired to make a confidence statement that applies simultaneously to all of the p differences \( \alpha_0 - \alpha_i \) (1 ≤ i ≤ p), we shall regard our problem as exactly two generator designs. (See (3.4).) By taking unions of replications of these generator designs, at least one of which has \( \alpha_0 > 0 \), we obtain an implementable BTIB design. The problem of determining how
many generator designs exist for arbitrary given \((p, k)\) is an open problem (it can be shown that there are a finite number), which appears to be very formidable; we comment on this problem in Section 6. In the sequel we consider only implementable BTIB designs.

Suppose that for given \((p, k)\) there are \(n\) generator designs \(D_i\) \((1 \leq i \leq n)\). Let \(\lambda_0^i, \lambda_1^i\) be the design parameters associated with \(D_i\), and let \(b_i\) be the number of blocks required by \(D_i\). Then a BTIB design \(D = \bigcup_{i=1}^{n} f_i D_i\) obtained by forming unions of \(f_i \geq 0\) replications of \(D_i\) has the design parameters \(\lambda_0 = \sum_{i=1}^{n} f_i \lambda_0^i, \lambda_1 = \sum_{i=1}^{n} f_i \lambda_1^i\) and requires \(b = \sum_{i=1}^{n} f_i b_i\) blocks. The set of \(D_i\) with \(f_i > 0\) will be referred to as the support of \(D\).

The following is another example of generator designs (generalizing (3.3)). As noted before, for each \(p > 2, k = 2\) there are exactly two generator designs.

From these generator designs, implementable BTIB designs of the type \(D = f_0 D_0 \cup f_1 D_1\) can be constructed for \(f_0 \geq 1, f_1 \geq 0\); the corresponding design parameters for \(D\) are \(\lambda_0 = f_0, \lambda_1 = f_1, b = f_0 p + f_1 p(p-1)/2\).

We now consider several methods of constructing generator designs for \(k \geq 3\); this list is not exhaustive.

**Method I:** The preceding example suggests the following method of constructing a class of generator designs: For given \((p, k)\), a generator design \(D_m\) will have \(m + 1\) plots in each block assigned to the control treatment; the \(p\) test treatments are assigned to the remaining \(k - m - 1\) plots of the \(b_m\) blocks \((0 < m < k - 2)\) in such a way as to form a BIB design. The generator design \(D_{k-1}\) contains no control treatments; it consists of a RB (BIB) design between the \(p\) test treatments if \(p = k(p > k)\). Thus for \((p, k) = (3, 3)\) we have the following three generator designs in this class.

\[
D_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
\]  

**Method II:** Starting with a BIB design between \(t > p\) treatments in \(b\) blocks, one can relabel the treatments \(p + 1, p + 2, \ldots, t\) zeros to obtain a new BTIB design with possibly an additional block or blocks, each one of the latter containing only one test treatment or only the control treatment. After deleting all of these one-treatment blocks, and identifying the support of the resulting BTIB design, one obtains the desired generator design(s). For example, we consider the following BIB design for \((t, k, b) = (7, 3, 7)\):

\[
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \end{bmatrix}
\]  

By replacing the sevens by zeros, one obtains a generator design for \((p, k, b) = (6, 3, 7)\) with \(\lambda_0 = 1, \lambda_1 = 1\). This process can be continued by then replacing the sixes by zeros to obtain a generator design for \((p, k, b) = (5, 3, 7)\) with \(\lambda_0 = 2, \lambda_1 = 1\); continuing, one can then replace the fives by zeros to obtain a generator design for \((p, k, b) = (4, 3, 7)\) with \(\lambda_0 = 3, \lambda_1 = 1\). Finally, replacing the fours by zeros, one obtains the union of the two generator designs \(D_0\) and \(D_1\) of (3.5) with a block containing all zeros. The BTIB designs obtained in this way for \(k = 3, b = 7\) are

\[
\begin{align*}
p = 6, b = 7 & (\lambda_0 = 1, \lambda_1 = 1): \\
& \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 2 & 5 & 3 & 4 \\ 3 & 6 & 5 & 4 & 6 & 5 & 6 \end{bmatrix} (3.7a) \\
p = 5, b = 7 & (\lambda_0 = 2, \lambda_1 = 1): \\
& \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 3 & 4 & 0 & 2 & 3 \\ 3 & 5 & 4 & 5 & 2 & 4 & 5 \end{bmatrix} (3.7b) \\
p = 4, b = 7 & (\lambda_0 = 3, \lambda_1 = 1): \\
& \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 2 \\ 3 & 3 & 4 & 1 & 2 & 4 & 4 \end{bmatrix} (3.7c)
\end{align*}
\]

**Remark 3.2:** From (3.7a) it is clear that every BIB design involving \(t\) treatments yields a BTIB design with \(p = t - 1\) test treatments.

**Remark 3.3:** It is well known (e.g., John 1964) that unequal replicate designs having the same properties as BIB designs can be constructed. Such designs are therefore BTIB designs. An example of such a generator design for \((p, k, b) = (5, 3, 8)\) with \(\lambda_0 = 2, \lambda_1 = 1\) is given by

\[
\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 & 5 & 5 & 5 & 5 \\ 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \end{bmatrix}
\]  

**Method III:** Consider a group-divisible partially balanced incomplete block (GD-PBIB) design with two associate classes between \(t\) treatments in blocks of size \(k\). The association scheme of such a GD-PBIB design can be represented in the form of an \(m \times n\) array (with \(mn = t\)). Any two treatments in the same row of the array are first associates, and those in
different rows are second associates. Suppose that \( m \geq k \); one can then take \( p = m \) and relabel the entries in \( n_1 > 0 \) columns of the array by 1, 2, \ldots, \( p \) and the entries in the remaining \( n_2 = n - n_1 > 0 \) columns by zeros, thus obtaining a BTIB design. As with Method II, such a design may not be a generator design and may contain some blocks that must be deleted. After deleting such blocks, a BTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the previous two methods. If \( n \geq k \), one can take \( p = n \) and then relabel the entries in \( m_1 > 0 \) rows of the array by 1, 2, \ldots, \( p \) and the entries in \( m_2 = m - m_1 > 0 \) rows by zeros, thus obtaining a BTIB design, possibly with blocks that must be deleted.

To see the use of this method, consider the GD-PBIB design \( \#R20 \) (for \( k = 3, t = 12, m = 4, n = 3, b = 36 \)) in the monograph by Bose, Clatworthy, and Shrikhande (1954), which has the following association scheme:

\[
\begin{array}{ccc}
1 & 5 & 9 \\
2 & 6 & 10 \\
3 & 7 & 11 \\
4 & 8 & 12 \\
\end{array}
\]

By relabeling the treatments 5 through 8 by 1 through 4, and 9 through 12 by zeros, one obtains the union of a BTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining BTIB design consists of the following:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 \\
2 & 3 & 4 & 4 \\
\end{array}
\] (3.9a)

2 replications of

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 \\
\end{array}
\] (3.9b)

2 replications of

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 3 & 3 \\
3 & 4 & 4 & 4 \\
\end{array}
\] (3.9c)

1 replication of

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 \\
2 & 4 & 4 & 3 \\
\end{array}
\] (3.9d)

Thus the designs given by (3.9a) through (3.9d) are generator designs for \( p = 4, k = 3 \); the first three designs are obtainable by Method I, while the fourth is not. Using the definition of inadmissibility of a design as given in Section 5, we shall see that the design of (3.7c) obtained for \( (p, k, b) = (4, 3, 7) \) by Method II is inadmissible wrt the design of (3.9d), which has \( \lambda_0 = 2, \lambda_1 = 2 \).

Method IV: Suppose that for given \( (p, k) \) we have a generator design \( D_1 \) with \( \lambda_0 > 0 \). Then a new generator design \( D_2 \) for the same \( (p, k) \) can be obtained by taking a “complement” of \( D_1 \) in the following way: Separate the blocks of \( D_1 \) in different sets so that each block in a given set has zero assigned in an equal number of plots (0 times, 1 time, etc.). For example, consider the design (3.7c) the blocks of which can be separated into three sets as follows:

\[
D_1 = \begin{bmatrix}
1 & 1 & 2 & 0 & 0 & 0 \\
2 & 1 & 2 & 3 & 0 & 0 \\
4 & 3 & 3 & 4 & 1 & 2 \\
\end{bmatrix}
\]

For each set of \( D_1 \) write its “complementary” set (with zero assigned in the same number of plots) so that the union of that set with its complementary set forms a generator design; if \( r_{ij} = 0 \) or 1 \( (1 \leq i \leq p) \) then that union is simply a generator design that can be constructed by Method I. These complementary sets in the present example are

\[
\begin{array}{cccc}
1 & 1 & 2 & 0 & 0 & 0 \\
2 & 3 & 3 & 1 & 1 & 2 \\
3 & 4 & 4 & 2 & 4 & 4 \\
\end{array}
\]

by taking their union we obtain the generator design \( D_2 \) given by (3.9d). The \( b \)-values for \( D_1 \) and its complement \( D_2 \) are not in general equal, although in the present example they are.

4. JOINT CONFIDENCE STATEMENTS

4.1 Expressions for Estimates

We first give the expressions for the BLUE \( \hat{\alpha}_0 - \hat{\alpha}_i \) of \( \alpha_0 - \alpha_i \) \( (1 \leq i \leq p) \). Let \( T_i \) denote the sum of all observations obtained with the \( i \)-th treatment \( (0 \leq i \leq p) \), and let \( B_j \) denote the sum of all observations in the \( j \)-th block \( (1 \leq j \leq b) \). Denote \( B^* = \sum_{i=1}^{b} r_{ij} B_j \) and let \( Q_i = k T_i - B^* \) \( (0 \leq i \leq p) \).

Then

\[
\hat{\alpha}_0 - \hat{\alpha}_i = \frac{\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0 (\lambda_0 + p \lambda_1)} \quad (1 \leq i \leq p). \quad (4.1)
\]

Also,

\[
\operatorname{var}(\hat{\alpha}_0 - \hat{\alpha}_i) = \tau^2 \sigma^2 \quad (1 \leq i \leq p) \quad (4.2)
\]

where

\[
\tau^2 = \frac{k(\lambda_0 + \lambda_1)}{\lambda_0 (\lambda_0 + p \lambda_1)}, \quad (4.3)
\]

and

\[
\rho = \operatorname{corr}(\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}) = \frac{\lambda_1}{\lambda_0 + \lambda_1}, \quad (i_1 \neq i_2; 1 \leq i_1, i_2 \leq p). \quad (4.4)
\]
Table 1. Analysis of Variance Table for BTIB Designs

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments (adjusted)</td>
<td>$Q_i^2((p-1)(\sum \alpha_i^2)+(\sum \alpha_i\beta_i)/p)/k(p+1)/\alpha_i/p\beta_i$</td>
<td>$p$</td>
</tr>
<tr>
<td>Blocks</td>
<td>$\sum \frac{b_i}{p-k}^2 - \frac{\sum \alpha_i^2}{P}$</td>
<td>$b-1$</td>
</tr>
<tr>
<td>Error (by subtraction)</td>
<td>$N - p - b$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$N - \frac{\sum \alpha_i^2}{b}$</td>
<td>$N - 1$</td>
</tr>
</tbody>
</table>

The expressions for (4.1) through (4.4) are derived in the Appendix.

An unbiased estimate $s^2$ of $\sigma^2$ based on $v = N - p - b$ degrees of freedom (df) can be computed as $SS_{error}/(N - p - b)$ where $SS_{error}$ can be obtained by subtraction (as in BIB designs) from Table 1. The expressions in the table are derived in the Appendix; $G$ and $H$ denote the sum and sum of squares, respectively, of all observations. We note that if $\lambda_0 = \lambda_1 = \lambda$ (say), then $SS_{treatments (adjusted)}$ reduces to $\sum Q_i^2/k\alpha(p + 1)$ (i.e., the same expression as for a BIB design); this latter expression thus holds for any completely balanced design (such as (3.2), which is not a BIB design).

Remark 4.1: For many BTIB designs we have $r_{ij} > 1$, and thus within-block replication occurs. For such designs the sum of squares (SS) for error can be partitioned into SS due to “pure error” and SS due to “interaction,” and this decomposition can be used in testing the additivity assumption or the assumption of block-to-block variance homogeneity. Such tests are not pursued in this paper.

Remark 4.2: It is easily shown for BTIB designs that the $\hat{\alpha}_i$ ($1 \leq i \leq p$) have a constant variance $\eta^2\sigma^2$ (say), a common correlation $= \gamma$ (say), and we have the relationship

$$\text{var}[^{\hat{\alpha}_i - \hat{\alpha}_j}] = 2(1 - \gamma)\eta^2\sigma^2$$

$$= 2(1 - \rho)\eta^2\sigma^2 \quad (i \neq j; \ 1 \leq i, j \leq p).$$

Thus the relative precision of the estimators $\hat{\alpha}_0 - \hat{\alpha}_i$ ($1 \leq i \leq p$) for the MCC problem wrt the estimators $\hat{\alpha}_i - \hat{\alpha}_j$ ($i \neq j, \ 1 \leq i, j \leq p$) for the pairwise comparisons among the $p$ test treatments is given by

$$\frac{\text{var}[^{\hat{\alpha}_n - \hat{\alpha}_i}]}{\text{var}[^{\hat{\alpha}_0 - \hat{\alpha}_i}]} = 2(1 - \rho).$$

4.2 Confidence Statements

Joint 100$(1 - \alpha)$ percent confidence intervals for the $\alpha_0 - \alpha_i$ ($1 \leq i \leq p$) and for the $\alpha_i - \alpha_j$ ($i \neq j, \ 1 \leq i, j \leq p$) are given below.

I. One-sided confidence intervals. When $\sigma^2$ is unknown, the joint one-sided confidence intervals are given by

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}} \quad (1 \leq i \leq p). \quad (4.7)$$

In (4.7) $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ denotes the upper equicoordinate $\alpha$ point of the $p$-variate equicorrelated central $t$ distribution with common correlation $\rho$, and with df $v$ (as defined by Dunnett and Sobel 1954); for tables of $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ see Krishnaiah and Armitage (1966).

When $\sigma^2$ is known, the joint one-sided confidence intervals are obtained by replacing $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ in (4.7) by $z(a)\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ where $z(a)\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ for $v = \infty$ denotes the upper equicoordinate $\alpha$ point of the $p$-variate equicorrelated standard normal distribution with common correlation $\rho$. For tables of $z(a)\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ see Gupta, Nagel and Panchapakesan (1973). Two other relevant references are Gupta (1963) and Milton (1963).

II. Two-sided confidence intervals. When $\sigma^2$ is unknown the joint two-sided confidence intervals are given by

$$\alpha_0 - \alpha_i \in [\hat{\alpha}_0 - \hat{\alpha}_i \pm t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}] \quad (1 \leq i \leq p). \quad (4.8)$$

In (4.8) $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ denotes the upper $\alpha$ point of the distribution of $\max\{|t_1|, \ldots, |t_p|\}$ where $\{t_1, \ldots, t_p\}$ has a $p$-variate equicorrelated central $t$ distribution with common correlation $\rho$, and with df $v$ (Dunnett and Sobel 1954); for tables of $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ see Hahn and Hendrickson (1971). See also, Krishnaiah, and Armitage (1970), who have tabulated $(t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}})^2$.

When $\sigma^2$ is known, the joint two-sided confidence intervals are obtained by replacing $t_{v}^{(p)}\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ in (4.8) by $z(a)\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ where $z(a)\eta \sigma \sqrt{\frac{k\alpha}{p\beta_i}}$ for $v = \infty$ denotes the upper $\alpha$ point of the distribution of $\max\{|Z_1|, \ldots, |Z_p|\}$ and $(Z_1, \ldots, Z_p)$ has a $p$-variate equicorrelated standard normal distribution with common correlation $\rho$. Hahn and Hendrickson's tables with $v = 60$ can be used here to obtain a conservative approximation to $v = \infty$.

III. Pairwise comparisons between test treatments. In Remark 4.2 we noted that the $\hat{\alpha}_i$ ($1 \leq i \leq p$) have a constant variance and equal correlation when BTIB designs are used. Thus Tukey's procedure can be easily modified as described in Miller (1966, pp. 41-42) to provide 100$(1 - \alpha)$ percent confidence intervals for $\alpha_i - \alpha_j$ ($i \neq j, \ 1 \leq i, j \leq p$). These are
coefficient \( P > 1 - a \) with the smallest \( b \) is said to be making the following definition: BTIB designs with unequal \( k \) values. We start by for given \((p, k)\); however, as pointed out in Remark 5.2, our results can be extended to the comparison of the BTIB design employed only through \( T \) and \( p \). This fact will facilitate comparisons between BTIB designs. In these comparisons we mainly restrict consideration to BTIB designs with possibly unequal \( b \) values.

5. THE CLASS OF ADMISSIBLE DESIGNS

5.1 Optimal and Admissible Designs

In this section we propose a rationale for choosing a design from a set of competing BTIB designs. For simplicity of exposition we consider here the case of one-sided confidence intervals; \( \sigma^2 \) is assumed to be known. In Remark 5.2 we point out how our results extend to the case of two-sided confidence intervals and/or to \( \sigma^2 \) unknown.

We limit consideration to confidence intervals of the form \( [\bar{x}_0 - \bar{x}_i \geq \hat{x}_0 - \hat{x}_i - d \ (1 \leq i \leq p)] \) where \( d > 0 \) is a specified “yardstick” associated with the common width of the confidence intervals. The probability \( P \) associated with this joint confidence statement can be written as

\[
P = \Pr[\bar{x}_0 - \bar{x}_i \geq \hat{x}_0 - \hat{x}_i - d \ (1 \leq i \leq p)]
\]

\[
= \Pr[Z_i \leq d/\sigma \ (1 \leq i \leq p)]
\]

\[
= \int_{-\infty}^{\infty} \Phi(...) \left(x \sqrt{\sigma^2 + d/\sigma^2} \right) d\Phi(x)
\]

(5.1)

where \((Z_1, \ldots, Z_p)\) has a \( p \)-variate equicorrelated standard normal distribution with common correlation \( \rho \), and \( \Phi(...) \) denotes the standard univariate normal distribution function. Note that for given \( p \) and specified \( d/\sigma \) the probability \( P \) of (5.1) depends on the BTIB design employed only through \( \tau \) and \( \rho \). This fact will facilitate comparisons between BTIB designs. In these comparisons we mainly restrict consideration to BTIB designs with possibly unequal \( b \) values for given \((p, k)\); however, as pointed out in Remark 5.1, our results can be extended to the comparison of BTIB designs with unequal \( k \) values. We start by making the following definition:

Definition 5.1: For given \((p, k)\) and specified \( d/\sigma \) the BTIB design that achieves a joint confidence coefficient \( P \geq 1 - \alpha \) with the smallest \( b \) is said to be optimal for that value of \( 1 - \alpha \).

To determine the optimal BTIB design for given \((p, k), 1 - \alpha \) and specified \( d/\sigma \), one would proceed as follows: Find the design that for given \((p, k, b)\) and \( d/\sigma \) maximizes \( P \), and then vary \( b \) to find the smallest \( b \) for which the maximum \( P \) is \( \geq 1 - \alpha \).

In the search for the optimal design for given \((p, k)\), it is desirable to eliminate from consideration certain designs that are uniformly dominated by other designs and hence cannot be optimal for any \( d/\sigma \) or \( 1 - \alpha \). The definition of such an inadmissible design follows.

Definition 5.2: If for given \((p, k)\) we have two BTIB designs \( D_1 \) and \( D_2 \) with parameters \((b_1, \tau_1, \rho_1)\) and \((b_2, \tau_2, \rho_2)\) with \( b_1 \leq b_2 \), and if for every \( d \) and \( \sigma \), \( D_1 \) yields a confidence coefficient \( P \) at least as large as (larger than) that yielded by \( D_2 \) when \( b_1 < b_2 \) (\( b_1 = b_2 \)), then we say that \( D_1 \) is inadmissible wrt \( D_2 \). If a design is not inadmissible, then it is admissible. If \( b_1 = b_2 \), \( \tau_1 = \tau_2 \), and \( \rho_1 = \rho_2 \), then we say that \( D_1 \) and \( D_2 \) are equivalent.

For given \((p, k)\) one would limit consideration to all admissible designs in the search for optimal designs. Furthermore, if for given \((p, k)\) we have two or more equivalent designs, then only one of such designs need be considered. It is easy to verify that a necessary and sufficient condition for BTIB designs \( D_1 \) and \( D_2 \) to be equivalent is that \((\tau_0^{(i)}), \lambda_0^{(i)}, b_1) = (\tau_0^{(i)}, \lambda_0^{(i)}, b_2) \) where \((\lambda_0^{(0)}, \lambda_0^{(1)}, b_i)\) corresponds to \( D_i (i = 1, 2) \). For an example of equivalent designs for \((p, k, b) = (4, 3, 7)\) we have the designs of (3.2) and (3.9d), both of which have \( \lambda_0 = \lambda_1 = 2 \). Equivalent designs can sometimes provide flexibility to the experimenter without changing the confidence coefficient. For example, the experimenter might prefer the design of (3.9d) to the design of (3.2) if (say) the control treatments are more readily available than any of the test treatments. (See also the example of \( D_1 \cup D_2 \) vs. \( D_1 \) in the beginning of Section 5.2.)

The following theorem gives a characterization of inadmissibility that is easy to verify.

Theorem 5.1: For given \((p, k)\) consider two BTIB designs \( D_1 \) and \( D_2 \) with parameters \((b_1, \tau_1, \rho_1)\) and \((b_2, \tau_2, \rho_2)\), respectively. Design \( D_2 \) is inadmissible wrt design \( D_1 \) if and only if \( b_1 < b_2 \), \( \tau_1 \leq \tau_2 \) and \( \rho_1 \geq \rho_2 \) with at least one inequality strict.

Proof of sufficiency: From (5.1) we see that as \( \tau \) decreases for fixed \( d, \sigma \) and \( \rho \) \( \rho \) increases for fixed \( d \), \( \sigma \), and \( \tau \), the confidence coefficient \( P \) increases. The monotonicity wrt \( \sigma \) follows from Slepian’s inequality.

Proof of necessity: Suppose that the confidence coefficient associated with \( D_1 \) is larger than the confidence coefficient associated with \( D_2 \) for every \( d \) and \( \sigma \). Then \( \tau_1 \leq \tau_2 \) \( (\rho_1 \geq \rho_2) \) follows from letting \( d \uparrow \infty \).
For an application of this theorem consider the following two BTIB designs for \((p, k) = (4, 3)\):

\[
\begin{align*}
D_1 &= \{0 0 0 0 1 1 1 2, 1 1 1 2 0 2 3 3, 2 4 4 3 3 4 4\}, \\
D_2 &= \{0 0 0 0 1 1 2 3, 1 1 2 4 2 4 4 4\}.
\end{align*}
\]

For these designs \(\lambda_{01}^1 = \lambda_{02}^1 = 2\), \(\lambda_{11}^1 = \lambda_{12}^1 = 2\), and \(b_1 = 7 < b_2 = 8\). Hence \(\tau_1^2 = \tau_2^2 = \frac{3}{4}\) and \(\rho_1 = \rho_2 = \frac{1}{2}\), and thus both \(D_1\) and \(D_2\) yield the same \(P\) for every \(d\) and \(\sigma\); however, \(D_2\) is inadmissible wrt \(D_1\) because \(D_2\) requires a larger total number of observations than does \(D_1\).

**Remark 5.1:** Definitions 5.1 and 5.2 can be extended to permit comparison of BTIB designs having unequal \(k\) values. Such comparisons would be of interest to the experimenter who is faced with the choice of block size, subject to the restriction that the common block size \(k < p + 1\). In this case, for given \(p\) and specified \(d/\sigma\) the BTIB design that achieves a joint confidence coefficient \(P \geq 1 - \alpha\) with the smallest \(N = kb\) is said to be *optimal* for that value of \(1 - \alpha\). If this more general definition of optimality is used, the characterization of inadmissibility given by Theorem 5.1 would be modified as follows: For given \(p\) consider two BTIB designs \(D_1\) and \(D_2\) with parameters \((b_1, k_1, \tau_1, \rho_1)\) and \((b_2, k_2, \tau_2, \rho_2)\), respectively. Design \(D_2\) is inadmissible wrt \(D_1\) if and only if \(N_1 = k_1 b_1 \leq N_2 = k_2 b_2\), \(\tau_1 \leq \tau_2\), and \(\rho_1 \geq \rho_2\) with at least one inequality strict.

For an interesting application of this more general definition see Remark 2.3 in Bechhofer and Tamhane (1980a). It was shown there that if one considers only one counterexample, namely,

\[
D_1 = \{0 0 0 0 1 1 1 2, 1 1 1 2 0 2 3 3, 2 4 4 3 3 4 4\}, \\
D_2 = \{0 0 0 0 1 1 2 3, 1 1 2 4 2 4 4 4\}.
\]

Both \(D_1\) and \(D_2\) have \(\tau^2 = \tau_{01}^1 = \frac{3}{4}\), \(\rho = \frac{1}{2}\) and therefore achieve the same \(P\). However, \(D_2\) for \(k = 4\) has \(N = 32\), while \(D_1\) for \(k = 3\) has \(N = 30\). Thus although each design is admissible for its own \(k\) value, \(D_2\) is inadmissible wrt \(D_1\).

**Remark 5.2:** The inadmissibility characterization given in Theorem 5.1 applies equally well to the case of joint two-sided confidence intervals; then the monotonicity wrt \(P\) follows from Šidák’s (1968) results. (Of course, the optimal designs might be different in the one-sided and two-sided cases for the same \((p, k)\) and \(d/\sigma\).) The same general ideas carry over for unknown \(\sigma^2\), except that then one would have to specify the expected common “width” of the confidence intervals.

### 5.2 Strongly (S-) Inadmissible and Combination (C-) Inadmissible Designs

The candidates for an optimal design for given \((p, k)\) will be all admissible BTIB designs that can be constructed by forming unions of replications of all known generator designs for that given \((p, k)\). In this section we give three rules that can be used to reduce further the number of generator designs that must be used for given \((p, k)\) and arbitrary \(b\) to construct all admissible BTIB designs; this set of generator designs is called the *minimal complete class of generator designs*, and is formally characterized in Definition 5.6.

If the union of two or more generator designs yields an equivalent generator design, then we choose to eliminate the latter design from consideration and thereby maintain more flexibility for our construction of designs involving larger numbers of blocks. Thus, for example, for \(p = 4, k = 4\) the designs

\[
D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 3 & 4 & 3 & 4 & 4 & 4 \end{bmatrix}, \\
D_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \\
D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 3 & 3 \\ 4 & 4 & 3 & 3 & 4 & 4 \end{bmatrix}
\]

are all generator designs, but \(D_3\) is equivalent to \(D_1 \cup D_2\); hence we choose to retain only \(D_1\) and \(D_2\) but not \(D_3\). This leads us to the following definition.

**Definition 5.3:** Suppose that for given \((p, k)\) we have \(n \geq 2\) BTIB generator designs \(D_i (1 \leq i \leq n)\), no two of which are equivalent, and no one of which is equivalent to the union of replications of one or more of the other generator designs. Then \(\{D_i (1 \leq i \leq n)\}\) is referred to as the set of *nonequivalent generator designs*.

It would be tempting to eliminate any inadmissible generator designs from the set of nonequivalent generator designs. However, it is not in general true for given \((p, k)\) that if design \(D\) is inadmissible, then every
design $D \cup D'$ is also inadmissible. Thus, for example, for $p = 4$, $k = 4$ the BTIB designs

\[
D_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 \\
1 & 2 & 3 & 4 & 3 & 4 & 4 & 4
\end{pmatrix},
\]

\[
D_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4
\end{pmatrix},
\]

which are unions of generator designs, have $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 2$, $\lambda_2^{(1)} = 4$, $\lambda_3^{(1)} = 0$, and $b_1 = b_2$, and it is easy to verify that $D_2$ is inadmissible wrt $D_1$. However, $D_2 \cup D_3$ is admissible wrt $D_1$. Hence in this case it would not be desirable to eliminate $D_2$ from our set of admissible designs.

In some cases it is possible to identify those generator designs that always yield inadmissible or equivalent designs when unions of them are taken with other designs. This can be done using the concept of strong (S-) inadmissibility and combination (C-) inadmissibility, which are defined as follows.

**Definition 5.4:** If for given $(p, k)$ we have two BTIB designs $D_1$ and $D_2$ (not necessarily generator designs), we say that $D_2$ is S-inadmissible wrt $D_1$ if $D_2$ is inadmissible wrt $D_1$, and if for any arbitrary BTIB design $D_3$ we have that $D_2 \cup D_3$ is inadmissible wrt $D_1 \cup D_3$.

Thus S-inadmissibility implies inadmissibility but not vice versa. An easily verifiable sufficient condition for S-inadmissibility of a BTIB design is given in the following theorem.

**Theorem 5.2:** A sufficient condition for a BTIB design $D_2$ to be S-inadmissible wrt a BTIB design $D_1$ with the same $(p, k)$ is that $b_1 \leq b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} \geq \lambda_1^{(2)}$ with at least one inequality being strict; here $(\lambda_0^{(1)}, \lambda_1^{(1)}, b_1)$ is associated with $D_i$ ($i = 1, 2$).

**Proof:** If $\lambda_1^{(1)} = \lambda_1^{(2)}$ and $b_1 < b_2$, then the result is obvious. If $\lambda_1^{(1)} > \lambda_1^{(2)}$, then the result follows from the fact that for fixed $\lambda_0$ the parameter $\tau^2$ of (4.3) is a decreasing function of $\lambda_1$, and $\rho$ of (4.4) is an increasing function of $\lambda_1$.

As an illustration of the use of Theorem 5.1 we note that for $p = 3$, $k = 3$ the designs $D_0$ and $D_1$ of (3.5) have $\lambda_0^{(0)} = 2$, $\lambda_0^{(1)} = 1$, $b_0 = 3$, and $\lambda_1^{(1)} = 2$, $\lambda_1^{(0)} = 0$, $b_1 = 3$, respectively. Hence, $D_1$ is S-inadmissible wrt $D_0$.

There are certain BTIB designs that are not S-inadmissible as defined before but that can be deleted without any loss from our set of generator designs. The identification of such designs requires the concept of combination (C-) inadmissibility, which is more general than S-inadmissibility.

**Definition 5.5:** Suppose that for given $(p, k)$ we have $n \geq 2$ generator designs $D_i (1 \leq i \leq n)$, which are nonequivalent, and none of which is S-inadmissible. The designs $D, D', D''$ described later are constructed from the designs in the set $\{D_i (1 \leq i \leq n)\}$. Consider a BTIB design $D$, and an arbitrary BTIB design $D'$. If for every $D'$ there exists a BTIB design $D''$ such that $D \cup D'$ is either inadmissible wrt $D''$ or equivalent to $D''$, and $D$ is not included in $D''$, then we say that $D$ is C-inadmissible wrt the set $\{D_i (1 \leq i \leq n)\}$.

**Remark 5.3:** If a design $D_i$ is a member of a set $\{D_1, \ldots, D_a\}$ that contains only generator designs that are nonequivalent and none of which is S-inadmissible, and if $D_i$ is C-inadmissible wrt that set, then $D_i$ can be deleted from the set, and we shall say that $D_i$ is C-inadmissible wrt the set $\{D_j (j \neq i, 1 \leq i \leq n)\}$.

**Remark 5.4:** We point out some critical distinctions between S-inadmissible and C-inadmissible designs. First, we note that Theorem 5.2 provides an easy way of checking whether certain BTIB designs are S-inadmissible wrt certain other BTIB designs for that $(p, k)$. On the other hand, in order to identify a C-inadmissible design it is necessary to examine every different elementary combination of generator designs, and in some cases higher order combinations, and show that each such combination leads to inadmissible or equivalent designs. Examples of C-inadmissible designs and proofs of their C-inadmissibility are given in Bechhofer and Tamhane (1979b, 1980a,b), and Bechhofer, Tamhane, and Mykytyn (1980).

We also note that unions of certain designs with C-inadmissible designs may be admissible, but each such admissible design is equivalent to some other design not involving that C-inadmissible design. Such a possibility cannot arise with an S-inadmissible design.

Finally we point out that if a design is identified as being S-inadmissible under the sufficient condition of Theorem 5.2, then that design can be permanently deleted without loss, even if it is not known whether the set $\{D_i (1 \leq i \leq n)\}$ contains all generator designs for given $(p, k)$. This is in contrast to the situation concerning a C-inadmissible design, which is defined wrt the set $\{D_i (1 \leq i \leq n)\}$. A design can be C-inadmissible wrt $\{D_i (1 \leq i \leq n)\}$, but not so wrt $\{D_i (1 \leq i \leq n + 1)\}$ where this new set contains the $n$ original designs plus one additional one and consists of $n + 1$ designs that are nonequivalent, none of which is S-inadmissible. Thus a C-inadmissible design
cannot be eliminated unless it is known that \( \{D_i\} \) contains all nonequivalent and non-S-inadmissible generator designs for the particular \((p, k)\) of interest. For \( p \geq 2, k = 2 \) and \( p = 3, k = 3 \) these sets are given by (3.4) and \( D_0, D_2 \) of (3.5), respectively. For \((p, k) = (4, 3) \) and \((5, 3) \), and \((p, k) = (4, 4) \) this problem is considered in Bechhofer and Tamhane (1979b) and (1980a), respectively; for these cases we do not know whether we have enumerated all the required generator designs in each set, but we conjecture that we have done so.

We are now led to our final definition.

**Definition 5.6:** If the set \( \{D_1, \ldots, D_n\} \) contains all generator designs for given \((p, k)\), and if \( \{D_0, \ldots, D_m\} \) with \( m \leq n \) is the subset that contains all nonequivalent, non-S-inadmissible and non-C-inadmissible generator designs, then the latter set will be referred to as a **minimal complete class** of generator designs for given \((p, k)\). (This class will not be unique if one or more of the generator designs in the class can be replaced by an equivalent generator design; if this happens, the resulting class will serve equally well in the search for an optimal design.) The designs in the minimal complete class will serve as building blocks for all BTIB designs which will be of interest to us in our search for the optimal design.

We illustrate Definition 5.6 by giving in Table 2 our conjectured minimal complete class of generator designs for \( p = 4, k = 3 \). We have prepared corresponding tables for \((p, k) = (5, 3), (6, 3), (4, 4), (5, 4), \) and \((6, 4)\), and using these tables we have computed catalogs of admissible designs for each \( b \) and also optimal designs for selected \( d/\alpha \) and \( 1 - \alpha \). These will appear elsewhere.

**Note added in proof:** When this paper was in galley proof the authors received a personal communication from William I. Notz of the Department of Statistics, Purdue University. Notz has proved analytically that the five generator designs in Table 2 do not constitute a minimal complete class for \( p = 4, k = 3 \), but that these five along with a sixth generator design that he has constructed \((b = 8, \lambda_0 = 1, \lambda_1 = 3)\) do constitute a minimal complete class; he has also proved that the conjectured minimal complete class of generator designs for \( p = 6, k = 3 \) (given in Bechhofer and Tamhane 1980b) is indeed a minimal complete class. Proofs of these results and other related ones will be given elsewhere.

### 5.3 Relationship to Other Optimality Criteria

Kiefer (along with many others) has studied extensively the problem of optimal design in a series of papers starting with Kiefer (1958). The three main criteria considered by Kiefer are \( D-, A-, \) and \( E\)-optimality, which correspond, respectively, to minimizing the determinant, trace, and the maximum eigen value of the variance-covariance matrix of the BLUE of the parameter vector of interest which in our case is \( (\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)' \). The eigen values (neglecting constant proportionality factors) of the variance-covariance matrix of \( (\tilde{\alpha}_0 - \tilde{\alpha}_1, \ldots, \tilde{\alpha}_0 - \tilde{\alpha}_p)' \) can be obtained from (4.2) through (4.4); they are \( \lambda_0^{-1} \) of multiplicity \( p - 1 \), and \( \lambda_0^{-1} \) of multiplicity 1. Thus for given \((p, k, b)\) the three criteria can be stated as

(a) **D-optimality:** minimize \( \lambda_0^{-1}(\lambda_0 + p\lambda_1)^{-1} \), that is, maximize \( \lambda_0 \);  
(b) **A-optimality:** minimize \( \lambda_0^{-1} + (p - 1)(\lambda_0 + p\lambda_1)^{-1} \);  
(c) **E-optimality:** minimize \( \lambda_0^{-1} \), that is, maximize \( \lambda_0 \).

Although these criteria are simpler than ours, it should be kept in mind that they refer to an *ellipsoidal* joint confidence region for \( (\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)' \). In the case of MCC such a confidence region is of less interest than the "rectangular" confidence regions that we have proposed, and are commonly used. Hence we do not consider the Kiefer criteria further here.

### 6. CONCLUDING REMARKS AND DIRECTIONS OF FUTURE RESEARCH

In this paper we introduced a new general class of incomplete block designs that are appropriate for use in the MCC problem. We refer to these as balanced treatment incomplete block (BTIB) designs. The basic results concerning the structure of such designs are derived, and the properties of the relevant estimates obtained with such designs are given. Admissibility and inadmissibility of these designs are
defined, and these criteria are used to eliminate inferior designs. In the search for optimal designs it suffices to restrict consideration to admissible designs. It is shown how the concepts of $S$-inadmissibility and $C$-inadmissibility can be used to obtain a minimal complete class of generator designs from which catalogs of admissible designs can be constructed.

The combinatorial problem of constructing all BTIB designs for given $(p, k, b)$, and the procedure for choosing an optimal design from such a set, are not solved in the present paper. However, some methods of design construction are given. The aforementioned problems are related in the sense that to solve the optimization part completely one must have constructed most, if not all, generator designs for given $(p, k)$; the problem of determining how many generator designs exist for arbitrary $(p, k)$, and then enumerating them, appears to be a very formidable one. Alternatively, the problem of construction of BTIB designs for arbitrary $(p, k, b)$ can be set up in the manner of Foody and Hedayat (1977, Lemma 4.1), which is suitable for a solution on a computer. However, there is no guarantee that all BTIB designs can be generated in this way. Also, the magnitude of difficulty of our problem is substantially greater than theirs, and therefore a solution by way of this route looks rather remote at this stage.

For $p > 2$, $k = 2$ and $p = 3$, $k = 3$ the situation is very simple since it is necessary to consider essentially only two generator designs for each $(p, k)$ case; these cases are considered in detail in Bechhofer and Tamhane (1979a) and the optimal design is given for a large range of the useful $(p, k, b)$- and $d/a$-values. For $p = 4$, 5, 6 and $k = 3, 4$ it is possible to enumerate most of the generator designs; these cases are considered in Bechhofer and Tamhane (1979b, 1980a, 1980b), Bechhofer, Tamhane, and Mykytyn (1980) along with the associated optimal designs.

7. ACKNOWLEDGMENTS

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8. APPENDIX

Proof of Theorem 3.1. For given $(p, k, b)$ consider an arbitrary design with the incidence matrix $\{r_{ij}\}$. Then it is well known (see Eq. (3.1) of Kiefer 1958) that the information matrix $M = \{m_{ij}\}$ of $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)$ is given by

$$m_{ij} = \begin{cases} r_{i} - \frac{1}{k} \sum_{j=1}^{b} r_{ij} & (i = j) \\ \frac{\lambda_{ij}}{k} & (i \neq j) \end{cases}$$

(A.1)

Note that $M$: $(p+1) \times (p+1)$ is a singular matrix and $\sum_{i=0}^{p} m_{ii} = 0$ for each $i$. We require the information matrix of $U\alpha = (\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)$ where $U = \{u_{ij}\}$: $p \times (p+1)$ is given by

$$u_{ij} = \begin{cases} 1 & (i_1 = 1, i_2 = 1, \ldots, p) \\ -1 & (i_2 = i_1 + 1, i_1 = 1, \ldots, p) \\ 0 & \text{otherwise}. \end{cases}$$

To avoid computing the generalized inverse of $M$, we shall employ the following method given to obtain the desired information matrix. Let $Q$: $p \times (p+1)$ be any matrix the rows of which form $p$ orthogonal contrasts. Then we can write $U$ as $U = PQ$ where $P$: $p \times p$ is a nonsingular matrix. The information matrix of $Q\alpha$ is given by $QM^TQ^t$. Therefore, the variance-covariance matrix (except for the common factor $\sigma^2$) of $U\alpha$ is given by $P(QM^TQ)^{-t}P^t$. Hence the information matrix $M^*$ of $U\alpha$ is given by

$$M^* = [P(QM^TQ)^{-t}P^t]^{-1}$$

$$= [P^{-1}(QM^T)^{-1}]^{-1}$$

$$= V'MV$$

(A.2)

where $V = QP^{-1}$ satisfies $UV = I$, with $I$ being the $p \times p$ identity matrix. It can be easily verified that $V = \{v_{ij}\}$: $(p+1) \times p$ is given by

$$v_{ij} = \begin{cases} 0 & (i_1 = i_2 + 1, i_2 = 1, \ldots, p), \\ 1 & \text{otherwise}. \end{cases}$$

Substituting for $V$ in (A.2), the information matrix $M^* = \{m^*_{ij}\}$ of $U\alpha$ can be written as

$$m^*_{ij} = \sum_{g=0}^{p} \sum_{h=0}^{g} m_{gh} \quad (i_1, i_2 = 1, \ldots, p)$$

(A.3)

where (A.3) follows from the fact that the rows and columns of $M$ sum to zero. For a design to be BTIB, the matrix $M^*$ must be completely symmetric (i.e., all
diagonal elements of $M^*$ must be equal, and all off-
diagonal elements must be equal). Therefore, we have

$$m_{11} = m_{22} = \cdots = m_{pp}$$

and

$$m_{12} = m_{13} = \cdots = m_{p-1,p}.$$ 

Using expression (A.1) for $m_{i1i2}$ in (A.4) implies (3.1). Denoting the common value of the $\lambda_0 (1 \leq i \leq p)$ by $\lambda_0$, and the common value of the $\lambda_{ij} (i_1 \neq i_2; 1 \leq i_1, i_2 \leq p)$ by $\lambda_1$, we see that

$$m_{i1i2} = -\lambda_1/k \quad (i_1 \neq i_2; 1 \leq i_1, i_2 \leq p)$$

and

$$m_{ii} = (\lambda_0 + (p - 1)\lambda_1)/k \quad (1 \leq i \leq p).$$

Thus

$$M^* = \left(\left\{\lambda_0 + p\lambda_1\right\}I - \lambda_1 J\right)/k \quad (A.5)$$

where $J: p \times p$ is a matrix consisting only of 1's. The inverse of $M^*$ is the desired variance-covariance matrix of $\hat{U}$; the expressions for (4.2) and (4.4) are then easily obtained from $(M^*)^{-1}$.

**Derivation of the expressions for the estimates.**

The normal equations for the least squares estimates (BLUE) $\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j$ of $\mu, \alpha_i, \beta_j$, respectively, $(0 \leq i \leq p, 1 \leq j \leq b)$ are

$$N\hat{\mu} + \sum_{i=0}^{p} r_i \hat{\alpha}_i + \sum_{j=1}^{b} \beta_j = G \quad (A.6)$$

$$r_i\hat{\mu} + r_i \hat{\alpha}_i + \sum_{j=1}^{b} r_{ij} \hat{\beta}_j = T_i \quad (0 \leq i \leq p) \quad (A.7)$$

$$k\hat{\mu} + \sum_{i=0}^{p} r_{ij} \hat{\alpha}_i + k \hat{\beta}_j = B_j \quad (1 \leq j \leq b) \quad (A.8)$$

where $T_i$ and $B_j$ are defined in Section 4.1, and $G = \sum_{i=0}^{p} T_i = \sum_{j=1}^{b} B_j$. Substituting

$$\hat{\mu} + \hat{\beta}_j = \left(B_j - \sum_{i=0}^{p} r_{ij} \hat{\alpha}_i\right)/k \quad (A.9)$$

from (A.8) into (A.7), we obtain

$$r_i \hat{\alpha}_i + \frac{1}{k} \sum_{j=1}^{b} r_{ij} B_j - \frac{1}{k} \sum_{k=0}^{p} \hat{\alpha}_k \sum_{j=1}^{b} r_{kj} r_{ij} = T_i. \quad (A.10)$$

Now $\sum_{j=1}^{b} r_{ij} B_j = B^*_i$ and for $h \neq i$ we have that $\sum_{j=1}^{b} r_{ij} r_{ij} = \lambda_0 (\lambda_1)$ if only one of $h$ and $i = 0$ (if both $h$ and $i \neq 0$). From (A.10) for $i = 0$ we obtain

$$r_0 \hat{\alpha}_0 + \frac{B^*_0}{k} - \frac{1}{k} \sum_{j=1}^{b} r_{0j}^2 + \frac{\lambda_0}{k} \sum_{h=1}^{p} \hat{\alpha}_h = T_0. \quad (A.11)$$

Using the fact that $\sum_{j=1}^{b} \hat{\alpha}_j = -\hat{\alpha}_0$ and that $r_0 - 1/k \geq \sum_{j=1}^{b} r_{0j}^2 = m_{00} = p\lambda_0/k$, from (A.11) we obtain

$$\hat{\alpha}_0 = \frac{Q_0}{p+1}\lambda_0. \quad (A.12a)$$

In the same way we obtain

$$\hat{\alpha}_i = \frac{Q_i + (\lambda_0 - \lambda_1)\hat{\alpha}_0}{(\lambda_0 + p\lambda_1)} \quad (1 \leq i \leq p). \quad (A.12b)$$

Combining (A.12a) and (A.12b), we obtain (4.1).

**Derivation of the formula for the adjusted treatment sum of squares in the analysis of variance table (Table 1).** Following the Scheffe (1959) notation, let $S_\Omega$ denote the minimum error sum of squares (SS) under the assumptions $\Omega$ of Section 2, and let $S_\omega$ denote the minimum error SS under $\omega = H \cap \Omega$ where the hypothesis is $H: \alpha_i = 0 \quad (0 \leq i \leq p)$. Then the SS of treatments adjusted for blocks is given by

$$S_{\text{treat(adj.)}} = S_\omega - S_\Omega. \quad (A.13)$$

Now

$$S_\Omega = SS_{\text{error}}$$

$$= \sum_{i=0}^{p} \sum_{j=1}^{b} \sum_{h=1}^{k} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2 I_{ijk} \quad (A.14)$$

where $I_{ijk} = 1$ if the $ith$ treatment is assigned to the $hth$ plot of the $jth$ block $(0 \leq i \leq p, 1 \leq j \leq b, 1 \leq h \leq k)$ and $= 0$ otherwise. Substituting $\hat{\beta}_j$ from (A.9) into (A.14) and expanding and simplifying the resulting square, we obtain

$$S_\Omega = H - \frac{1}{k} \sum_{j=1}^{b} B_j^2 + \frac{p}{k} \sum_{i=0}^{p} r_i \hat{\alpha}_i^2$$

$$- \frac{1}{k} \sum_{j=1}^{b} \left(\sum_{i=0}^{p} r_{ij} \hat{\alpha}_i\right)^2 - Q \quad (A.15)$$

where $H = \sum_{i=0}^{p} \sum_{j=1}^{b} \sum_{h=1}^{k} y_{ijk}^2 L_{ijk}$ and

$$Q = \frac{2}{k} \sum_{i=0}^{p} \hat{\alpha}_i Q_i.$$  

Under $\omega$, we have $\hat{\mu} = G/N$ and $\hat{\beta}_j = B_j/k - \hat{\mu}$; hence

$$S_\omega = H - \frac{1}{k} \sum_{j=1}^{b} B_j^2 \quad (A.16)$$

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Subtracting (A.15) from (A.16), we have from (A.13) that
\[
SS_{\text{treat(adj.)}} = Q + \frac{1}{k} \sum_{j=1}^{b} \left( \sum_{i=0}^{p} r_{ij} \hat{a}_{i} \right)^{2} - \sum_{i=0}^{p} r_{i} \hat{a}_{i}^{2} = Q - \sum_{i=0}^{p} \left( r_{i} - \frac{1}{k} \sum_{j=1}^{b} r_{ij}^{2} \right) \hat{a}_{i}^{2} + \frac{2}{k} \sum_{i=1}^{p} \sum_{i_{1}=i+1}^{p} \hat{a}_{i_{1}} \hat{a}_{i_{2}} \hat{a}_{i_{12}} = Q - \frac{p \lambda_{0} \hat{a}_{0}^{2}}{k} - \frac{\{ \lambda_{0} + (p - 1) \lambda_{1} \}}{k} \sum_{i=1}^{p} \hat{a}_{i}^{2} + \frac{2 \lambda_{0} \hat{a}_{0}}{k} \sum_{i=1}^{p} \hat{a}_{i} + \frac{2 \lambda_{1}}{k} \left( \sum_{i=1}^{p} \hat{a}_{i} \right)^{2} + \frac{(p + 2) \lambda_{0} \hat{a}_{0}^{2}}{k} = Q - \frac{(p + 2) \lambda_{0} \hat{a}_{0}^{2}}{k} - \frac{(\lambda_{0} + p \lambda_{1})}{k} \sum_{i=1}^{p} \hat{a}_{i}^{2} + \frac{\lambda_{1}}{k} \left( \sum_{i=1}^{p} \hat{a}_{i} \right)^{2} = Q - \frac{(p + 2) \lambda_{0} \hat{a}_{0}^{2}}{k} - \frac{(\lambda_{0} + p \lambda_{1})}{k} \sum_{i=1}^{p} \hat{a}_{i}^{2}. \quad (A.17)
\]

In the preceding equations we have used the relations
\[
r_{i} - \frac{1}{k} \sum_{j=1}^{b} r_{ij}^{2} = m_{ii} \quad (\text{from (A.1)})
\]
where
\[
m_{00} = p \lambda_{0} / k, \quad m_{ii} = \left\{ \lambda_{0} + (p - 1) \lambda_{1} / k \right\} \quad (1 \leq i \leq p),
\]
and \( \sum_{i=1}^{p} \hat{a}_{i} = - \hat{a}_{0} \). Substituting in (A.17) for \( \hat{a}_{0} \) from (A.12a) and for \( \hat{a}_{i} \) (1 \( \leq i \leq p \)) from (A.12b) and after some tedious algebra, we obtain the expression for \( SS_{\text{treat(adj.)}} \) as given in Table 1. The other SS expressions and the df in the table are obtained in a straightforward way.

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REFERENCES


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