# Newsvendor-Type Models with Decision-Dependent Uncertainty

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April 15, 2009

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**Abstract**

Models for decision-making under uncertainty use probability distributions to represent variables whose values are unknown when the decisions are to be made. Often the distributions are estimated with observed data. Sometimes these variables depend on the decisions but the dependence is ignored in the decision maker's model, that is, the decision maker models these variables as having an exogenous probability distribution independent of the decisions, whereas the probability distribution of the variables actually depend on the decisions. It has been shown in the context of revenue management problems that such modeling error can lead to systematic deterioration of decisions as the decision maker attempts to refine the estimates with observed data. Many questions remain to be addressed. Motivated by the revenue management, newsvendor, and a number of other problems, we consider a setting in which the optimal decision for the decision maker's model is given by a particular quantile of the estimated distribution, and the empirical distribution is used as estimator. We give conditions under which the estimation and control process converges, and show that although in the limit the decision maker's model appears to be consistent with the observed data, the modeling error can cause the limit decisions to be arbitrarily bad.

## 1 Introduction

### 1.1 Quantiles as Solutions

Various decision problems have optimal solutions that are given by a particular quantile of a probability distribution. The following are examples of such problems:

**Classical Newsvendor Problem** A seller has to choose the amount $q$ of inventory to obtain at the beginning of a selling season. The decision is made only once — there is no opportunity to replenish inventory during the selling season. The demand $D$ during the selling season is a nonnegative random variable with probability distribution $F$. The cost of obtaining inventory is $c$ per unit. The product is sold at a given price $r$ per unit during the selling season, and at the end of the season unsold inventory has a salvage value of $v$ per unit. The seller wants to choose the amount $q$ of inventory to maximize the objective function

$$z(q) \quad := \quad r \int_0^\infty \min\{q, \xi\} \, dF(\xi) + v \int_0^q (q - \xi) \, dF(\xi) - cq. \tag{1}$$

If $v < c < r$, then any $q^*$ that satisfies

$$F(x) \quad \leq \quad \frac{r-c}{r-v} \quad \text{for all } x < q^* \qquad \text{and} \qquad F(x) \quad \geq \quad \frac{r-c}{r-v} \quad \text{for all } x > q^*$$

is an optimal amount of inventory to obtain at the beginning of the selling season. That is, the set of optimal solutions is given by the set of $\gamma$-*quantiles* of the distribution $F$, which can be written as

$$\Psi_\gamma(F) \quad := \quad \{q \in \mathbb{R} \; : \; F(x) \leq \gamma \text{ for all } x < q \text{ and } F(x) \geq \gamma \text{ for all } x > q\}, \tag{2}$$

where $\gamma = (r-c)/(r-v)$. Note that $\Psi_\gamma(F)$ is a nonempty closed interval for all $\gamma \in (0,1)$.

**Revenue Management Problem** (Littlewood, 1972) A seller has $Q$ units of inventory to sell. Each unit is sold for a price of $r_1$ or $r_2$, with $r_1 > r_2$. During the first phase of the selling season, customers arrive who will buy the product at price $r_2$, but not at price $r_1$. During the second phase of the selling season, customers arrive who will buy the product at price $r_1$. The demand $D_2$ for product at price $r_2$ is a random variable with distribution $F_2$, and the demand $D_1$ for product at price $r_1$ is a random variable with distribution $F_1$, independent of anything that happened during the first phase of the selling season. The seller wants to choose the amount $\ell$ of inventory to reserve for the high price customers, that is, the amount $Q - \ell$ of inventory to be made available at the low price $r_2$, to maximize the objective function

$$\begin{aligned} z(\ell) \quad := \quad & r_1 \int_0^{Q-\ell} \int_0^\infty \min\{Q-y, x\} \, dF_1(x) \, dF_2(y) + r_1 \int_0^\infty \min\{\ell, x\} \, dF_1(x) \, [1 - F_2(Q-\ell)] \\ & + r_2 \int_0^\infty \min\{Q-\ell, y\} \, dF_2(y). \end{aligned}$$

Note that all inventory that is not sold during the first phase is available in the second phase. The set of optimal solutions is given by $\Psi_\gamma(F_1)$, where $\gamma = 1 - r_2/r_1$.

**Call Center Staffing Problem** (Harrison and Zeevi, 2005; Bassamboo and Zeevi, 2009) A manager has to choose the number $b$ of staff members to employ at a call center. Calls arrive at the call center at an arrival rate which has distribution $F$, that is, $F(\lambda)$ is the fraction of time that the arrival rate is less than or equal to $\lambda$. Each staff member serves customers at a rate $\mu$. Waiting customers abandon the system at rate $\gamma$ per waiting customer. Thus, while there are $Q$ customers waiting, the total abandonment rate is $\gamma Q$. The manager pays a cost of $c$ per staff member per unit time, and a penalty of $p$ per customer who abandons the system. The manager wants to choose the number $b$ of staff members to minimize the average cost per unit time. Consider a stochastic fluid version of the process described above (the adjective "stochastic" indicates that the call arrival rate $\lambda$ still has distribution $F$). Then, if $\lambda < \mu b$, all the customers are served, and there are no abandonments, so that the rate at which cost is incurred is $cb$. If $\lambda \geq \mu b$, then customers abandon the system at total rate $\gamma Q = \lambda - \mu b$, so that the rate at which cost is incurred is $cb + p(\lambda - \mu b)$. Thus the objective for the stochastic fluid model is to minimize

$$z(b) \quad := \quad cb + p \int_{\mu b}^{\infty} (\lambda - \mu b) \, dF(\lambda).$$

The set of optimal solutions is given by $\Psi_\gamma(F)/\mu$, where $\gamma = (p\mu - c)/(p\mu)$.

**Operating Room Booking Problem** (Olivares et al., 2008) A decision maker has to choose the amount $q$ of time to book in an operating room (OR) for a surgery case. It is not known in advance exactly how long the surgical procedure will take, but the distribution $F$ of the duration $D$ of similar procedures is known. There is a booking cost $c$ per unit time for the OR, even if the OR is not occupied for the entire booked time $q$. If the OR is occupied for a time longer than $q$, then there is a penalty $p > c$ per unit time that the OR is occupied beyond the booked time, including the overtime paid to the OR staff and the penalty levied by the hospital for using the OR beyond the booked time. The decision maker wants to choose the amount $q$ of booked time to minimize

$$z(q) \quad := \quad \int_0^{\infty} \left[ C_o \max\{q - \xi, 0\} + C_u \max\{\xi - q, 0\} \right] \, dF(\xi)$$

where $C_o := c$ denotes the overage cost and $C_u := p - c$ denotes the underage cost. Then it follows that the set of optimal solutions is given by $\Psi_\gamma(F)$, where $\gamma = C_u/(C_o + C_u) = (p - c)/p$.

## 1.2 Estimation and Modeling Error

In applications, the underlying distribution $F$ is not known, and is usually estimated with observed data. Suppose that $k$ observations have been collected. Then, it is natural to use the current estimate $\hat{H}^k$ as given input, and choose a decision $q^k \in \Psi_\gamma(\hat{H}^k)$ that is optimal for the estimate. Often the situation is part of repeating operations: An estimate is constructed, a decision is made, new data are observed, the estimate is updated, a decision is made, and so on.

Another feature common in applications is that the model used by the decision maker is often incorrect or misspecified, that is, there does not exist a value of the estimate such that the resulting model of the decision maker is correct. In many cases the decision maker is not aware that the model is structurally incorrect. For example, as will be demonstrated below, when the decision maker uses a learning procedure to fit the incorrect model to observed data, the incorrect model with the fitted quantities often appears to be consistent with the observed data, so that the modeling error can easily go unnoticed, that is, the decision maker may be lured into a false sense of security that the model employed is correct. In some cases the decision maker is aware that the model is structurally incorrect, but does not know a model with correct structural form for the problem under consideration. Even if the decision maker would know a correct model, he may choose to use a simpler model to make the calculations tractable.

In the setting considered in this paper, not only is the estimate $\hat{H}^k$ not equal to some unknown true $F$, but there may not exist an exogenous distribution $F$ such that the model gives a true description of the decision problem under consideration (although, as will be shown, there may exist an $F$ that is consistent with the observed process). One situation in which such a phenomenon occurs is when the demand for the retailer's product may depend on the initial amount $q$ of inventory. As an example, consider a movie rental store. Many customers browse the shelves to decide which movie to rent. Such customers may be more motivated to rent a movie which has many empty cases displayed on the shelves, indicating that many copies of the movie are checked out, and thus that the movie is popular. As another example, consider an artist who produces a work of which several copies are going to be made and sold. Each copy indicates the number of the copy and the total number of copies, and thus the total number of copies have to be chosen and fixed in advance of any sales. The scarcer the work is, the higher the perceived value of each copy; equivalently, the smaller the total number of copies, the higher the demand at each fixed price. Giri et al. (1996) and Benkherouf et al. (2001) consider the so-called inventory model with stock-dependent demand, in which the demand rates depend on the inventory levels. Balakrishnan et al. (2004) consider a situation in which high inventory stimulates demand — for example, in the superstore, tall stacks of a product impact its visibility, which can lead more customers to buy the displayed products. They present various reasons for evidence of the dependence between the inventory level and customer demand. Dana and Petruzzi (2001) study pricing and inventory policy where demand depends on both price and inventory level.

In the situations discussed above, there is not a single exogenous distribution $F$ of the demand for the newsvendor's product independent of the chosen inventory level $q$, and thus the newsvendor model is an incorrect model. Similarly, the revenue management model described earlier may be incorrect, because the distribution of the demand for product at the high price may depend on the amount $Q - \ell$ of product made available at the low price. Also, the call center staffing model described above may be incorrect, because the call arrival rate (or the call reneging rate) may depend on the waiting times, and thus on the number of servers chosen. In the operating room booking problem, the surgeon may work faster if he knows that the operating room has been booked for a short amount of time only, and thus the distribution of time in the operating room may depend on the booked time.

## 1.3 Iterative Decision Process

Taking into account the possibility of modeling error, next we give a general description of an *iterative decision process* (IDP) involving estimation, decision making, and data collection. Let $\hat{H}^k$ denote the estimate used in iteration $k$, which may be a real number, a real vector, a probability distribution on the real line, a continuous function on the real line, or any other quantity of interest. For any estimate $\hat{H}$, let $\Psi(\hat{H})$ denote the set of optimal decisions for the (possibly incorrect) model of the decision maker using estimate $\hat{H}$ as input (e.g., the set $\Psi_\gamma(\hat{H})$ of $\gamma$-quantiles of $\hat{H}$ for some $\gamma \in (0, 1)$). Let $q^k \in \Psi(\hat{H}^k)$ denote the decision in iteration $k$, which may also be any quantity of interest. Let $\mathcal{F}^k$ denote the $\sigma$-algebra generated by the history $q^0, X^1, q^1, \ldots, X^k, q^k$ up to iteration $k$. Let $G(q^k, \cdot)$ denote the conditional probability distribution of the next observed data $X^{k+1}$, given $\mathcal{F}^k$, that is, the conditional probability distribution of the observed data depends on the history of the process only through the most recent decision. For any finite sequence $X^1, \ldots, X^k$ of observed data, let $\phi(X^1, \ldots, X^k)$ denote the resulting estimate. Sometimes $\phi$ is referred to as the forecasting method. Next we summarize the IDP.

**Iterative decision process (IDP)**

Initialization: Select initial estimate $\hat{H}^0$.

For $k = 0, 1, 2, \ldots$, repeat the following steps:

Step 1: Choose a decision $q^k \in \Psi(\hat{H}^k)$ optimal for the model.

Step 2: Observe $X^{k+1}$ which has distribution $G(q^k, \cdot)$.

Step 3: Compute updated estimate $\hat{H}^{k+1} = \phi(X^1, \ldots, X^{k+1})$.

## 1.4 Consequences of Modeling Error

Modeling error can result in poor and even systematically deteriorating performance of the system as the decision maker attempts to improve the fitted model with observed data. Cachon and Kok (2007) analyze a situation in which the newsvendor model is used with an incorrect assumption of a fixed salvage value, while the salvage value actually depends on the number of salvaged items. It is shown that estimates for the salvage value exist that are consistent with the observed data but when such estimates are used in the newsvendor model the resulting solutions are not optimal. Cachon and Kok (2007) do not analyze learning and convergence for their setting, but rather propose an estimation method that takes the dependency into account. Cooper et al. (2006) consider the behavior of the dynamical process involving data collection, estimation, and decision making in the revenue management setting in which the revenue manager uses an incorrect model. It is shown that the decisions may deteriorate systematically as the revenue manager attempts to improve the estimates with observed data, leading to a phenomenon that is called the spiral-down effect.

Many alternative approaches besides the iterative decision process described above exist. Approaches to optimization when appropriate probability distributions are not known include robust optimization and min-max approaches. Some approaches separate learning and optimization. For example, Besbes and Zeevi (2006) propose an approach for network revenue management problems, where in the first phase the learning procedure approximates the demand function using a chosen set of experimental prices. Thereafter, in the second phase, the optimization procedure follows using the estimate obtained with the learning procedure, i.e., the process does not alternate between learning and optimization. However, it can be costly to have such an experimentation phase, and in such situations it would be natural for the revenue manager to alternate between estimation and optimization as in the iterative decision process. Other approaches combine estimation and optimization. For example, Liyanage and Shanthikumar (2005) propose a method, called "operational statistics", that integrates the estimation and optimization into one task. The expected profit given by the inventory decision operational statistic is higher than the expected profit given by the traditional approach. In addition, the procedure is consistent in the sense that the decision converges to the true optimal solution. However, it requires that the observed demands be independent and identically distributed, which does not hold in the setting studied in this paper.

### 1.5 Goals of the Paper

In this paper, we study the dynamic behavior of the estimates and decisions for problems with quantile solutions where the unknown distributions of observed data depend on the decisions, but this dependence is not taken into account correctly in the models used for decision making. The decision maker uses an iterative decision process for repeated estimation and decision making. We establish sufficient conditions for convergence of the estimated distribution, the decision, and the corresponding objective value. Furthermore, we show that if the decision maker uses an erroneous model, then the decisions resulting from the iterative decision process can be arbitrarily bad compared with the solution of the correct model.

This paper complements the work started in Cooper et al. (2006), who studied some consequences of modeling error for the revenue management problem described in Section 1.1. Convergence results were provided for some specific forecasting techniques, but, as pointed out in the conclusion of that paper, many questions remain unanswered, such as the behavior of the dynamic process if the empirical distribution is used to generate forecasts. This paper addresses that gap. In addition, we provide analytical results comparing the limit point resulting from the iterative decision process with the solution of the correct model under some conditions.

The remainder of the paper is organized as follows. In Section 2 we discuss some issues related to convergence of quantiles, which lead to the more general notion of stability of stochastic optimization problems. In Section 3 we investigate the asymptotic behavior of the IDP. Section 4 provides numerical experiments that verify our analytical convergence results in the newsvendor framework. Section 5 provides comparisons between the true optimal decision and the decision obtained under modeling error, and between the corresponding objective function values. We offer concluding remarks and discussions in Section 6.

## 2 Stability Issues

### 2.1 Convergence of Quantiles

Some natural questions are raised when the IDP discussed in Section 1.3 is considered, such as whether the sequences of estimates and decisions converge, and if so, how the limit decisions compare with the optimal solution for the correct model. Several difficulties arise in this context. First, note that the sequence $X^1, X^2, \ldots$ of observed data is neither an independent sequence nor are the observations identically distributed (unless very specific initial conditions are chosen). Consider the newsvendor setting, in which the estimate $\hat{H}$ is a probability distribution on $\mathbb{R}$, and $\Psi_\gamma(\hat{H}) \neq \varnothing$, for a specific value of $\gamma \in (0,1)$, is the set of decisions that are optimal for the model used by the decision maker. Suppose that $\{G(q^k, \cdot)\}$ converges, say in the sense of weak convergence, to a limit distribution, say $G^\infty$, as $k \to \infty$. If $\phi$ is a "good" forecasting method, then $\{\hat{H}^k\}$ also weakly converges to $G^\infty$ as $k \to \infty$. (For example, it was shown in Cooper et al. (2006) that the empirical distribution is a good forecasting method in this sense.)

However, the following example shows that it may not follow that the distance between $\Psi_\gamma(G(q^k, \cdot))$ and $\Psi_\gamma(\hat{H}^k)$ becomes small, even in a weak sense. For all $k$ such that $\max\{1/\gamma, 1/(1-\gamma)\} < k \leq \infty$, let $G^k(\cdot)$ and $\hat{H}^k(\cdot)$ be distributions with support on $\{0,1\}$ that assign masses equal to $\gamma + 1/k$ and $\gamma - 1/k$ respectively to 0. Then for all $k > \max\{1/\gamma, 1/(1-\gamma)\}$, $\sup\{|G^\infty(x) - G^k(x)| : x \in \mathbb{R}\} = \sup\{|G^\infty(x) - \hat{H}^k(x)| : x \in \mathbb{R}\} = 1/k \to 0$ as $k \to \infty$, which implies that both $\{G^k\}$ and $\{\hat{H}^k\}$ converge (uniformly and thus also weakly) to $G^\infty$ as $k \to \infty$. However, $\Psi_\gamma(G^k) = \{0\}$ and $\Psi_\gamma(\hat{H}^k) = \{1\}$ for all $k$, and thus the distance between $\Psi_\gamma(G^k)$ and $\Psi_\gamma(\hat{H}^k)$ does not become small. In other words, convergence of forecasts does not imply convergence of decisions.

In some special cases, stronger results hold. For example, suppose that $\{G^k\}$ and $\{\hat{H}^k\}$ converge pointwise on a dense set to $G^\infty$ as $k \to \infty$, and suppose that $\Psi_\gamma(G^\infty)$ is a singleton, say $\Psi_\gamma(G^\infty) = \{x^\infty\}$. Then, it is possible to show that for all $k$ sufficiently large, the Hausdorff distance[1] between $\Psi_\gamma(G^k)$ and $\Psi_\gamma(\hat{H}^k)$ is small, and both $\Psi_\gamma(G^k)$ and $\Psi_\gamma(\hat{H}^k)$ are close to $\Psi_\gamma(G^\infty) = \{x^\infty\}$.

As another special case, suppose that the decision maker's model is correct, that is, although the true distribution $G$ is unknown to the decision maker, it does not depend on the decision $q$. Then the sequence $\{X^k\}$ of observed quantities is an i.i.d. sequence, with distribution $G$. If the decision maker uses a good forecasting method, such as the empirical distribution, to estimate $G$, then the sequence $\{\hat{H}^k\}$ converges weakly to $G$ as $k \to \infty$ and hence, by Lemma 4 in Cooper et al. (2006), it follows that any choice $q^k \in \Psi_\gamma(\hat{H}^k)$ satisfies $q^k \to \Psi_\gamma(G)$ as $k \to \infty$ (however, $\Psi_\gamma(\hat{H}^k)$ may be much smaller than $\Psi_\gamma(G)$ even for large $k$).

## 2.2 General Stability Concepts

The discussion in Section 2.1 suggests the need to impose some regularity conditions in order to ensure convergence of the decisions when the estimates converge. In the stochastic optimization literature, such a property is known as *stability*. In what follows we briefly review these ideas, and then apply them to the newsvendor setting.

[1] The Hausdorff distance $d_{\mathrm{H}}(X,Y)$ between two subsets $X$ and $Y$ of a metric space with metric $d$ is defined by $d_{\mathrm{H}}(X,Y) := \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}$. In words, $d_{\mathrm{H}}(X,Y)$ is small if all points in $X$ are close to $Y$ and all points in $Y$ are close to $X$.

Consider general stochastic optimization problems of the form:

$$\max_{q \in \mathcal{Q}} \mathbb{E}_P[R(q, \xi)] \tag{3}$$

where $\mathcal{Q}$ is a subset of $\mathbb{R}^n$, $\xi$ is a real-valued random variable with distribution $P$, and $R : \mathcal{Q} \times \mathbb{R} \mapsto \mathbb{R}$. Consider the solution set of (3) as a function of the probability distribution $P$ of $\xi$, which for the moment we assume does not depend on $q$. Let $\mathbb{D}$ denote the set of distribution functions on $\mathbb{R}$, let $2^{\mathcal{Q}}$ denote the collection of subsets of $\mathcal{Q}$, and let $\Psi : \mathbb{D} \mapsto 2^{\mathcal{Q}}$ be given by

$$\Psi(P) \quad := \quad \arg\max_{q \in \mathcal{Q}} \left\{ \mathbb{E}_P[R(q, \xi)] \; = \; \int_{\mathbb{R}} R(q, \xi) dP(\xi) \right\}. \tag{4}$$

Solving (3) directly may be hard because the expectation may not be easily computed. In such cases, we can approximate problem (3) with

$$\max_{q \in \mathcal{Q}} \int_{\mathbb{R}} R(q, \xi) dP_N(\xi),$$

where $P_N$ is an approximation of $P$. For example, a sample average approach (i.e., solve the problem $\max_{q \in \mathcal{Q}} \frac{1}{N} \sum_{j=1}^{N} R(q, \xi_j)$ where $\xi_1, \ldots, \xi_N$ is an i.i.d. sample from $P$) can be used. In this case, $P_N$ is the empirical distribution corresponding to the sample. The resulting solution set — henceforth called the *empirical decision* — can be represented as $\Psi(P_N)$.

The following is an example of an often used stability condition for problem (3). For any $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, let $d_{\mathcal{S}}(x, S) := \inf_{y \in S} \|x - y\|$ denote the distance between $x$ and $S$, and let $d$ denote a metric on $\mathbb{D}$. Consider a fixed $P \in \mathbb{D}$. Then the stability condition is that there exist $\kappa, \delta > 0$ such that for all $F \in \mathbb{D}$ with $d(F, P) < \delta$ and for all $q_F \in \Psi(F)$ it holds that

$$d_{\mathcal{S}}(q_F, \Psi(P)) \quad \leq \quad \kappa d(F, P). \tag{5}$$

More generally, stability refers to continuity properties of the optimal value function and the solution mapping $\Psi$ when both are regarded as mappings on a certain set of probability measures (see Rachev and Römisch 2002). The stability condition (5) is a specific kind of those addressed in Rachev and Römisch (2002). Stability in stochastic optimization can also be defined in other ways (Römisch 2003). When $F$ in (5) is chosen as a forecasting distribution obtained from observations from $P$, the stability condition (5) provides a consistent decision, in the sense that the empirical decisions converge to the set of true optimal decisions as long as the forecasting method is good — in other words, as long as the approximating measure $P_N$ converges to $P$ as we collect more data. Moreover, the distance between the empirical decisions and the set of optimal decisions can be quantified in terms of the distance between $P_N$ and $P$, as long as the constant $\kappa$ is known.

### 2.3 Stability Conditions for the Newsvendor Model

In this section we discuss stability issues for the newsvendor model without modeling error. The choice of metric on $\mathbb{D}$ is important when studying the stability condition (5) as well as when using such a condition to show convergence of decisions. Consider the first order Wasserstein metric, defined as

$$d_{\mathcal{W}_1}(H,F) \quad := \quad \int_{(0,1)} \left|H^{-1}(u) - F^{-1}(u)\right| du \quad = \quad \int_{\mathbb{R}} |H(x) - F(x)|\, dx \tag{6}$$

where $F^{-1}(u) := \min\{x \,:\, F(x) \geq u\}$ denotes the smallest $u$-quantile of $F$.

Proposition 1 below presents the stability conditions under this metric. The proposition gives an explicit value for $\kappa$ in (5), which will be useful in our analysis. Note that the result is valid for discrete distributions, which is the case we shall consider for the remainder of the paper. First we state Lemma 1, the proof of which is given in the Online Appendix.

**Lemma 1** *Consider problem* (3)*, and the solution mapping* $\Psi$ *defined in* (4)*. Suppose that there exists an* $L > 0$ *such that*

$$\left|R(q,\xi) - R(q,\tilde{\xi})\right| \quad \leq \quad L \left\|\xi - \tilde{\xi}\right\|$$

*for all* $q, \xi, \tilde{\xi} \in \mathbb{R}$*. Then, for any distributions* $P$ *and* $F$ *on* $\mathbb{R}$*, and any* $\tilde{q} \in \Psi(F)$*, it holds that*

$$d(\tilde{q}, \Psi(P)) \quad \leq \quad \overline{\phi}_P^{-1}(2L d_{\mathcal{W}_1}(P,F))$$

*where* $\overline{\phi}_P^{-1}(y) := \sup\{\tau \,:\, \phi_P(\tau) \leq y\}$*.*

**Proposition 1** *Consider the newsvendor problem* (1)*. Assume that the distribution* $F$ *has support on the integers, so that* $F(x) = \sum_{j=0}^{\lfloor x \rfloor} \pi_j$ *where* $\pi_j := F(j) - \lim_{x \uparrow j} F(x)$*, and let* $H \in \mathbb{D}$ *be any distribution on* $\mathbb{R}$*. Let* $\Psi_\gamma(F) = [\underline{q^*}, \overline{q^*}]$*, let* $L := r - v$*, and let* $A_F := \min\{(r-c) - (r-v)F(\underline{q^*} - 1),\ (r-v)F(\overline{q^*}) - (r-c)\}$*. Then, for any* $\tilde{q} \in \Psi_\gamma(H)$ *it holds that*

$$d(\tilde{q}, \Psi_\gamma(F)) \quad \leq \quad \frac{2L}{A_F} d_{\mathcal{W}_1}(H,F). \tag{7}$$

**Proof**

We follow the analysis in Römisch (2003). Let $\vartheta(F)$ denote the optimal objective value of (1). Consider the growth function $\phi_F : \mathbb{R}_+ \mapsto \mathbb{R}_+$ given by

$$\phi_F(\tau) \quad := \quad \inf\left\{\vartheta(F) - \mathbb{E}_F[R(q,\xi)] \;:\; d(q, \Psi_\gamma(F)) \geq \tau,\ q \in \mathbb{R}\right\}.$$

Lemma 1 shows that if there exists an $\hat{L}$ such that the function $R(q,\xi)$ satisfies

$$\left|R(q,\xi)-R(q,\tilde{\xi})\right| \quad\leq\quad \hat{L}|\xi-\tilde{\xi}| \tag{8}$$

for all $q,\xi,\tilde{\xi}\in\mathbb{R}$, then for any $H\in\mathbb{D}$, and any $\tilde{q}\in\Psi_\gamma(H)$, it holds that

$$d(\tilde{q},\Psi_\gamma(F)) \quad\leq\quad \overline{\phi}_F^{-1}(2\hat{L}d_{\mathcal{W}_1}(H,F)), \tag{9}$$

where $\overline{\phi}_F^{-1}(y) := \sup\{\tau\,:\,\phi_F(\tau)\leq y\}$. Therefore, it suffices to show that (i) the function $R(q,\xi)$ satisfies (8) with $\hat{L}=r-v$, and (ii) $\overline{\phi}_F^{-1}(\eta)\leq\eta/A_F$ for any $\eta\geq 0$.

Recall that the objective function is given by

$$R(q,\xi) \;=\; r\min\{q,\xi\}+v\max\{q-\xi,0\}-cq \;=\; (r-v)\min\{q,\xi\}-(c-v)q. \tag{10}$$

Clearly, $R$ is concave in both in $q$ and $\xi$ and satisfies (8) with $\hat{L}=L:=r-v$. Moreover,

$$\begin{aligned}\mathbb{E}_F[R(q,\xi)] &= (r-v)\sum_{j=0}^{\lfloor q\rfloor} j\pi_j+(r-v)q\sum_{j=\lfloor q\rfloor+1}^{\infty}\pi_j-(c-v)q\\ &= (r-v)\sum_{j=0}^{\lfloor q\rfloor} j\pi_j-(r-v)q\sum_{j=0}^{\lfloor q\rfloor}\pi_j+(r-c)q.\end{aligned}$$

Note that the function $\mathbb{E}_F[R(\cdot,\xi)]$ is concave piecewise linear. Its subdifferential $\partial\mathbb{E}_F[R(q,\xi)]$ is given by

$$\partial\mathbb{E}_F[R(q,\xi)] \quad=\quad \left[(r-c)-(r-v)F(q),\ (r-c)-(r-v)\lim_{x\uparrow q}F(x)\right]. \tag{11}$$

Concavity of the objective function implies that an optimal solution $q^*$ must satisfy $0\in\partial\mathbb{E}_F[R(q^*,\xi)]$, and thus

$$\lim_{x\uparrow q^*}F(x) \quad\leq\quad \frac{r-c}{r-v} \quad=\quad \gamma \qquad\text{and}\qquad F(q^*) \quad\geq\quad \frac{r-c}{r-v} \quad=\quad \gamma,$$

i.e., $q^*\in\Psi_\gamma(F)=[\underline{q^*},\overline{q^*}]$. Since $F$ has support on the integers, it follows that $\underline{q^*}$ and $\overline{q^*}$ are integers, and the objective function $\mathbb{E}_F[R(\cdot,\xi)]$ changes linearly between $\underline{q^*}-1$ and $\underline{q^*}$ and also between $\overline{q^*}$ and $\overline{q^*}+1$, with slopes $(r-c)-(r-v)\lim_{x\uparrow\underline{q^*}}F(x)=(r-c)-(r-v)F(\underline{q^*}-1)>0$ and $(r-c)-(r-v)F(\overline{q^*})<0$ respectively. Thus, for $\tau\in[0,1]$, $\phi_F(\tau)$ is given by

$$\phi_F(\tau) \quad=\quad \tau\,\min\left\{(r-c)-(r-v)F(\underline{q^*}-1),\ (r-v)F(\overline{q^*})-(r-c)\right\}.$$

Let

$$A_F \quad:=\quad \min\left\{(r-c)-(r-v)F(\underline{q^*}-1),\ (r-v)F(\overline{q^*})-(r-c)\right\} \quad>\quad 0.$$

Then $\phi_F(\tau) = A_F\tau$ for $\tau \in [0,1]$. Moreover, concavity of $R$ implies that $\phi_F(\tau) \geq A_F\tau$ for $\tau \geq 1$. Thus, $\overline{\phi}_F^{-1}(\eta) \leq \eta/A_F$ for any $\eta \geq 0$. □

**Remark:** When $d_{\mathcal{W}_1}(H,F) < \delta_F := A_F/(2L)$, then inequality (7) actually yields a stronger result. Suppose that both $H$ and $F$ has support on the integers. Then $\Psi_\gamma(H)$ and $\Psi_\gamma(F)$ are intervals with integer endpoints. If $d_{\mathcal{W}_1}(H,F) < \delta_F$, then the right side of (7) is less than one, and thus $d(\tilde{q}, \Psi_\gamma(F)) = 0$ for all $\tilde{q} \in \Psi_\gamma(H)$. Hence, $\Psi_\gamma(H) \subseteq \Psi_\gamma(F)$ for all $H \in \mathbb{D}$ with support on the integers such that $d_{\mathcal{W}_1}(H,F) < \delta_F$. Such a conclusion agrees with the results in Shapiro and Homem-de-Mello (2000) and Homem-de-Mello (2008) — in those papers, it is shown that, in case of piecewise linear convex stochastic optimization problems with discrete distributions, the solution obtained with the sample average approximation problem belongs to the set of true optimal solutions if the sample size is large enough. Proposition 1 specializes to the newsvendor model but on the other hand it quantifies the error in terms of the distance between the distributions.

It follows from Proposition 1 that if distribution $F$ has support on the integers, then there exists $\kappa = 2L/A_F$ such that (5) holds with $d = d_{\mathcal{W}_1}$ for any $\delta > 0$. That is, a global stability condition holds for the newsvendor problem — in fact, it is easy to see from the proof of Proposition 1 that such global stability holds more generally for problems with piecewise linear concave (or convex) objective functions for which (8) holds if the distribution is discrete. Thus, the empirical inventory decision derived by the newsvendor model converges to the optimal decision as more data are collected if the forecast distribution $\hat{H}^k$ converges to the true distribution (assuming there is a single true distribution) under the first order Wasserstein metric. Moreover, the error can be quantified since the stability constant $\kappa$ is given explicitly.

# 3 Asymptotic Behavior of the IDP

In this section, we study the convergence of the IDP described in Section 1.2 for settings in which newsvendor-type decisions are used, that is, $\Psi(\hat{H}^k) = \Psi_\gamma(\hat{H}^k)$ for some $\gamma \in (0,1)$, and the empirical distribution is used to construct $\hat{H}^k$, for cases in which the family of actual distributions $G(\cdot,\cdot)$ has structure as specified.

## 3.1 Empirical Distribution

As mentioned, we study the dynamics of the IDP when the empirical distribution is used as a forecasting method. The empirical distribution function $\hat{H}^k$ constructed with $k$ observations is given by

$$\hat{H}^k(x) \quad := \quad \frac{1}{k}\sum_{j=1}^{k} \mathbb{I}_{\{X^j \le x\}}$$

where $X^j$'s are the observations. Note that we can write $X^{j+1} = G^{-1}(q^j, U^{j+1})$, where $\{U^j\}_{j=1}^{\infty}$ is an independent sequence of uniform $[0,1]$ random variables (w.p.1, $\Psi_{U^{j+1}}(G(q^j,\cdot))$ is a singleton). As before, at iteration $j$ the decision maker chooses some $q^j \in \Psi_\gamma(\hat{H}^j)$. We indicate the chosen element by $\psi(\hat{H}^j)$. We can write the empirical distribution at iteration $k+1$ in terms of the empirical distribution at iteration $k$ as follows:

$$\begin{aligned}
\hat{H}^{k+1}(x) &= \frac{1}{k+1}\sum_{j=1}^{k+1} \mathbb{I}_{\{X^j \le x\}} \quad = \quad \frac{k}{k+1}\left(\frac{1}{k}\sum_{j=1}^{k} \mathbb{I}_{\{X^j \le x\}}\right) + \frac{1}{k+1}\mathbb{I}_{\{X^{k+1} \le x\}} \\
&= \frac{k}{k+1}\hat{H}^k(x) + \frac{1}{k+1}\mathbb{I}_{\{G^{-1}(q^k,U^{k+1}) \le x\}} \\
&= \left(1 - \frac{1}{k+1}\right)\hat{H}^k(x) + \frac{1}{k+1}\left[\mathbb{P}\left[G^{-1}(q^k, U^{k+1}) \le x | \mathcal{F}^k\right]\right. \\
&\qquad \left. + \left(\mathbb{I}_{\{G^{-1}(q^k,U^{k+1}) \le x\}} - \mathbb{E}\left[\mathbb{I}_{\{G^{-1}(q^k,U^{k+1}) \le x\}} | \mathcal{F}^k\right]\right)\right] \\
&= \hat{H}^k(x) + \frac{1}{k+1}\left[G(\psi(\hat{H}^k), x)\right. \\
&\qquad \left. + \left(\mathbb{I}_{\{G^{-1}(\psi(\hat{H}^k)),U^{k+1}) \le x\}} - \mathbb{E}\left[\mathbb{I}_{\{G^{-1}(\psi(\hat{H}^k),U^{k+1}) \le x\}} | \mathcal{F}^k\right]\right) - \hat{H}^k(x)\right]
\end{aligned} \tag{12}$$

The form of (12) allows us to use stochastic approximation results to study the dynamics of the IDP. Note however that the iterates of (12) are elements of the space $\mathbb{D}$ of distribution functions on $\mathbb{R}$. Stochastic approximation convergence results have been established in different spaces. For example, convergence results in $\mathbb{R}^m$ are studied in Bertsekas and Tsitsiklis (1996) and Kushner and Yin (1991). Revesz (1973) and Walk (1977) deal with Hilbert spaces. Walk (1978) considers the space $D[0,1]$ of real-valued functions on $[0,1]$ that are right continuous with left limits, endowed with Skorohod's $J_1$-topology. Although the results in Walk (1978) can in principle be adapted to the present setting, the assumptions in that work are difficult to verify.

In the following sections we restrict ourselves to the case of distributions with finite support. More specifically, we make the following assumption:

**Assumption 1** *There exists a finite subset of* $\mathbb{R}$*, denoted by* $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$*, such that each member of the family of distributions* $\{G(q,\cdot) : q \in \mathbb{R}\}$ *has support on* $\mathcal{X}$*.*

Such a restriction allows us to study the convergence of the iterates $\hat{H}^k$ by using results for stochastic approximation in $\mathbb{R}^n$, which we review in the Online Appendix.

### 3.2 The Contraction Case

In this section we consider the case where the family of distributions $\{G(q, \cdot) : q \in \mathbb{R}\}$ possesses a certain contraction property, which will be described later. Let $\mathbb{D}_\mathcal{X}$ denote the set of distribution functions with support on $\mathcal{X}$. Without loss of generality, assume that $x_1 < \cdots < x_n$. Then, each $P \in \mathbb{D}_\mathcal{X}$ can be written as $P(x) = p_1 \mathbb{I}_{\{x_1 \le x < x_2\}} + p_2 \mathbb{I}_{\{x_2 \le x < x_3\}} + \cdots + p_{n-1} \mathbb{I}_{\{x_{n-1} \le x < x_n\}} + \mathbb{I}_{\{x_n \le x\}}$ where $p_i = P(x_i)$, $i = 1, \ldots, n$ are the cumulative probabilities corresponding to the $x_i$-values. Based on such a representation, we can define mappings from $\mathbb{D}_\mathcal{X}$ to $\mathbb{R}^{2n}$ in various ways. One example is $\mathcal{M}(P) := (x_1, p_1, x_2, p_2, \ldots, x_n, p_n = 1)$, that is, a pairwise list of $x$-values and their corresponding cumulative probabilities. Another example of such a mapping is

$$\mathcal{T}(P) = (x_1, 0, x_2, p_1(x_2 - x_1), \ldots, x_j, p_{j-1}(x_j - x_{j-1}), \ldots, x_n, p_{n-1}(x_n - x_{n-1})), \tag{13}$$

that is, a pairwise list of $x$-values and their corresponding integrated areas under the distribution function curve.

Recall the first order Wasserstein metric $d_{\mathcal{W}_1}$ defined in (6). The proposition below shows that the Wasserstein distance between two distribution functions with finite support is the same as the $l_1$ distance between the $\mathcal{T}$ mappings of those two distribution functions.

**Proposition 2** *Let $P$ and $F$ be distributions in $\mathbb{D}_\mathcal{X}$. Let $d_{l_1^{2n}}$ be the metric in $\mathbb{R}^{2n}$ defined by the $l_1$ norm. The mapping $\mathcal{T} : \mathbb{D}_\mathcal{X} \mapsto \mathbb{R}^{2n}$ defined in* (13) *is distance-preserving with respect to the first order Wasserstein metric, i.e., $d_{l_1^{2n}}(\mathcal{T}(P), \mathcal{T}(F)) = d_{\mathcal{W}_1}(P, F)$.*

**Proof**

Let us write $\mathcal{T}(F)$ as $(x_1, 0, x_2, f_1(x_2 - x_1), \ldots, x_j, f_{j-1}(x_j - x_{j-1}), \ldots, x_n, f_{n-1}(x_n - x_{n-1}))$, where $f_i = F(x_i)$. Then, $d_{l_1^{2n}}(\mathcal{T}(P), \mathcal{T}(P)) = \sum_{i=1}^{n-1} |p_i - f_i|(x_{i+1} - x_i)$. Since for $x_i \le x < x_{i+1}$ we have $P(x) = p_i$ and $F(x) = f_i$, $i = 1, \ldots, n$, it follows that $d_{\mathcal{W}_1}(P, F) = \int_\mathbb{R} |P(x) - F(x)| dx = \sum_{i=1}^{n-1} |p_i - f_i|(x_{i+1} - x_i)$. Therefore, $d_{\mathcal{W}_1}(P, F) = d_{l_1^{2n}}(\mathcal{T}(P), \mathcal{T}(F))$. □

Proposition 2 enables us to measure distances between probability distributions by measuring distances between vectors in $\mathbb{R}^{2n}$. Next, define $\Phi : \mathbb{D}_\mathcal{X} \mapsto \mathbb{D}_\mathcal{X}$ as

$$\Phi(P)(x) \ := \ G(\psi(P), x). \tag{14}$$

The above definition allows us to write the recursive formula (12) for the stochastic iterates as

$$\hat{H}^k(x) \ = \ \left(1 - \frac{1}{k}\right) \hat{H}^{k-1}(x) + \frac{1}{k} \left[ (\Phi(\hat{H}^{k-1}))(x) + W^k(x) \right] \tag{15}$$

where $W^k$ is given by

$$W^k(x) \;:=\; \mathbb{I}_{\{G^{-1}(\psi(\hat{H}^{k-1})),U^k)\le x\}} - \mathbb{E}\left[\mathbb{I}_{\{G^{-1}(\psi(\hat{H}^{k-1})),U^k)\le x\}}|\mathcal{F}^{k-1}\right]. \tag{16}$$

In the following, we use $\mathcal{T}$ defined in (13) to represent distribution functions in $\mathbb{D}_{\mathcal{X}}$ as vectors. Let $p_i^k := \mathbb{P}(G^{-1}(\psi(\hat{H}^{k-1}), U^k) \le x_i|\mathcal{F}^{k-1})$, $i = 1, \ldots, n$. Then we have

$$\mathcal{T}(\Phi(\hat{H}^{k-1})) \;=\; (x_1, 0, x_2, p_1^k(x_2 - x_1), \ldots, x_n, p_{n-1}^k(x_n - x_{n-1})). \tag{17}$$

Moreover,

$$\mathcal{T}(\mathbb{I}_{\{G^{-1}(\psi(\hat{H}^{k-1})),U^k)\le x\}}) \;=\; (x_1, 0, x_2, 0, \ldots, x_{j+1}, (x_{j+1} - x_j), ...x_n, (x_n - x_{n-1})) \tag{18}$$

with probability $p_j^k - p_{j-1}^k$, $j = 1, \ldots, n$.

Our goal is to use the above vector representation to extend Proposition OA–1 in the Online Appendix to the space $\mathbb{D}_{\mathcal{X}}$. We state some assumptions and a result that will be used in the sequel.

**Assumption 2** *The point-to-set mapping $\Psi_\gamma(G(q,\cdot))$ defined in* (2) *has a singleton fixed point $q^\bullet$, i.e., $\Psi_\gamma(G(q^\bullet,\cdot)) = \{q^\bullet\}$.*

**Assumption 3** *There exists an $\alpha \in [0, 1)$ such that*

$$d_{\mathcal{W}_1}(G(q,\cdot), G(q^\bullet,\cdot)) \;\le\; \alpha\,|q - q^\bullet|$$

*for all $q \in \mathbb{R}$, where $q^\bullet$ is the fixed point in Assumption 2. That is, $G(q,\cdot)$ is a pseudo-contraction with respect to $q$ in the first order Wasserstein metric.*

**Assumption 4** *The solution-set mapping $\Psi_\gamma(\cdot)$ defined in* (2) *is stable with respect to $G(q^\bullet,\cdot)$ under the first-order Wasserstein metric, i.e., there exists a $\kappa < 1/\alpha$, where $\alpha$ is the constant in Assumption 3, such that for any $F \in \mathbb{D}$ and any $q_F \in \Psi_\gamma(F)$, it holds that*

$$d_{\mathcal{S}}(q_F, \Psi_\gamma(G(q^\bullet,\cdot))) \;\le\; \kappa\, d_{\mathcal{W}_1}(F, G(q^\bullet,\cdot)).$$

Let us verify whether Assumptions 1–4 hold. Recall that the solution of the newsvendor model is given by the set of quantiles of the underlying distribution. In the following analysis we assume that the smallest quantile is always chosen for the sake of of the IDP, i.e., $\psi(P) = P^{-1}(\gamma)$ where $P^{-1}(u) := \min\{x : P(x) \ge u\}$.

In order to establish some structure on the dependence of the underlying distribution $G$ on the decisions $q$, we shall assume in the sequel that $G(q,\cdot)$ can be written as $\tilde{G}(\mu(q),\cdot)$, where $\tilde{G}$ is a distribution parameterized by one argument (e.g., its mean) and $\mu$ is a function of the decision $q$, so the distribution changes depending on the control.

Within this setting, the mapping $\Phi$ in (14) is written as

$$\Phi(P)(x) \;=\; G(P^{-1}(\gamma), x) \;=\; \tilde{G}(\mu(P^{-1}(\gamma)), x)$$

where $\gamma = \frac{r-c}{r-v}$.

Consider a family of distributions $G$ indexed by the integers $0, \ldots, Q$ with support contained in $\{0, \ldots, Q_1\}$, with $Q_1 \geq Q$. Suppose also that the function $\mu(q)$ is given by the integral part of $\tilde{\mu}(q)$, where $\tilde{\mu}$ is some function such that $\tilde{\mu}(q) \leq Q$ for all $q \in \mathbb{R}$. Then Assumption 1 holds in this case, with $\mathcal{X} = \{0, \ldots, Q_1\}$.

An example of the situation described above is when $\tilde{G}$ belongs to the *location-scale family*, as defined below.

**Definition 1** *The* location-scale family *is a set of probability distributions parameterized by a location parameter $\lambda$ and a scale parameter $\sigma \geq 0$ such that, if $Y$ is a random variable whose probability distribution belongs to such a family, then $Y$ is of the form $Y = \lambda + \sigma Z$, where the distribution of $Z$ is also in the family.*

For example, suppose that $\tilde{G}(\lambda, \cdot)$ is the distribution of $Y = \lambda + \sigma Z$, where $Z$ has support on $\{0, \ldots, Q\}$ and $\sigma$ is a positive integer. If $\mu$ and $\tilde{\mu}$ are defined as before, then the support of $G(q, \cdot)$ is a subset of $\{0, \ldots, (\sigma+1)Q\}$.

Next, note that Assumption 2 holds if there exists $q^\bullet$ such that $G(q^\bullet)$ has a unique $\gamma$-quantile and $G^{-1}(q^\bullet, \gamma) = q^\bullet$ (i.e., $\tilde{G}^{-1}(\mu(q^\bullet), \gamma) = q^\bullet$). For the location family, this holds if

$$q^\bullet = \inf\left\{x : F_Z\left(\frac{x - \mu(q^\bullet)}{\sigma}\right) \geq \gamma\right\}$$

where $F_Z$ denotes the distribution of $Z$ in Definition 1. Then, the fixed point exists if $q^\bullet$ satisfies

$$\frac{q^\bullet - \mu(q^\bullet)}{\sigma} \;=\; F_Z^{-1}(\gamma). \tag{19}$$

As an example, let $\mu(q) := \min\{\lfloor\sqrt{q}\rfloor, Q\}$. Then, since the function $q - \mu(q)$ is nondecreasing and takes on every integer value, it can be easily seen that there exists $q^\bullet$ such that

$$q^\bullet - \lfloor\sqrt{q^\bullet}\rfloor \;=\; \sigma F_Z^{-1}(\gamma). \tag{20}$$

We check now Assumption 3. Note that for the location family we have

$$G^{-1}(q, u) - G^{-1}(\tilde{q}, u) \;=\; \mu(q) - \mu(\tilde{q}) \tag{21}$$

for any $q,\ \tilde{q} \in \mathbb{R}$. It follows that Assumption 3 holds if the function $\mu$ and the point $q^\bullet$ have the property that there exists $\alpha \in (0,1)$ such that, for each $q \in \mathbb{R}$,

$$|\mu(q) - \mu(q^\bullet)| \ \le \ \alpha(q - q^\bullet). \tag{22}$$

For example, let $\mu(q) := \min\{\lfloor\sqrt{q}\rfloor, Q\}$, and suppose that $\mu(q^\bullet) = \mu(q^\bullet - 1) = \mu(q^\bullet + 1)$. Then, it is easy to see that (22) holds with $\alpha = 1/2$.

Finally, consider Assumption 4. Proposition 1 ensures that the newsvendor problem is stable with respect to the first order Wasserstein metric. Thus, by taking the distribution $G(q^\bullet)$ as the reference distribution, the stability constant $\kappa$ is equal to $2(r-v)/A_{q^\bullet}$, where $A_{q^\bullet} = \min\{(r-c) - (r-v)G(q^\bullet, q^\bullet - 1),\ (r-v)G(q^\bullet, q^\bullet) - (r-c)\}$. Thus, Assumption 4 holds as long as $\kappa$ is not too large (more specifically, less than $1/\alpha$, where $\alpha$ is the constant in (22)).

**Lemma 2** *If Assumptions 2-4 hold, then $\Phi$ in (14) is a* pseudo-contraction *under the first order Wasserstein metric, i.e., there exist a $\theta \in [0,1)$ and a $P^\bullet \in \mathbb{D}_\mathcal{X}$ such that $d_{\mathcal{W}_1}(\Phi(P), P^\bullet) \le \theta d_{\mathcal{W}_1}(P, P^\bullet)$. Moreover, $P^\bullet = G(q^\bullet)$, where $q^\bullet$ is the fixed point in Assumption 2.*

**Proof**

We have

$$\begin{aligned} d_{\mathcal{W}_1}(\Phi(P), G(q^\bullet)) = &\int_0^1 \left|G^{-1}(\psi(P), u) - G^{-1}(q^\bullet, u)\right| du && \text{(from (6))} \\ \le\ & \alpha\,|\psi(P) - q^\bullet| && \text{(by Assumption 3)} \\ =\ & \alpha\,|\psi(P) - \psi(G(q^\bullet))| && \text{(by Assumption 2)} \\ \le\ & \alpha\kappa d_{\mathcal{W}_1}(P, G(q^\bullet)). && \text{(by Assumption 4)} \end{aligned}$$

The result follows by taking $\theta := \alpha\kappa < 1$. □

The following theorem shows that the forecasts constructed with the IDP described in Section 1.2 converge under the first order Wasserstein metric, and the limit point of the decisions is the fixed point in Assumption 2.

**Theorem 1** *Consider the stochastic recursion given by* (12)*. Suppose that Assumptions 1–4 hold. Then $d_{\mathcal{W}_1}(\hat{H}^k, G(q^\bullet)) \to 0$ as $k \to \infty$ with probability 1, where $q^\bullet$ is the fixed point given in Assumption 2. Moreover, $q^k \to q^\bullet$ with probability 1.*

**Proof**

By Assumption 1, the distribution of demand in any iteration $k$ belongs to $\mathbb{D}_{\mathcal{X}}$. Thus, $\hat{H}^k$ has a vector representation in $\mathbb{R}^n$, for which we will use the mapping $\mathcal{T}$ defined in (13). We prove the convergence of the sequence $\{\hat{H}^k\}$ as $k \to \infty$ using the results in Proposition OA–1 in the Online Appendix, applied to the sequence $\{\mathcal{T}(\hat{H}^k)\}$. In that case, following (17), recursion (15) becomes

$$\mathcal{T}(\hat{H}^k(x)) \quad = \quad \left(1 - \frac{1}{k}\right)\mathcal{T}(\hat{H}^k(x)) + \frac{1}{k}\left[\mathcal{T}(\Phi(\hat{H}^{k-1})) + \tilde{W}^k(x)\right] \tag{23}$$

where $\tilde{W}^k(x)$ is defined as a random vector such that

$$\tilde{W}^k(x) = (0, 0, 0, -p_1^k(x_2 - x_1), \ldots, 0, (x_{j+1} - x_j)(1 - p_j^k), \ldots, 0, (x_n - x_{n-1})(1 - p_{n-1}^k)) \tag{24}$$

with probability $p_j^k - p_{j-1}^k$, $j = 1, \ldots, n$, where $p_0^k = 0$ and $p_n^k = 1$.

We check now the conditions of Proposition OA–1:

**1.** In this case, $\beta^k = 1/k$, and thus $\sum_{k=0}^{\infty} \beta^k = \infty$ and $\sum_{k=0}^{\infty} (\beta^k)^2 < \infty$.

**2.** It follows from Proposition 2 and Lemma 2 that there exist an $\alpha \in [0, 1)$ and a $q^\bullet \in \mathbb{R}$ (which is the fixed point in Assumption 2) such that

$$\begin{aligned} d_{\ell_1^{2n}}(\mathcal{T}(\Phi(\hat{H}^k)), \mathcal{T}(G(q^\bullet))) \ = \ d_{\mathcal{W}_1}(\Phi(\hat{H}^k), G(q^\bullet)) \ &\leq \ \alpha d_{\mathcal{W}_1}(\hat{H}^k, G(q^\bullet)) \\ &= \ \alpha d_{\ell_1^{2n}}(\mathcal{T}(\hat{H}^k), \mathcal{T}(G(q^\bullet))). \end{aligned}$$

**3.** For each $k$ and $x$,

$$\begin{aligned} &\mathbb{E}\left[\tilde{W}^k(x) \mid \mathcal{F}^{k-1}\right] = \\ &= \sum_{j=1}^{n} \Big(0, 0, 0, -p_1^k(x_2 - x_1), \ldots, 0, (x_{j+1} - x_j)(1 - p_j^k), \ldots, 0, (x_n - x_{n-1})(1 - p_{n-1}^k)\Big)\,(p_j^k - p_{j-1}^k) \\ &= \Big(0, 0, 0, (x_2 - x_1)(1 - p_1^k), \ldots, 0, (x_{j+1} - x_j)(1 - p_j^k), \ldots, 0, (x_n - x_{n-1})(1 - p_{n-1}^k)\Big)\,(p_1^k) + \ldots + \\ &\quad \Big(0, 0, 0, (x_2 - x_1)(-p_1^k), \ldots, 0, (x_{j+1} - x_j)(-p_j^k), \ldots, , 0, (x_n - x_{n-1})(1 - p_{n-1}^k)\Big)\,(p_n^k - p_{n-1}^{k-1}) \\ &= (0, \ldots, 0). \end{aligned}$$

**4.** For all $k$ and $x$,

$$\begin{aligned} \mathbb{E}\left[\tilde{W}^k(x)^2 \mid \mathcal{F}^{k-1}\right] \ &= \ \sum_{j=1}^{n} \Big(0, 0, 0, -p_1^k(x_2 - x_1), \ldots, 0, (x_{j+1} - x_j)(1 - p_j^k), \ldots, \\ &\qquad\qquad , \ldots, 0, (x_n - x_{n-1})(1 - p_{n-1}^k)\Big)^2 (p_j^k - p_{j-1}^k) \\ &\leq \ \sum_{j=1}^{n-1} |x_{j+1} - x_j|^2 \qquad \text{(which is constant)} \end{aligned}$$

**5.** In this case, $Z^k = 0$ for all $k$.

Therefore, it follows from Proposition OA–1 that $d_{l_1^{2n}}(\mathcal{T}(\hat{H}^k), \mathcal{T}(G(q^\bullet))) \to 0$ with probability 1 as $k \to \infty$. Since $d_{l_1^{2n}}(\mathcal{T}(\hat{H}^k), \mathcal{T}(G(q^\bullet)) = d_{\mathcal{W}_1}(\hat{H}^k, G(q^\bullet))$, we have that $d_{\mathcal{W}_1}(\hat{H}^k, G(q^\bullet)) \to 0$ with probability 1 as $k \to \infty$. The last assertion then follows from Proposition 1. □

Theorem 1 shows that the IDP does converge under some conditions. We emphasize that the conditions we have derived are *sufficient* conditions; it is quite possible that convergence holds in more general settings. We shall see some numerical illustrations of this phenomenon in Section 4.

### 3.3 The Stochastically Monotonic Case

In this section we study the convergence of the IDP under the following assumption:

**Assumption 5** *The family of distributions* $\{G(q, \cdot) : q \in \mathbb{R}\}$ *is* stochastically decreasing *in* $q$*, i.e., if* $q_1 \le q_2$*, then* $G^{-1}(q_1, u) \ge G^{-1}(q_2, u)$ *for all* $u \in (0, 1)$*, or equivalently,* $G(q_1, x) \le G(q_2, x)$ *for all* $x \in \mathbb{R}$*.*

The assumption is relevant in situations where demand decreases as the order quantity increases. Some examples were discussed in Section 1.

Recall that the family of distributions $\{G(q) : q \in \mathbb{R}\}$ has support on a set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Suppose that Assumption 2 holds, and let $x_i$ be the point in $\mathcal{X}$ such that $x_i = q^\bullet$. Then, by definition of $q^\bullet$, we have

$$G(q^\bullet, x_i) \quad \ge \quad \gamma \quad \text{and} \quad G(q^\bullet, x_{i-1}) \quad < \quad \gamma.$$

Our goal is to show that

$$\hat{\hat{H}}^k(x_{i-1}) \to G(q^\bullet, x_{i-1}) \tag{25}$$

$$\hat{\hat{H}}^k(x_i) \to G(q^\bullet, x_i), \tag{26}$$

since these equations imply that $q^k = \psi(\hat{\hat{H}}^k) = (\hat{H}^k)^{-1}(\gamma) = x_i$ for $k$ large enough, i.e., $q^k \to q^\bullet$.

To show (25)–(26), consider the dynamics of the bi-variate process $h^k := (h_1^k, h_2^k) := (\hat{H}^k(x_{i-1}), \hat{H}^k(x_i))$. It follows from (12) that

$$\hat{H}^{k+1}(x) \quad = \quad \hat{H}^k(x) + \frac{1}{k+1}\left[\mathbb{I}_{\{G^{-1}(q^k, U^{k+1}) \le x\}} - \hat{H}^k(x)\right].$$

Let $S^k := \Big(\mathbb{I}_{\{G^{-1}(q^k, U^{k+1}) \le x_{i-1}\}} - \hat{H}^k(x_{i-1}),\ \mathbb{I}_{\{G^{-1}(q^k, U^{k+1}) \le x_i\}} - \hat{H}^k(x_i)\Big)$ denote the direction of movement of the process $\{h^k\}$. Then, we have

$$s^k := \mathbb{E}[S^k \,|\, \mathcal{F}^k] \ = \ \Big(G(q^k, x_{i-1}) - \hat{H}^k(x_{i-1}),\ G(q^k, x_i) - \hat{H}^k(x_i)\Big)$$

$$= \left(G(q^k, x_{i-1}) - h_1^k,\ G(q^k, x_i) - h_2^k\right). \tag{27}$$

Define now the function

$$L(y_1, y_2) \ := \ \frac{1}{2}\left[(G(q^\bullet, x_{i-1}) - y_1)^2 + (G(q^\bullet, x_i) - y_2)^2\right]. \tag{28}$$

Clearly,

$$\nabla L(y_1, y_2) \ = \ (y_1 - G(q^\bullet, x_{i-1}),\ y_2 - G(q^\bullet, x_i))^T.$$

We aim to prove convergence of $\{h^k\}$ by using a classical convergence result for stochastic approximation, based on the existence of a Lyapunov function (see Proposition OA–2 in the Online Appendix), which ensures that $\nabla L(h^k) \to 0$. It is easy to see that all assumptions for that result are immediately satisfied, except for the so-called pseudo-gradient property, which states that there exists some constant $c > 0$ such that

$$c\|\nabla L(h^k)\|^2 \ \leq \ -\nabla L(h^k)^T s^k \quad \forall\, k. \tag{29}$$

Rather than working with the standard scalar product, it will be convenient to work with a $M$-product, defined as $\langle x, y\rangle_M = x^T M y$ for a positive definite matrix $M$. In particular, we shall use the matrix

$$M \ := \ \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}, \quad \text{where} \quad r \ := \ \Delta_1/\Delta_2, \tag{30}$$

with $\Delta_1 := \gamma - G(q^\bullet, x_{i-1})$, $\Delta_2 := G(q^\bullet, x_i) - \gamma$. Definition (30) assumes that $\Delta_1 \geq \Delta_2$; if the opposite relation holds, we can define $M = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ with $s := \Delta_2/\Delta_1$ and the analysis is very similar. Thus, we assume without loss of generality that

$$\Delta_1 \ \geq \ \Delta_2. \tag{31}$$

We shall also impose the following assumption:

**Assumption 6** *There exists an $\varepsilon > \Delta_1$ such that*

$$G(q, x_i) - G(q, x_{i-1}) \quad \geq \quad \varepsilon$$

*for all $q \in \mathbb{R}$.*

Assumption 6 ensures that the value $x_i = q^\bullet$ can be generated with positive probability in any iteration, i.e., regardless of the value of the current iterate $q^k$. Lemma 3 shows that (29) is satisfied under the $M$-norm. The proof is presented in the Online Appendix.

**Lemma 3** *Suppose that Assumptions 5 and 6 hold. Then, there exists $c > 0$ such that*

$$\nabla L(h^k)^T M s^k \quad \leq \quad -c \|\nabla L(h^k)\|_M^2 \quad \forall\, k.$$

We summarize the above discussion in the following result.

**Theorem 2** *Consider the stochastic recursion given by* (12)*. Suppose that Assumptions 2, 5 and 6 hold. Then $q^k \to q^\bullet$ as $k \to \infty$ with probability 1, where $q^\bullet$ is the fixed point given in Assumption 2.*

**Remark:** It is possible that Assumption 6 can be relaxed, but some form of it is necessary. To see that, consider the case where the distributions $G(\lambda, \cdot)$ are indexed by $\lambda \in \{1, \ldots, 2n\}$ and are all point masses (i.e., mass one on a single point), with $G(1, \cdot)$ =mass on the value $2n - 1$, $G(2n, \cdot)$ =mass on the value 1, $G(\lambda, \cdot)$ =mass on the value $n$ for all $\lambda \in \{2, \ldots, 2n - 1\}$. Let $\mu(\cdot)$ be defined as $\mu(q) = 1$ for $q \leq 1$, $\mu(q) = 2n$ for $q \geq 2n$, and $\mu(q) = \lfloor q \rfloor$ otherwise. Clearly, we have $q^\bullet = n$ for any $\gamma \in (0, 1)$. Also, $G$ satisfies Assumption 5. Suppose now $\gamma = 1/2$, and that the initial $q$ is $q^1 = 2n - 1$. Then, it is clear that the IDP will produce iterates $q^1 = 2n - 1$, $q^2 = 1$, $q^3 = 2n - 1$, $q^4 = 1$, etc., i.e., the process oscillates and never converges.

## 4 Numerical Experiments

In this section we present numerical experiments to illustrate the theoretical results derived in Section 3. More specifically, we simulate the iterative procedure described in Section 1.2 for three families of distributions, check convergence, and compare the limit point with the theoretical results in Section 3. Since those results were derived for the case of distributions with finite support, we "discretize" the support and "truncate" the tails if necessary. For completeness, though, we also present results for the original distributions, i.e. without discretization or truncation. For example, we show convergence results for the truncated and discretized normal distribution and (continuous) normal distribution. Although we have not proved whether the stochastic approximation approach can be applied to the space of distribution functions with infinite support, the numerical results suggest that even in that case the procedure converges to a fixed point solution, as proven in Theorem 1 for the finite support case.

When a distribution function belongs to the location-scale family described in Definition 1, the limit point can be computed analytically. To see this, let $Z_\lambda$ be a random variable with distribution $G_\sigma(\lambda, \cdot)$ belonging to the location-scale family with a location parameter $\lambda = \mu(q)$ and a scale parameter $\sigma$. Then, as seen in (19), the fixed point $q^\bullet = G^{-1}(q^\bullet, \gamma)$ satisfies

$$q^\bullet \;=\; \mu(q^\bullet) + \sigma G_1^{-1}(0, \gamma).$$

Consider first the contraction case discussed in Section 3.2. Let $\mu(q) := \sqrt{\alpha q}$ (with $0 < \alpha < 1$) for the case of continuous distributions. Then, we have a closed form solution for $q^\bullet$, since it as a root of a quadratic equation. We obtain

$$q^\bullet = \left( \frac{\sqrt{\alpha} + \sqrt{\alpha + 4\sigma G_1^{-1}(0,\gamma)}}{2} \right)^2 . \tag{32}$$

In the case of truncated and discretized distributions, we take $\mu(q) := \min\{\lfloor \sqrt{\alpha q} \rfloor, Q\}$, where $Q$ is the number of points in the discretization of $Z_0$. In that case, $q^\bullet$ is found in a similar fashion to (20).

For the numerical experiments, we set $r = 700$, $c = 100$, $v = 80$, $\alpha = 0.99$, and consider two cases for the demand distribution: normal and uniform, which belong to location-scale family. Figure 1 depicts the decisions at each iteration of the sequential procedure, for the case of normal distributions.[2] As expected, larger variance leads to slower convergence.

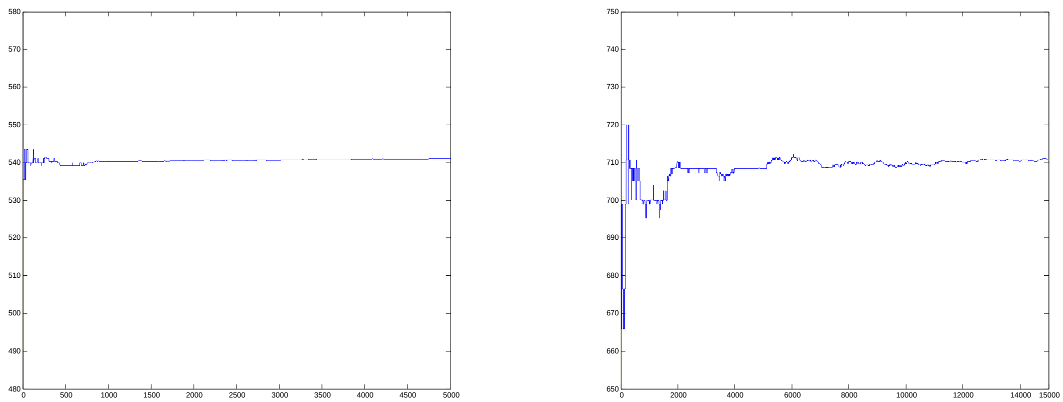


Figure 1: Convergence of iterative inventory decision when demand follows a normal distribution with $\sigma$=10 and 100, $\mu(q) = \sqrt{\alpha q}$.

Tables 1 and 2 compare the numerical results from the simulation with the theoretical ones for the case of normal and uniform distributions.

We also conducted experiments for the Poisson distribution. Let $\tilde{G}(\lambda, \cdot)$ be a Poisson distribution with parameter $\lambda$. Then we have

$$q^\bullet = \tilde{G}^{-1}(\mu(q^\bullet), \gamma) = \min\{x : \tilde{G}(\mu(q^\bullet), x) \geq \gamma\} = \min\left\{ x : \sum_{n=1}^{x} \frac{e^{-\mu(q^\bullet)}\mu(q^\bullet)^n}{n!} \geq \gamma \right\}$$

We can find $q^\bullet$ satisfying the above expression by using numerical methods. For the case of the parameters $r$, $c$, $v$, and $\alpha$ listed above, we obtained $q^\bullet = 6$. The simulation also yielded $q^N = 6$. For

[2]For this graph, we used $\lambda = C + \mu(q)$ for some constant $C$ in order to facilitate the depiction of convergence.

| $\sigma$ | Normal($q^\bullet$) | Normal($q^N$) | TnD Normal($q^\bullet$) | TnD Normal($q^N$) |
|---|---|---|---|---|
| 10 | 23.29 | 23.37 | 18 | 18 |
| 50 | 102.50 | 102.26 | 79 | 79 |
| 100 | 198.89 | 198.33 | 153 | 153 |

Table 1: Theoretical ($q^\bullet$) and experimental ($q^N$) limit values, for normal distribution, $\mu(q) = \sqrt{\alpha q}$. "TnD" indicates a truncated and discretized distribution.

| $\delta$ | Uniform ($q^\bullet$) | Uniform ($q^N$) | D Uniform ($q^\bullet$) | D Uniform($q^N$) |
|---|---|---|---|---|
| 50 | 54.09 | 54.24 | 54 | 54 |
| 100 | 103.68 | 103.61 | 104 | 104 |
| 200 | 201.21 | 201.38 | 202 | 202 |

Table 2: Theoretical ($q^\bullet$) and experimental ($q^N$) limit values, for uniform distribution, $\mu(q) = \sqrt{\alpha q}$. "D" indicates a discretized distribution and $\delta$ is the half-width.

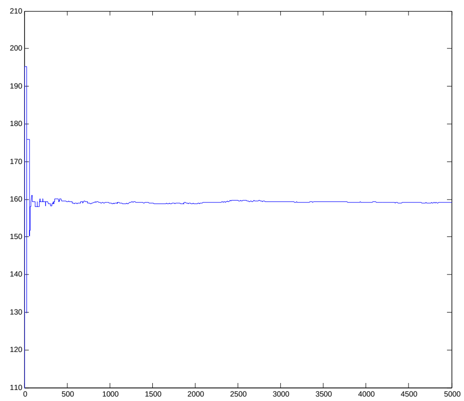
a truncated Poisson distribution — obtained by truncating the upper tail at 95% — we obtained $q^\bullet = 5$, and the simulation also yielded $q^N = 5$.

Next, we consider the case of stochastically decreasing distributions discussed in Section 3.3. Let $\mu(q) = 300 - q$ ($\mu(q) = \lfloor 300 - q \rfloor$ in the discretized case). As before, we use (19) and (20) to determine $q^\bullet$ analytically. Figure 2 depicts the decisions at each iteration of the sequential procedure, for the case of normal distributions. Tables 3 and 4 compare the numerical results from the simulation with the theoretical ones for the case of normal and uniform distributions (both the original and the truncated/discretized versions).

The numerical results above suggest that the conclusions of Theorem 1 are valid not only for finite support distributions, but also, for infinite support distributions. Therefore, we confirm that

| $\sigma$ | Normal($q^\bullet$) | Normal($q^N$) | TnD Normal($q^\bullet$) | TnD Normal($q^N$) |
|---|---|---|---|---|
| 10 | 159.24 | 159.32 | 157 | 157 |
| 50 | 196.21 | 196.37 | 185 | 185 |
| 100 | 242.43 | 242.27 | 220 | 220 |

Table 3: Theoretical ($q^\bullet$) and experimental ($q^N$) limit values, for normal distribution, $\mu(q) = 300 - q$. "TnD" indicates a truncated and discretized distribution.



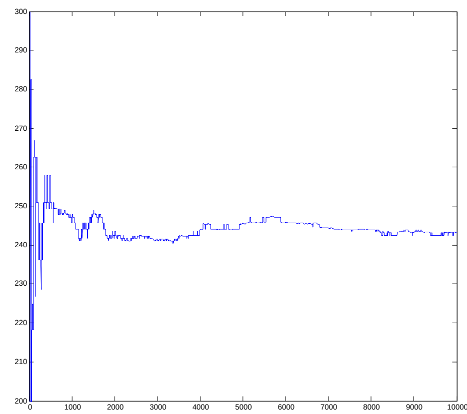



Figure 2: Convergence of iterative inventory decision when demand follows a normal distribution with $\sigma$=10 and 100, $\mu(q) = 300 - q$.

| $\delta$ | Uniform ($q^{\bullet}$) | Uniform ($q^N$) | D Uniform ($q^{\bullet}$) | D Uniform($q^N$) |
|---|---|---|---|---|
| 50 | 173.39 | 173.06 | 173 | 173 |
| 100 | 196.77 | 196.44 | 197 | 197 |
| 200 | 243.55 | 243.32 | 244 | 244 |

Table 4: Theoretical ($q^{\bullet}$) and experimental ($q^N$) limit values, for uniform distribution, $\mu(q) = 300 - q$. "D" indicates a discretized distribution and $\delta$ is the half-width.

a fixed point solution is the limit of sequences of the decisions in the iterative procedure. In the next section, we compare the limit point with the true optimal solution.

## 5 Optimal Decision vs. Iterative Decision

The results presented in the previous sections demonstrate that, under appropriate conditions, the IDP described in Section 1.2 converges to the fixed point $q^\bullet$ of a certain function — more specifically, we have $q^\bullet = \psi(G(q^\bullet))$, with $\psi$ as defined earlier. When the distribution $G$ is exogenous to the system, we simply have $q^\bullet = \psi(G)$, which is the optimal solution to the problem. When $G$ depends on the decisions $q$, however, the fixed point may not be optimal.

It is no surprise that the limit point $q^\bullet$ is not optimal in case of dependency — after all, the main thrust of this paper is the study of the situation where the decision maker uses an incorrect model in which such dependency is ignored. It is natural then to ask, how does the limit point $q^\bullet$ compare with the solution obtained in case the dependency between the random components of the model and the decisions is known? The answer to such a question yields the *price of modeling error*.

In this section we provide a comparison between the true optimal solution and the limit point $q^\bullet$ in terms of the objective function in the newsvendor framework. Although the convergence results provided in the previous sections are valid under the assumption that the demand distribution has finite support, in the discussion that follows we assume that the demand distribution is continuous with support in $\mathbb{R}$. This facilitates the analysis and provides more insightful results on the impact of using an incorrect model within the context of the IDP. Note that the numerical experiments of Section 4 have already suggested that convergence holds under more general assumptions than those made in Theorem 1.

Let $\xi$ denote the demand random variable, and assume its distribution belongs to a location family with location parameter $\lambda$ and scale parameter $\sigma$. As before, we assume that the demand depends on the order quantity $q$ through the location parameter, so that $\lambda = \mu(q)$. Let $G(q, \cdot)$ and $g_q(\cdot)$ denote respectively be the cumulative distribution function and the probability density function of $\xi$. As described earlier, $r$ is the unit selling price, $c$ is the unit production cost, and $v$ is the unit salvage value. The revenue function is given by (10), hence the expected revenue is given by

$$\mathbb{E}[R(q, \xi)] \;=\; \int_{\mathbb{R}} (r \min(\xi, q) - cq + v \max(q - \xi, 0)) g_q(\xi) d\xi. \tag{33}$$

Let $q^*$ be an optimal solution, i.e.,

$$q^* \in \text{argmax}_{q\in\mathbb{R}}\mathbb{E}[R(q,\xi)]. \tag{34}$$

We write $\xi$ as $\xi = Z + \mu(q)$ where $Z$ is a random variable that does not depend on $q$. The random variable $Z$ can be chosen in such a way that its distribution (call it $F$) has the same scale parameter as $\xi$ but with location parameter equal to 0, so that $\mathbb{E}(\xi) = \mathbb{E}(Z) + \mu(q) = \mu(q)$ and $Var(\xi) = Var(Z) = \sigma^2$. In that case, we have $G(q,x) = F(x - \mu(q))$ and $g_q(x) = f(x - \mu(q))$, where $f$ is the density of $F$. By performing the change of variables $x := \xi - \mu(q)$, we rewrite (33) as

$$\mathbb{E}[R(q,\xi)] = (r-c)q - (r-v)(q-\mu(q))F(q-\mu(q)) + (r-v)\int_{-\infty}^{q-\mu(q)} x\, f(x)dx. \tag{35}$$

Throughout the section we assume that $\mu(q)$ is a nondecreasing differentiable concave function. It can be easily seen that the expected revenue function is also concave in $q$ (due to the concavity of $\mu$). Given that the expected revenue function is concave in $q$ and the problem is unconstrained, the optimal decision $q^*$ can be computed by solving $\frac{\partial \mathbb{E}[R(q,\xi)]}{\partial q} = 0$. We obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial R(q)}{\partial q}\right] &= \mathbb{E}\left[\frac{\partial((r-c)q)}{\partial q}\mathbb{I}_{\{\xi \geq q\}} + \frac{\partial(rD - cq + v(q-\xi))}{\partial q}\mathbb{I}_{\{\xi < q\}}\right] \\
&= \mathbb{E}\left[\frac{\partial((r-c)q)}{\partial q}\mathbb{I}_{\{x+\mu(q)\geq q\}} + \frac{\partial(r(x+\mu(q)) - cq + v(q-(\mu(q)+x)))}{\partial q}\mathbb{I}_{\{x+\mu(q)<q\}}\right] \\
&= \mathbb{E}\left[(r-c)\mathbb{I}_{\{x+\mu(q)\geq q\}} + \left((r-v)\frac{\partial \mu(q)}{\partial q} + v - c\right)\mathbb{I}_{\{x+\mu(q)<q\}}\right] \\
&= (r-c) - (r-v)\left(1 - \frac{\partial \mu(q)}{\partial q}\right)F(q-\mu(q)) \;=\; 0.
\end{aligned} \tag{36}$$

If follows from (36) that $q^*$ satisfies

$$F(q-\mu(q)) = \frac{r-c}{(r-v)(1-\mu'(q))} = \frac{\gamma}{(1-\mu'(q))}$$

where $\gamma = \frac{r-c}{r-v}$.

Recall from Section 4 that the limit (denoted by $q^\bullet$) of the iterative decision process satisfies

$$F(q^\bullet - \mu(q^\bullet)) = \gamma. \tag{37}$$

On the other hand, as shown in (36), the true optimal solution (denoted by $q^*$) satisfies

$$F(q^* - \mu(q^*))(1-\mu'(q^*)) = \gamma. \tag{38}$$

Proposition 3 shows how much the limit inventory decision resulting from the IDP deviates from the optimal order quantity. The results are stated in terms of the critical ratio $\gamma$ of the newsvendor model. Below we use the notation $h(x) = \Theta(g(x))$ if there exist constants $c$ and $C$ such that $c < \frac{h(x)}{g(x)} < C$ as $x \to \infty$.

**Proposition 3** *Suppose that Assumption 3 holds. Let $q^\bullet$ and $q^*$ be points satisfying equations* (37) *and* (38). *Assume that (i) $\mu(\cdot)$ is increasing and concave, $\mu'(q) = \Theta(q^\delta)$ where $\delta < 0$; (ii) $F$ is differentiable and concave on $[\inf_{\gamma\in(0,1)}(q^\bullet - \mu(q^\bullet)), \sup_{\gamma\in(0,1)}(q^* - \mu(q^*)]$; (iii) $f(q) = o(\frac{\mu'(q)}{q})$ where $f = F'(\cdot)$. Then, we have*

1. *$q^* \geq q^\bullet$ for all $\gamma \in (0,1)$;*
2. *As $\gamma \to 1$, both $q^\bullet$ and $q^*$ go to $\infty$;*
3. *As $\gamma \to 1$, $\frac{q^*}{q^\bullet} \to \infty$.*

**Proof**

We begin with the first statement. Since $F$ belongs to a location family, then it follows from Assumption 3 that $\mu$ is a contraction. Therefore, $\mu'(\cdot) < 1$. It is immediate from (37) and (38) that $q^\bullet - \mu(q^\bullet) \leq q^* - \mu(q^*)$ since $0 < (1 - \mu'(q^*)) < 1$ holds (note that $\mu(\cdot)$ is increasing function). Since $q - \mu(q)$ is increasing in $q$, it follows that $q^* \geq q^\bullet$ for all $\gamma \in (0,1)$. As $\gamma \to 1$, it is immediate from (37) and (38) that both $q^*$ and $q^\bullet$ go to $\infty$.

Let $x^\bullet = q^\bullet - \mu(q^\bullet)$ and $x^* = q^* - \mu(q^*)$. As $\gamma \to 1$, it follows from (ii) that

$$f(x^*) \;\leq\; \frac{F(x^*) - F(x^\bullet)}{x^* - x^\bullet} \;\leq\; f(x^\bullet). \tag{39}$$

It follows from (37) and (38) that

$$F(x^*) - F(x^\bullet) \;=\; \frac{\gamma}{1 - \mu'(q^*)} - \gamma \;=\; \frac{\mu'(q^*)\gamma}{1 - \mu'(q^*)}.$$

Thus,

$$\frac{\mu'(q^*)\gamma}{1 - \mu'(q^*)} \frac{1}{f(x^\bullet)} \;\leq\; x^* - x^\bullet \;\leq\; \frac{\mu'(q^*)\gamma}{1 - \mu'(q^*)} \frac{1}{f(x^*)} \tag{40}$$

and since $\mu(q^*) - \mu(q^\bullet) \geq 0$, we have

$$\frac{\mu'(q^*)\gamma}{1 - \mu'(q^*)} \frac{1}{f(x^\bullet)} \;\leq\; x^* - x^\bullet \;\leq\; q^* - q^\bullet. \tag{41}$$

Moreover, since we assume that $0 < \mu'(q) < 1$, we have

$$\frac{\mu'(q)}{f(x)} \leq \frac{\mu'(q)}{(1 - \mu'(q))f(x)}. \tag{42}$$

By dividing both sides of (41) by $q^\bullet$, and combining with (42), we obtain

$$\frac{\mu'(q^*)}{q^\bullet f(x^\bullet)} \leq \frac{q^*}{q^\bullet} - 1. \tag{43}$$

We prove that $\frac{q^*}{q^\bullet} \to \infty$ by showing the left hand side of (43) goes to $\infty$. Indeed, suppose that $\frac{\mu'(q^*)}{q^\bullet f(x^\bullet)} < \infty$, which is equivalent to $\frac{\mu'(q^*)\mu'(q^\bullet)}{q^\bullet \mu'(q^\bullet) f(x^\bullet)} < \infty$. By assumption (iii), we have $\frac{\mu'(q^\bullet)}{q^\bullet f(x^\bullet)} \to \infty$ as $\gamma \to 1$. Together, the two inequalities imply $\frac{\mu'(q^*)}{\mu'(q^\bullet)} \to 0$. Finally, under assumption (ii) we have $\mu'(q) = \Theta(q^\delta)$ and thus $\frac{\mu'(q^*)}{\mu'(q^\bullet)} \to 0$, which in turn implies that $\frac{q^*}{q^\bullet} \to \infty$. □

We turn now to a comparison between the expected revenues at $q^\bullet$ and $q^*$. By combining expressions (35), (37) and (38), we obtain

$$\mathbb{E}[R(q^\bullet, \xi)] = (r-c)\mu(q^\bullet) + (r-v)\int_{-\infty}^{q^\bullet - \mu(q^\bullet)} x f(x) dx$$

and

$$\mathbb{E}[R(q^*, \xi)] = (r-c)(q^* - \frac{q^* - \mu(q^*)}{1-\mu'(q^*)}) + (r-v)\int_{-\infty}^{q^* - \mu(q^*)} x f(x) dx.$$

Figure 3 shows how the expected revenue function changes as $\gamma$ goes to 1. As shown in the graph, the difference between the fixed point and the optimal solution increases as $\gamma$ goes 1. Proposition 4 below formalizes that result.

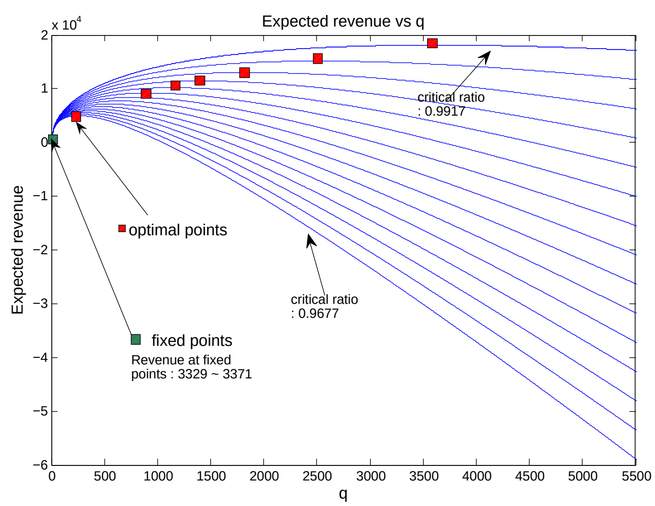



Figure 3: Expected revenue function for different values of the critical ratio $\gamma$

**Proposition 4** *Under the assumptions of Proposition 3,* $\frac{\mathbb{E}[R(q^*)]}{\mathbb{E}[R(q^\bullet)]} \to \frac{\mu(q^*) - \mu'(q^*)q^*}{(1-\mu'(q^*))\mu(q^\bullet)}$ *as* $\gamma \to 1$*.*

**Proof**

As shown in Proposition 3, both $q^\bullet$ and $q^*$ go to infinity as $\gamma \to 1$. Since $\int_{-\infty}^{q-\mu(q)}(r-v)xf(x)dx \to \mathbb{E}[Z] = 0$ as $q \to \infty$, we have

$$\frac{\mathbb{E}[R(q^*)]}{\mathbb{E}[R(q^\bullet)]} \to \frac{(r-c)(q^* - \frac{q^*-\mu(q^*)}{1-\mu'(q^*)})}{(r-c)\mu(q^\bullet)} = \frac{(q^* - \frac{q^*-\mu(q^*)}{1-\mu'(q^*)})}{\mu(q^\bullet)} \quad \text{as } \gamma \to 1. \tag{44}$$

□

**Remark:** Suppose we choose $\mu(q) = \sqrt{\alpha q}$, where $0 < \alpha < 1$. Then, since $\mu'(q^*) = \frac{\sqrt{\alpha}}{2\sqrt{q^*}}$, the numerator of the right-hand side of (44) becomes

$$\frac{\mu(q^*) - q^*\mu'(q^*)}{1-\mu'(q^*)} = \frac{\sqrt{\alpha q^*} - q^*\frac{\sqrt{\alpha}}{2\sqrt{q^*}}}{1 - \frac{\sqrt{\alpha}}{2\sqrt{q^*}}} = \frac{2\sqrt{\alpha}q^* - q^*\sqrt{\alpha}}{2\sqrt{q^*} - \sqrt{\alpha}} = \frac{q^*\sqrt{\alpha}}{2\sqrt{q^*} - \sqrt{\alpha}}.$$

Thus, we have $\frac{\mathbb{E}[R(q^*)]}{\mathbb{E}[R(q^\bullet)]} = \Theta(\frac{\sqrt{q^*}}{\sqrt{q^\bullet}})$. Since $\frac{q^*}{q^\bullet} \to \infty$ by Proposition 3, it follows that $\frac{\mathbb{E}[R(q^*)]}{\mathbb{E}[R(q^\bullet)]} \to \infty$.

## 6 Conclusions

The probability distributions used in stochastic optimization models usually are estimated with observed data. Often the estimates are updated when new data become available. Sometimes the observed data depend on the decisions made, and such dependence is not known to the decision maker, or the decision maker chooses for the sake of tractability not to incorporate the dependence into the model. Even if the decision maker attempts to incorporate the dependence into the model, the exact structure of the dependence may not be known to the decision maker and may thus be incorporated incorrectly. The resulting process is an iterative decision process in which the decision maker uses a misspecified model to make decisions, and estimated model parameters are updated as new data become available.

One may ask whether the decision maker would be able to detect the dependence of the distribution on the decision. Figure 4 indicates why it may be hard to do so with the observed data. The graph on the left side is a scatter plot indicating how the distribution of the observed quantity depends on the decision. The graph on the right side is a scatter plot of the observed quantity versus the chosen decision resulting from the iterative decision process when the same dependence as in the left graph holds. The data generated by the IDP may not facilitate detection of the dependence. In applications, selection of a variety of decisions just to learn about some possible dependence may be too costly. Also, as mentioned before, even if the dependence can be detected, the decision maker may still not know the correct structure of the dependence.

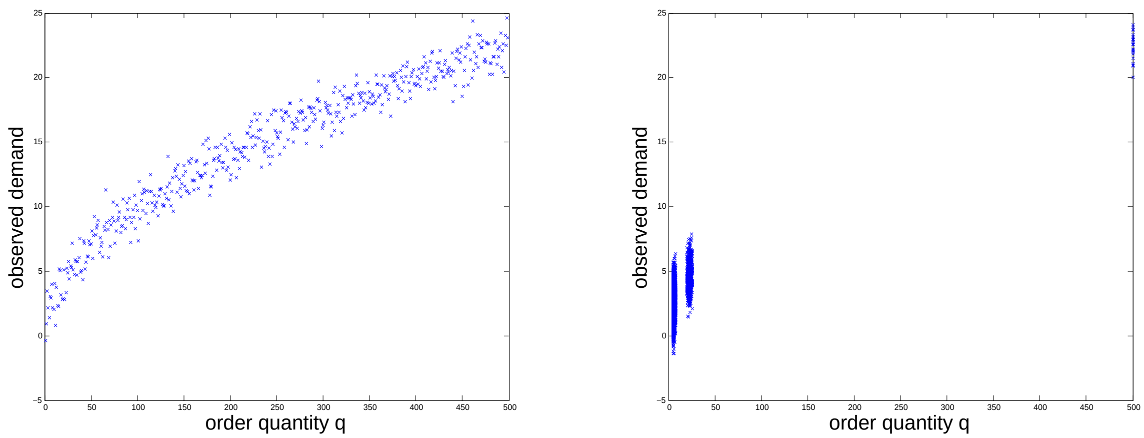



Figure 4: The left figure shows a scatter plot indicating how the distribution of the observed quantity depends on the decision. The right figure shows a scatter plot of the observed quantity versus the chosen decision resulting from the IDP.

This paper considers the case in which the empirical distribution is used as estimator, and the chosen solution for the decision maker's model is given by a particular quantile of the distribution. We provide sufficient conditions for the estimates and the resulting decisions to converge. In the limit the observed data appear to be consistent with the decision maker's model, that is, the data do not indicate that the decision maker is using an incorrect model. However, as shown, the limit decision may be arbitrarily bad compared with the true optimal solution.

Complex systems may be prone to modeling error and unpredictable behavior of iterative decision processes. However, in this paper we consider some of the simplest settings in operations research, and show that even in such simple settings modeling error may have severe long term consequences.

One may be motivated to consider estimating the underlying distribution with an approach other than building the empirical distribution. The use of the empirical distribution is common in practice and has a sound foundation — indeed, the Glivenko-Cantelli theorem establishes that with i.i.d. data, the empirical distribution converges w.p.1 to the distribution of the data in the vertical distance $d_{\mathcal{V}}(H,F) := \sup_{x\in\mathbb{R}} |H(x) - F(x)|$ and, moreover, it was shown in Cooper et al. (2006) that if the sequence of conditional distributions of the data converges weakly, then the empirical distribution converges weakly to the same limit w.p.1. However, in either case the empirical distribution may not converge in the horizontal distance $d_{\mathcal{H}}(H,F) := \sup_{u\in(0,1)} \left|H^{-1}(u) - F^{-1}(u)\right|$. The latter metric is useful when quantiles are of interest, since it may lead to a proof of convergence of estimates and quantiles under more general conditions than those presented in this paper. Further research may consider estimators that have more robust properties in terms of the horizontal distance.

One may attempt to reduce the possibility of modeling error by using data-driven or model-free methods. These methods are aimed at optimizing the objective as a function of the decisions by directly using observed pairs of decisions and objective values, without intermediate estimation of an objective function. A well-known such method is response surface methodology (Kleijnen et al., 2004). A related approach that constructs local approximations to the objective function based on noisy observations of objective values to optimize the objective is given in Bharadwaj and Kleywegt (2005, 2008). Related future research includes a data-driven method that combines ideas from operational statistics (Liyanage and Shanthikumar, 2005) and kernel density estimation techniques in order to optimize the objective directly, based on observed objective values.

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# Online Appendix for the Paper "Newsvendor-Type Models with Decision-Dependent Uncertainty"

## Online Appendix

### OA–1 General Results for Stochastic Approximation

The following results on the convergence of stochastic approximation iterations are described in Bertsekas and Tsitsiklis (1996). They give sufficient conditions for the convergence of the stochastic iterates in $\mathbb{R}^m$, one under a condition called pseudo-contraction and the other for a Lyapunov function.

**Proposition OA–1** *Consider a random sequence $\{(Y^k, W^k, Z^k)\}_{k=0}^{\infty}$ in $\mathbb{R}^m$ satisfying*

$$Y^{k+1} \quad = \quad (1-\beta^k)Y^k + \beta^k\Big(F(Y^k) + W^k + Z^k\Big). \tag{OA–1}$$

*Suppose that the following assumptions hold:*

1. *The deterministic nonnegative step size sequence $\{\beta^k\}_{k=0}^{\infty}$ satisfies $\sum_{k=0}^{\infty}\beta^k = \infty$ and $\sum_{k=0}^{\infty}(\beta^k)^2 < \infty$.*
2. *There exists an $y^*$ and an $\alpha \in [0,1)$ such that*
$$\left\|F(Y^k) - y^*\right\| \quad \le \quad \alpha\left\|Y^k - y^*\right\|$$
*for all $k$. (Such a function $F$ is called a pseudo-contraction. A contraction is a pseudo-contraction.)*
3. *For every $k$, $\mathbb{E}\left[W^k \mid \mathcal{F}^{k-1}\right] = 0$.*
4. *There exist constants $K_1, K_2 > 0$ such that*
$$\mathbb{E}\left[\|W^k\|^2 \mid \mathcal{F}^{k-1}\right] \quad \le \quad K_1 + K_2\|Y^k\|^2$$
*for all $k$.*
5. *There exists a random nonnegative sequence $\{\theta^k\}_{k=0}^{\infty}$ such that $\theta^k \to 0$ w.p.1 as $k \to \infty$, and such that*
$$\|Z^k\| \quad \le \quad \theta^k\left(\|Y^k\| + 1\right)$$
*for all $k$.*

*Then $Y^k \to y^*$ with probability 1 as $k \to \infty$.*

**Proposition OA–2** *Consider a random sequence $\{(Y^k, W^k, Z^k)\}_{k=0}^{\infty}$ in $\mathbb{R}^m$ satisfying*

$$Y^{k+1} \quad = \quad Y^k + \beta^k Z^k. \tag{OA–2}$$

*Suppose that the following assumptions hold:*

1. *The deterministic nonnegative step size sequence $\{\beta^k\}_{k=0}^{\infty}$ satisfies $\sum_{k=0}^{\infty} \beta^k = \infty$ and $\sum_{k=0}^{\infty}(\beta^k)^2 < \infty$.*
2. *There exists a function $L : \mathbb{R}^m \mapsto \mathbb{R}_+$ with the following properties:*
   (a) *There exists a constant $M > 0$ such that*
   $$\|\nabla L(y_1) - \nabla L(y_2)\| \quad \leq \quad M\|y_1 - y_2\|$$
   *for all $y_1, y_2 \in \mathbb{R}^m$.*
   (b) *There exists a positive constant c such that w.p.1, $\nabla L(Y^k)^T \mathbb{E}[Z^k \,|\, \mathcal{F}^k] \;\leq\; -c\|\nabla L(Y_k)\|^2$.*
   (c) *There exist constants $K_1, K_2 > 0$ such that w.p.1,*
   $$\mathbb{E}\left[\|Z^k\|^2 \,|\, \mathcal{F}^k\right] \quad \leq \quad K_1 + K_2\|\nabla L(Y^k)\|^2.$$

*Then, $\nabla L(Y^k) \to 0$ with probability 1 as $k \to \infty$.*

## OA–2 Auxiliary Results

**Lemma 1** *Consider problem (3), and the solution mapping $\Psi$ defined in (4). Suppose that there exists an $L > 0$ such that*

$$\left|R(q, \xi) - R(q, \tilde{\xi})\right| \quad \leq \quad L\left\|\xi - \tilde{\xi}\right\| \tag{OA–3}$$

*for all $q, \xi, \tilde{\xi} \in \mathbb{R}$. Then, for any distributions $P$ and $F$ on $\mathbb{R}$, and any $\tilde{q} \in \Psi(F)$, it holds that*

$$d(\tilde{q}, \Psi(P)) \quad \leq \quad \overline{\phi}_P^{-1}(2Ld_{\mathcal{W}_1}(P, F))$$

*where $\overline{\phi}_P^{-1}(y) := \sup\{\tau \,:\, \phi_P(\tau) \leq y\}$.*

**Proof** Let

$$d_{\mathcal{R}}(P,F) \;\; := \;\; \sup_{q\in\mathcal{Q}}\left|\int_{\mathbb{R}} R(q,\xi)\,dP(\xi) - \int_{\mathbb{R}} R(q,\xi)\,dF(\xi)\right|$$

and $\overline{\delta}_\bullet := d_{\mathcal{R}}(P,F)$. Then, taking $q^* \in \Psi(P)$ we have

$$2\overline{\delta}_\bullet \;\; \geq \;\; \overline{\delta}_\bullet + \mathbb{E}_F[R(\tilde{q},\xi)] - \mathbb{E}_P[R(q^*,\xi)] \tag{OA–4}$$

$$\geq \;\; \mathbb{E}_P[R(\tilde{q},\xi)] - \mathbb{E}_P[R(q^*,\xi)] \tag{OA–5}$$

$$\geq \;\; \phi_P(d(\tilde{q},\Psi(P)) \tag{OA–6}$$

where (OA–4) and (OA–5) hold by the definition of $d_{\mathcal{R}}$, whereas (OA–6) holds by the definition of $\phi_P$ stated in Proposition 1.

On the other hand, condition (OA–3) implies that

$$d_{\mathcal{R}}(P,F) \;\; \leq \;\; L d_{\mathcal{FM}_1}(P,F) \tag{OA–7}$$

where $d_{\mathcal{FM}_1}$ is the first-order Fortet-Mourier metrics (see Rachev (1991)). It follows from (OA–6) and (OA–7) that

$$\phi_P(d(\tilde{q},\Psi(P)) \;\; \leq \;\; 2d_{\mathcal{R}}(P,F) \;\; \leq \;\; 2L d_{\mathcal{FM}_1}(P,F).$$

Furthermore, Rachev (1991) has shown that, for any $p \geq 1$,

$$d_{\mathcal{FM}_p}(P,F) \;\; \leq \;\; (1 + \mathbb{E}_P[\|\xi\|]^p + \mathbb{E}_F[\|\xi\|]^p)^{\frac{p-1}{p}} d_{\mathcal{W}_p}(P,F).$$

In particular, for $p = 1$ we have

$$d_{\mathcal{FM}_1}(P,F) \;\; \leq \;\; d_{\mathcal{W}_1}(P,F)$$

and thus

$$\phi_P(d(\tilde{q},\Psi(P))) \;\; \leq \;\; 2L d_{\mathcal{W}_1}(P,F).$$

It follows that

$$d(\tilde{q},\Psi(P)) \;\; \leq \;\; \overline{\phi}_P^{-1}(2L d_{\mathcal{W}_1}(P,F)).$$

$\square$

**Lemma 3** *Suppose that Assumptions 5 and 6 hold. Then, there exists a $c > 0$ such that*

$$\nabla L(h^k)^T M s^k \;\; \leq \;\; -c\|\nabla L(h^k)\|_M^2 \quad \forall\, k.$$

**Proof** Note that

$$\begin{aligned}\nabla L(h^k)^T M s^k &= \left(h_1^k - G(q^\bullet, x_{i-1}),\ h_2^k - G(q^\bullet, x_i)\right) M \left(G(q^k, x_{i-1}) - h_1^k,\ G(q^k, x_i) - h_2^k\right)^T \\ &= \left[G(q^k, x_{i-1}) - h_1^k\right]\left[G(q^\bullet, x_{i-1}) - h_1^k\right] + r\left[G(q^k, x_i) - h_2^k\right]\left[G(q^\bullet, x_i) - h_2^k\right].\end{aligned}$$

Consider the following cases regarding the sign of $G(q^\bullet, x_{i-1}) - h_1^k$ and $G(q^\bullet, x_i) - h_2^k$:

1. $G(q^\bullet, x_{i-1}) - h_1^k \geq 0$

2. $G(q^\bullet, x_{i-1}) - h_1^k < 0$

a. $G(q^\bullet, x_i) - h_2^k \geq 0$

b. $G(q^\bullet, x_i) - h_2^k < 0$

The combinations of these cases will be denoted by 1a, 2a, 1b and 2b. A key observation concerning the above cases is that, since $q^k = \psi(\hat{H}^k) = (\hat{H}^k)^{-1}(\gamma)$ we have, in case 1,

$$h_1^k = \hat{H}^k(x_{i-1}) \leq G(q^\bullet, x_{i-1}) < \gamma \implies q^k \geq q^\bullet \implies G(q^k, x) \geq G(q^\bullet, x)\ \forall x, \tag{OA–8}$$

whereas in case b we have

$$h_2^k = \hat{H}^k(x_i) > G(q^\bullet, x_i) \geq \gamma \implies q^k \leq q^\bullet \implies G(q^k, x) \leq G(q^\bullet, x)\ \forall x. \tag{OA–9}$$

In both cases, the right-most implications follow from the assumption that $G(q)$ is stochastically decreasing in $q$.

We now analyze the four cases above:

Case 1b: From (OA–8) and (OA–9) we have that $q^k = q^\bullet$ and hence

$$\begin{aligned}&\left[G(q^k, x_{i-1}) - h_1^k\right]\left[G(q^\bullet, x_{i-1}) - h_1^k\right] + r\left[G(q^k, x_i) - h_2^k\right]\left[G(q^\bullet, x_i) - h_2^k\right] \\ &= \left[G(q^\bullet, x_{i-1}) - h_1^k\right]^2 + r\left[G(q^\bullet, x_i) - h_2^k\right]^2 \\ &\geq c\|\nabla L(h^k)\|_M^2\end{aligned}$$

for any $c \in (0, 1]$.

Case 2b: From (OA–9) we have that, for any $c \in (0, 1)$,

$$h_1^k > G(q^\bullet, x_{i-1}) = \frac{G(q^\bullet, x_{i-1}) - cG(q^\bullet, x_{i-1})}{1-c} \geq \frac{G(q^k, x_{i-1}) - cG(q^\bullet, x_{i-1})}{1-c}$$

and thus

$$G(q^k, x_{i-1}) - h_1^k \leq c\left[G(q^\bullet, x_{i-1}) - h_1^k\right] < 0.$$

Moreover, (OA–9) implies that $G(q^k, x_i) - h_2^k \leq G(q^\bullet, x_i) - h_2^k < 0$. It follows that

$$\begin{aligned}
&\left[G(q^k, x_{i-1}) - h_1^k\right]\left[G(q^\bullet, x_{i-1}) - h_1^k\right] + r\left[G(q^k, x_i) - h_2^k\right]\left[G(q^\bullet, x_i) - h_2^k\right] \\
&\geq\ c\left[G(q^\bullet, x_{i-1}) - h_1^k\right]^2 + r\left[G(q^\bullet, x_i) - h_2^k\right]^2 \\
&\geq\ c\|\nabla L(h^k)\|_M^2
\end{aligned}$$

for any $c \in (0,1)$.

Case 1a: From (OA–8) we have that, for any $c \in (0,1)$,

$$h_2^k \ \leq\ G(q^\bullet, x_i) \ =\ \frac{G(q^\bullet, x_i) - cG(q^\bullet, x_i)}{1-c} \ \leq\ \frac{G(q^k, x_i) - cG(q^\bullet, x_i)}{1-c}$$

and thus

$$G(q^k, x_i) - h_2^k \ \geq\ c\left[G(q^\bullet, x_i) - h_2^k\right] \ \geq\ 0.$$

Moreover, (OA–8) implies that $G(q^k, x_{i-1}) - h_1^k \geq G(q^\bullet, x_{i-1}) - h_1^k \geq 0$. It follows that

$$\begin{aligned}
&\left[G(q^k, x_{i-1}) - h_1^k\right]\left[G(q^\bullet, x_{i-1}) - h_1^k\right] + r\left[G(q^k, x_i) - h_2^k\right]\left[G(q^\bullet, x_i) - h_2^k\right] \\
&\geq\ \left[G(q^\bullet, x_{i-1}) - h_1^k\right]^2 + cr\left[G(q^\bullet, x_i) - h_2^k\right]^2 \\
&\geq\ c\|\nabla L(h^k)\|_M^2
\end{aligned}$$

for any $c \in (0,1)$.

Case 2a: We shall show that there exists a constant $c \in (0,1)$ such that

$$\frac{\left[G(q^k, x_{i-1}) - h_1^k\right]\left[G(q^\bullet, x_{i-1}) - h_1^k\right] + r\left[G(q^k, x_i) - h_2^k\right]\left[G(q^\bullet, x_i) - h_2^k\right]}{\left[G(q^\bullet, x_{i-1}) - h_1^k\right]^2 + r\left[G(q^\bullet, x_i) - h_2^k\right]^2} \ \geq\ c \qquad \text{(OA–10)}$$

for all $k$.

Suppose initially that $h_1^k = \hat{H}^k(x_{i-1}) \geq \gamma > G(q^\bullet, x_{i-1})$. This implies that $q^k < q^\bullet$ and hence $G(q^k, x) \leq G(q^\bullet, x)$ for all $x$. Moreover, since $h_1^k \geq \gamma$, we have that

$$G(q^\bullet, x_{i-1}) - h_1^k \ \leq\ G(q^\bullet, x_{i-1}) - \gamma \ =\ -\Delta_1.$$

Also, since $\gamma \leq h_1^k \leq h_2^k \leq G(q^\bullet, x_i)$, we have

$$G(q^\bullet, x_i) - h_2^k \ \leq\ G(q^\bullet, x_i) - \gamma \ =\ \Delta_2.$$

Finally, note that

$$|G(q^\bullet, x_{i-1}) - h_1^k| \ \leq\ |G(q^\bullet, x_{i-1}) - G(q^\bullet, x_i)| \ =\ \Delta_1 + \Delta_2$$

and

$$\begin{aligned} G(q^k, x_i) - h_2^k &\geq G(q^k, x_{i-1}) + \varepsilon - G(q^\bullet, x_{i-1}) + G(q^\bullet, x_{i-1}) - h_2^k \\ &\geq G(q^k, x_{i-1}) - G(q^\bullet, x_{i-1}) - (G(q^\bullet, x_i) - G(q^\bullet, x_{i-1})) + \varepsilon \\ &= (G(q^k, x_{i-1}) - h_1^k) - (G(q^\bullet, x_{i-1}) - h_1^k) - (\Delta_1 + \Delta_2 - \varepsilon) \end{aligned}$$

Thus, a lower bound on the minimum value of the term on the left-hand side of (OA–10) can be determined by solving the minimization problem

$$\min \frac{uv + rzw}{v^2 + rw^2} \tag{OA–11}$$

s.t.

$$u \leq v < 0 \tag{OA–12}$$

$$0 \leq w \ \leq \Delta_2 \tag{OA–13}$$

$$\Delta_1 \leq |v| \ \leq \Delta_1 + \Delta_2 \tag{OA–14}$$

$$z \geq \ u - v - (\Delta_1 + \Delta_2 - \varepsilon). \tag{OA–15}$$

We solve problem (OA–11)-(OA–15) analytically by solving for one variable at a time:

- For each feasible $u, v, w$, it is clear from (OA–15) that the optimal $z$ is given by $z^*(u,v) = u - v - (\Delta_1 + \Delta_2 - \varepsilon)$ since $w \geq 0$.

- For each feasible $u, v$, define the function $f(w) := (uv + rz^*(u,v)w)/(v^2 + rw^2)$. We claim that $\min\{f(w) : 0 \leq w \ \leq \Delta_2\}$ occurs at $w^* := \Delta_2$. Indeed, note that

$$f'(w) \ = \ \frac{rz^*(u,v)[v^2 - rw^2] - 2ruvw}{(v^2 + rw^2)^2},$$

  which implies that the sign of $f'(w)$ is given by the sign of the quadratic function $q(w) := z^*(u,v)[v^2 - rw^2] - 2uvw$. Since $z^*(u,v) < 0$ for all feasible $u, v$, it follows that $q(\cdot)$ is convex. Moreover, we have $q(\Delta_2) = z^*(u,v)[v^2 - r\Delta_2^2] - 2uv\Delta_2 < 0$ since $z^*(u,v) < 0$, $v^2 \geq r\Delta_2^2$ (from (OA–14), (30), and assumption 31), and $uv\Delta_2 > 0$ (from (OA–12)). Thus, $q(\cdot)$ — and therefore $f'(\cdot)$ — has negative sign on $[0, \Delta_2]$, which proves our claim.

- For each feasible $v$, define the function

$$g(u) \ := \ \frac{uv + rz^*(u,v)w^*}{v^2 + r(w^*)^2} \ = \ \frac{u(v + r\Delta_2) - r(v + \Delta_1 + \Delta_2 - \varepsilon)\Delta_2}{v^2 + r\Delta_2^2}$$

  It is clear from (30), (OA–12), and (OA–14) that $u \leq v \leq -\Delta_1 = -r\Delta_2 < 0$, so $g(\cdot)$ is minimized at $u^*(v) := v$.

- Finally, consider the function

$$h(v) \;:=\; \frac{u^*(v)v + rz^*(u^*(v),v)w^*}{v^2 + r(w^*)^2} \;=\; \frac{v^2 - r(\Delta_1+\Delta_2)\Delta_2}{v^2 + r\Delta_2^2} \;=\; 1 - \frac{r(\Delta_1+\Delta_2-\varepsilon)\Delta_2 + r\Delta_2^2}{v^2 + r\Delta_2^2}.$$

  It follows that the minimum of $h(\cdot)$ over the region (OA–14) is achieved at $v^* := -\Delta_1$.

It follows from the above developments that the optimal value of (OA–11)-(OA–15) is given by

$$\theta^* \;:=\; \frac{u^*(v^*)v^* + rz^*(u^*(v^*),v^*)w^*}{(v^*)^2 + r(w^*)^2} \;=\; \frac{\Delta_1^2 - r(\Delta_1+\Delta_2-\varepsilon)\Delta_2}{\Delta_1^2 + r\Delta_2^2}.$$

It follows that $\theta^* > 0$ if and only if $\Delta_1^2 > r(\Delta_1+\Delta_2-\varepsilon)\Delta_2$. Since $r = \Delta_1/\Delta_2$, the inequality holds if $\varepsilon > \Delta_2$, a condition that is satisfied by Assumption 6.

Now, suppose that $h_1^k = \hat{H}^k(x_{i-1}) < \gamma$. If $h_2^k = \hat{H}^k(x_i) \geq \gamma$, then we have $q^k = q^\bullet$ and thus the analysis is identical to that of Case 1b. Therefore, we only need to consider the case $\hat{H}^k(x_i) < \gamma$, which implies that $q^k > q^\bullet$ and hence $G(q^k,x) \geq G(q^\bullet,x)$ for all $x$. Moreover, since $h_2^k < \gamma$, we have that

$$G(q^\bullet,x_i) - h_2^k \;\;>\;\; G(q^\bullet,x_i) - \gamma \;\;=\;\; \Delta_2. \qquad \text{(OA–16)}$$

Also, since $\gamma > h_2^k \geq h_1^k \geq G(q^\bullet,x_{i-1})$, we have

$$0 \;\geq\; G(q^\bullet,x_{i-1}) - h_1^k \;>\; G(q^\bullet,x_{i-1}) - \gamma \;=\; -\Delta_1.$$

Finally, note that

$$G(q^\bullet,x_i) - h_2^k \;\leq\; G(q^\bullet,x_i) - G(q^\bullet,x_{i-1}) \;=\; \Delta_1 + \Delta_2$$

and

$$\begin{aligned} G(q^k,x_{i-1}) - h_1^k \;&\leq\; G(q^k,x_i) - \varepsilon - G(q^\bullet,x_i) + G(q^\bullet,x_i) - h_1^k \\ &\leq\; G(q^k,x_i) - G(q^\bullet,x_i) + (G(q^\bullet,x_i - G(q^\bullet,x_{i-1})) - \varepsilon \\ &=\; (G(q^k,x_i) - h_2^k) - (G(q^\bullet,x_i) - h_2^k) + (\Delta_1 + \Delta_2 - \varepsilon) \end{aligned}$$

Thus, the minimum value of the term on the left-hand side of (OA–10) can be determined by solving the minimization problem

$$\min \frac{uv + rzw}{v^2 + rw^2} \qquad \text{(OA–17)}$$

s.t.

$$0 \leq w \leq z \qquad \text{(OA–18)}$$

$$0 \geq v \;> -\Delta_1 \qquad \text{(OA–19)}$$

$$\Delta_2 < w \ \leq \Delta_1 + \Delta_2 \tag{OA–20}$$

$$u \leq \ z - w + (\Delta_1 + \Delta_2 - \varepsilon). \tag{OA–21}$$

As before, we can solve problem (OA–17)-(OA–21) analytically by solving for one variable at a time:

- For each feasible $v, z, w$, it is clear from (OA–21) that the optimal $u$ is given by $u^*(z, w) = z - w + (\Delta_1 + \Delta_2 - \varepsilon)$ since $v \leq 0$.

- For each feasible $z, w$, define the function $f(v) := (u^*(z, w)v + rzw)/(v^2 + rw^2)$. We claim that $\min\{f(v) : 0 \geq v \ \geq -\Delta_1\}$ occurs at $v^* := -\Delta_1$. Indeed, note that

  $$f'(v) \ = \ \frac{u^*(z, w)[rw^2 - v^2] - 2rzvw}{(v^2 + rw^2)^2},$$

  which implies that the sign of $f'(v)$ is given by the sign of the quadratic function $q(v) := u^*(z, w)[rw^2 - v^2] - 2rzvw$. Since $u^*(z, w) > 0$ for all feasible $z, w$, it follows that $q(\cdot)$ is strictly concave. Moreover, since $q(0) = u^*(z, w)rw^2 \geq 0$, we conclude that $q(\cdot)$ — and therefore $f'(\cdot)$ — has positive sign on $[-\Delta_1, 0]$, which proves our claim.

- For each feasible $w$, define the function

  $$g(z) \ := \ \frac{u^*(z, w)v^* + rzw}{(v^*)^2 + rw^2} \ = \ \frac{z(rw - \Delta_1) + (w - \Delta_1 - \Delta_2 - \varepsilon)\Delta_1}{\Delta_1^2 + rw^2}$$

  It is clear from (31), (30), (OA–18) and (OA–20) that $rw \geq r\Delta_2 = \Delta_1 > 0$, so $g(\cdot)$ is minimized at $z^*(w) := w$.

- Finally, consider the function

  $$h(w) \ := \ \frac{u^*(z^*(w), w)v^* + rz^*(w)w}{(v^*)2 + rw^2)} \ = \ \frac{rw^2 - (\Delta_1 + \Delta_2 - \varepsilon)\Delta_1}{\Delta_1^2 + rw^2} \ = \ 1 - \frac{(\Delta_1 + \Delta_2 - \varepsilon)\Delta_1 + \Delta_1^2}{\Delta_1^2 + rw^2}.$$

  It follows that the minimum of $h(\cdot)$ over the region (OA–20) is achieved at $w^* := \Delta_2$.

It follows from the above developments that the optimal value of (OA–17)-(OA–21) is given by

$$\theta^* \ := \ \frac{u^*(z^*(w^*), w^*)v^* + rz^*(w^*)w^*}{(v^*)^2 + r(w^*)^2} \ = \ \frac{r\Delta_2^2 - (\Delta_1 + \Delta_2 - \varepsilon)\Delta_1}{\Delta_1^2 + r\Delta_2^2}.$$

It follows that $\theta^* > 0$ if and only if $r\Delta_2^2 > (\Delta_1 + \Delta_2 - \varepsilon)\Delta_1$. Since $r = \Delta_1/\Delta_2$, the inequality holds if $\varepsilon > \Delta_1$, a condition that is satisfied by Assumption 6. □