A Line Search Exact Penalty Method Using Steering Rules

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Abstract

Line search algorithms for nonlinear programming must include safeguards to enjoy global convergence properties. This paper describes an exact penalization approach that extends the class of problems that can be solved with line search SQP methods. In the new algorithm, the penalty parameter is adjusted at every iteration to ensure sufficient progress in linear feasibility and to promote acceptance of the step. A trust region is used to assist in the determination of the penalty parameter (but not in the step computation). It is shown that the algorithm enjoys favorable global convergence properties. Numerical experiments illustrate the behavior of the algorithm on various difficult situations.

1 Introduction

Exact penalty methods have proved to be effective techniques for solving difficult nonlinear programs. They overcome the difficulties posed by inconsistent constraint linearizations [13] and are successful in solving certain classes of problems in which standard constraint qualifications are not satisfied [2, 3, 22, 20, 10, 18]. Despite their appeal, it has proved difficult to design penalty methods that perform well over a wide range of problems; the main difficulty lies in choosing appropriate values of the penalty parameter. Various approaches proposed in the literature update the penalty parameter only if convergence to an undesirable point appears to be taking place; see e.g. [28, 19] and the references therein. This can result in inefficient behavior and requires heuristics to determine when to change the penalty parameter.

Recently, a new strategy for updating the penalty parameter has been proposed in the context of trust region methods [6, 8]. In that approach, the penalty parameter is selected

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at every iteration so that sufficient progress toward feasibility and optimality is guaranteed, to first order. This requires that an auxiliary subproblem (a linear program) be solved in certain cases. A particular implementation of this approach has been incorporated into the active-set method of the KNITRO [9, 30] package and has proved to be effective in practice. The technique just mentioned relies on the fact that the optimization algorithm is of the trust region kind.

In this paper we describe *line search* penalty methods for nonlinear programming. Unlike trust region methods, which control the quality and length of the steps, line search methods can produce very large and unproductive search directions in the neighborhood of points where standard regularity conditions are not satisfied, and this situation can lead to failures that would not occur with a trust region method. By relaxing the constraints and ensuring that steady progress toward the solution is made, the proposed method enjoys the same type of global convergence properties as trust region methods. The global analysis thus shows that use of exact penalty methods can have a regularizing effect without a trust region or an explicit regularization term.

To achieve these goals, the algorithm solves a linear program with an auxiliary trust region that helps determine the adequacy of the penalty parameter. The algorithm is nevertheless a pure line search method because the step computation does not depend on the auxiliary trust region—only the choice of the penalty parameter depends on it. In fact the the auxiliary trust region radius may be fixed at an arbitrary value without affecting convergence properties.

The new algorithm incurs an additional cost compared with classical line search methods. At those iterations in which the penalty parameter must be adjusted, an auxiliary linear program must be solved and the SQP step must be recomputed one or more times using larger values of the penalty parameter. This extra cost may, however, not be significant because warms starts can be employed in the solution of these additional quadratic programs. Furthermore, the hope is that the new strategy yields savings in iterations and improves the robustness of the method. An attractive feature of the new algorithm is that it treats all problems (regular or deficient) equally and does not need to resort to special iterations when progress is not achieved.

In the next section, we present the new line search SQP algorithm, giving particular attention to the dynamic strategy for updating the penalty parameter. The convergence properties of the algorithm are analyzed in Section 3, and numerical experiments are reported in Section 4.

2 A Line Search Penalty Method

Penalty methods attack the general nonlinear programming problem

$$\begin{aligned} \text{minimize} \quad & f(x) && \text{(2.1a)} \\ \text{subject to} \quad & g_i(x) \geq 0, \quad i \in \mathcal{I}, && \text{(2.1b)} \\ & h_i(x) = 0, \quad i \in \mathcal{E} && \text{(2.1c)} \end{aligned}$$

by performing an unconstrained minimization of the exact penalty function

$$\phi_\pi(x) = f(x) + \pi\, v(x), \tag{2.2}$$

where $v(x)$ is a measure of constraint violation and π is the penalty parameter. The penalty approach proposed in this paper is applicable to a variety of line search methods; for concreteness we focus our discussion on sequential quadratic programming.

We use the 1-norm of the constraint violation as the measure of infeasibility for the nonlinear program (2.1). Thus the penalty function is defined by (2.2) with

$$v(x) = \sum_{i\in\mathcal{I}} [g_i(x)]^- + \sum_{i\in\mathcal{E}} |h_i(x)|, \tag{2.3}$$

where $[g_i(x)]^- = \max\{-g_i(x), 0\}$ is the negative part of $g_i(x)$.

It is well known (see for example [21]) that when multipliers exist, stationary points of the nonlinear problem (2.1) are also stationary points of the exact penalty function (2.2) for all sufficiently large values of π. Conversely, and more important from the standpoint of practical penalty methods, any stationary point of the exact penalty function (2.2) that is feasible for problem (2.1) is a stationary point of (2.1).

Exact penalty methods attempt to find stationary points of the nonlinear program (2.1) by minimizing the penalty function (2.2), and use the exogenous penalty parameter π as a control to promote feasibility. Two key questions arise: a) How can we find stationary points of the nonsmooth exact penalty function ϕ_π, for a fixed value of π? b) How should we update the penalty parameter π?

The first of these issues is well understood [13]. We can search for stationary points of the penalty function by taking steps based on a quadratic model of ϕ_π. To define this model we first construct the following piecewise linear model of the measure of constraint violation v at an iterate x_k:

$$m_k(d) = \sum_{i\in\mathcal{I}} [\nabla g_i(x_k)^T d + g_i(x_k)]^- + \sum_{i\in\mathcal{E}} |\nabla h_i(x_k)^T d + h_i(x_k)|. \tag{2.4}$$

Next, we define a piecewise quadratic model of ϕ_π at x_k as

$$q_k^\pi(d) = f(x_k) + \nabla f(x_k)^T d + \tfrac{1}{2} d^T W_k d + \pi\, m_k(d), \tag{2.5}$$

where W_k is a symmetric positive definite matrix that approximates the Hessian of the Lagrangian of the nonlinear program (2.1). We compute the search direction d_k by solving the problem

$$\underset{d}{\text{minimize}} \quad q_k^\pi(d). \tag{2.6}$$

In practice, we recast (2.6) as the smooth quadratic optimization program

$$\begin{aligned}
\underset{d,r,s,t}{\text{minimize}} \quad & f(x_k) + \nabla f(x_k)^T d + \tfrac{1}{2} d^T W_k d + \pi \textstyle\sum_{i\in\mathcal{E}} (r_i + s_i) + \pi \sum_{i\in\mathcal{I}} t_i & \text{(2.7a)}\\
\text{subject to} \quad & \nabla h_i(x_k)^T d + h_i(x_k) = r_i - s_i, \quad i \in \mathcal{E}, & \text{(2.7b)}\\
& \nabla g_i(x_k)^T d + g_i(x_k) \ge -t_i, \quad i \in \mathcal{I}, & \text{(2.7c)}\\
& r, s, t \ge 0. & \text{(2.7d)}
\end{aligned}$$

Once the solution d_k of problem (2.6) is found, a line search is performed in the direction d_k to ensure that sufficient decrease in the exact penalty function (2.2) is achieved at the new iterate.

The positive definiteness assumption on W_k is common in line search methods, where W_k is obtained through a quasi-Newton update or by adding (if necessary) a multiple of the identity to the Hessian of the Lagrangian of problem (2.1). Note that the quadratic subproblem (2.7) is always feasible, and we show in this paper that the introduction of the surplus variables r, s, t, together with the positive definiteness of W_k provide a regularization effect to the algorithm.

The second challenge in penalty methods concerns the selection of the penalty parameter. If π is too small, the penalty function (2.2) may be unbounded below, and the iterates will diverge unless the value of π is corrected in time. If π is too large, the efficiency of the penalty approach may be impaired [8]. The goal of this paper is to propose a dynamic strategy for updating the penalty parameter in a class of line search methods for nonlinear programming.

To describe this strategy, we denote the solution of (2.6) by $d_k(\pi)$ to stress its dependence on the penalty parameter. At iteration k, we first solve (2.6) using the current penalty parameter π_k, to obtain $d_k(\pi_k)$. We are content if the linearized constraints are satisfied, i.e., if $m_k(d_k(\pi_k)) = 0$. In this case, the penalty parameter is not changed and we define the search direction as $d_k \triangleq d_k(\pi_k)$. Thus, no regularization is needed in this case and the search direction d_k coincides with that computed by a classical SQP method (in which r, s, t are always zero).

On the other hand, if $m_k(d_k(\pi_k)) > 0$, we assess the adequacy of the current penalty parameter by computing the lowest possible violation of the linearized constraints in a neighborhood of the current iterate. This is done by solving the problem

$$\underset{d}{\text{minimize}} \quad m_k(d), \qquad \text{subject to} \quad \|d\|_\infty \leq \Delta_k, \tag{2.8}$$

where $\Delta_k > 0$ is given. This problem is equivalent to the linear program

$$\begin{aligned}
\underset{d,r,s,t}{\text{minimize}} \quad & \sum_{i\in\mathcal{E}}(r_i + s_i) + \sum_{i\in\mathcal{I}} t_i && \text{(2.9a)}\\
\text{subject to} \quad & \nabla h_i(x_k)^T d + h_i(x_k) = r_i - s_i, \;\; i \in \mathcal{E}, && \text{(2.9b)}\\
& \nabla g_i(x_k)^T d + g_i(x_k) \geq -t_i, \;\; i \in \mathcal{I}, && \text{(2.9c)}\\
& \|d\|_\infty \leq \Delta_k, && \text{(2.9d)}\\
& r, s, t \geq 0. && \text{(2.9e)}
\end{aligned}$$

We denote the solution of this problem by d_k^{LP}. A new penalty parameter $\pi_+ \geq \pi_k$ is now determined such that the solution $d_k(\pi_+)$ of problem (2.6) yields an improvement in linearized feasibility that is commensurate with that obtained by the step d_k^{LP}, as measured by the model m_k. This strategy is specified more precisely in Algorithm I, which is the new line search penalty method.

Algorithm I: Line Search Penalty SQP Method

Initial data: $x_1, \pi_1 > 0$, $\rho > 0$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \epsilon_1)$, $\tau \in (0, 1)$, $\eta \in (0, 1)$, and $0 < \Delta_{\min} \leq \Delta_1 \leq \Delta_{\max}$.

For $k = 1, 2, ...$

1. Find a search direction $d_k(\pi_k)$ by solving the subproblem (2.6) with $\pi = \pi_k$. If $d_k(\pi_k) = 0$ and $v(x_k) = 0$, STOP: x_k is a KKT point of problem (2.1).

2. If $m_k(d_k(\pi_k)) = 0$, set $\pi_+ = \pi_k$ and go to Step 6.

3. Solve the linear programming subproblem (2.8) to get d_k^{LP}. If

$$0 < m_k(0) = m_k(d_k^{\text{LP}}), \tag{2.10}$$

   STOP: x_k is an infeasible stationary point of the penalty function (2.2).

4. Update the penalty parameter:

   (a) If $m_k(d_k^{\text{LP}}) = 0$, find $\pi_+ \geq \pi_k + \rho$ and a corresponding vector $d_k(\pi_+)$ that solves (2.6), such that

$$m_k(d_k(\pi_+)) = 0. \tag{2.11}$$

   (b) Else set $\pi_+ = \pi_k$. If the following inequality does not hold

$$m_k(0) - m_k(d_k(\pi_+)) \geq \epsilon_1[m_k(0) - m_k(d_k^{\text{LP}})], \tag{2.12}$$

   then find $\pi_+ \geq \pi_k + \rho$ and a corresponding vector $d_k(\pi_+)$, such that (2.12) is satisfied.

5. If the following inequality does not hold

$$q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k(\pi_+)) \geq \epsilon_2 \pi_+[m_k(0) - m_k(d_k^{\text{LP}})], \tag{2.13}$$

   then increase π_+ (by at least ρ) as necessary, until the solution $d_k(\pi_+)$ of (2.6) satisfies (2.13).

6. Set $d_k = d_k(\pi_+)$, and let $0 < \alpha_k \leq 1$ be the first member of the sequence $\{1, \tau, \tau^2, \ldots\}$ such that

$$\phi_{\pi_+}(x_k) - \phi_{\pi_+}(x_k + \alpha_k d_k) \geq \eta \alpha_k [q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)]. \tag{2.14}$$

7. Set $\Delta_{k+1} \in [\Delta_{\min}, \Delta_{\max}]$.

8. Let $\pi_{k+1} = \pi_+$ and $x_{k+1} = x_k + \alpha_k d_k$.

It is worth emphasizing that Algorithm I is a line search method. The global convergence results established in the next section are based on the properties of line search methods together with the regularization effects of the penalty approach. The trust region constraint in (2.8) plays an indirect role, since it only influences the choice of the penalty parameter.

Note that since the Hessian W_k in the piecewise quadratic model (2.5) is positive definite, q_k^π is strictly convex. Therefore the solution of problem (2.6), which we denote by $d_k(\pi)$, is unique. The model $m(d)$ is convex, but not strictly convex.

The overall design of Algorithm I is based on the following three updating guidelines, which we call the *steering rules* and are an adaptation of the strategy given in [8] to the line search setting:

1. If it is possible to satisfy the linearized constraints in a neighborhood of the current iterate, we compute such a step. This is achieved by enforcing condition (2.11). In other words, if a classical SQP step exists and is not too long, we would like to use it.

2. If the linearized constraints are locally infeasible, we will be content with taking a step that achieves at least a fraction of the best possible local feasibility improvement. We impose this requirement through condition (2.12). Note that (2.12) could be satisfied with the current penalty parameter π_k, in which case no extra quadratic subproblems (2.6) need to be solved.

3. Not only feasibility but also improvement in the penalty function has to be commensurate with the improvement in feasibility obtained with d_k^{LP}. This is guaranteed by condition (2.13). We note that it is not necessary to re-solve problem (2.6) when condition (2.13) is violated because an appropriate value of the penalty parameter is readily computed. First, if $m_k(0) - m_k(d_k^{\text{LP}}) = 0$ then (2.13) is satisfied for any π because $d_k(\pi)$ is the minimizer of q_k^π. Otherwise, $m_k(0) > m_k(d_k^{\text{LP}})$, which implies that $m_k(0) > 0$. Let π_+ be the value of the penalty at the beginning of Step 5 of Algorithm I, let $d_+ = d_k(\pi_+)$ and define

$$\hat{\pi} = \frac{\frac{1}{2} d_+^T W_k d_+ + \nabla f(x_k)^T d_+}{(1-\epsilon_2) m_k(0) - m_k(d_+) + \epsilon_2 m_k(d_k^{\text{LP}})} + \rho. \tag{2.15}$$

   (Note that (2.11) or (2.12), together with the relations $0 < \epsilon_2 < \epsilon_1 \leq 1$, imply that the denominator is positive.) Then, by writing $\hat{d} = d(\hat{\pi})$ we have from (2.15) that

$$\begin{aligned} \epsilon_2 \hat{\pi} [m_k(0) - m_k(d_k^{\text{LP}})] &\leq -\nabla f(x_k)^T d_+ - \tfrac{1}{2} d_+^T W_k d_+ + \hat{\pi}[m_k(0) - m_k(d_+)] \\ &= q_k^{\hat{\pi}}(0) - q_k^{\hat{\pi}}(d_+) \\ &\leq q_k^{\hat{\pi}}(0) - q_k^{\hat{\pi}}(\hat{d}), \end{aligned}$$

   where the last inequality follows from the fact that $\hat{d}$ is the minimizer of $q_{\hat{\pi}}^k$. Therefore condition (2.13) is satisfied if the penalty parameter is given by (2.15).

We show in Lemma 3.5-(b) that if the penalty parameter is increased in Step 5 to satisfy (2.13), it will still satisfy conditions (2.11) and (2.12).

When the penalty parameter is updated in Algorithm I, the main cost is in the repeated solution of the quadratic program (2.6) for different values of π in Step 4(b); the linear program (2.8) is solved only once. We expect, however, that warm starts can greatly accelerate the solution of these quadratic programs and that the savings in total iterations will overcome any extra cost incurred in some iterations.

The choice of penalty parameter π_+ in Algorithm I ensures that d_k is a descent direction for the penalty function (see Lemma 3.5 (c)). Note that in the right-hand side of (2.14) we use the decrease in the piecewise quadratic model $q_k^{\pi_+}$, instead of the directional derivative of the penalty function. This accounts for possible kinks in the merit function near the current iterate that could be overlooked by a standard Armijo line search and result in jamming.

The constraint $\|d\|_\infty \leq \Delta_k$ in problem (2.8) is not a trust region in the usual sense, and the value of Δ_k is not critical to the performance of the algorithm. The function $m_k(d)$ is always bounded below and the radius Δ_k is used only to ensure that $||d_k^{\text{LP}}||$ is of reasonable size in case there is an unbounded ray of minimizers of $m_k(d)$. In fact, the radius Δ_k could be kept constant and the convergence properties of Algorithm I would not be affected. In practice, however, it may be advantageous to choose Δ_k based on local information of the problem, as discussed in Section 4.

3 Convergence analysis

In this section we study the global convergence properties of Algorithm I. We make the following assumptions about the sequence of iterates $\{x_k\}$ and the matrices W_k generated by the algorithm.

Assumptions I.

A1. The functions $f, g_i, i \in \mathcal{I}$, and $h_i, i \in \mathcal{E}$, are twice differentiable with bounded derivatives over a bounded convex set that contains the sequence $\{x_k\}$.

A2. The matrices W_k are uniformly positive definite and bounded above, i.e., there exist values $0 < \mu_{\min} < \mu_{\max}$ such that

$$\mu_{\min}\|p\|^2 \leq p^T W_k p \leq \mu_{\max}\|p\|^2, \tag{3.1}$$

for an $p \in \mathbb{R}^n$.

We denote the directional derivative of a function f at x in the direction p by $Df(x;p)$. A point x is said to be a stationary point of the penalty function if $D\phi_\pi(x;p) \geq 0$ for all directions p. A point $\hat{x}$ is called an *infeasible stationary point* for problem (2.1) if $v(\hat{x}) > 0$ and $Dv(\hat{x};p) \geq 0$ for all p. We say that problem (2.1) is *locally infeasible* if there is an infeasible stationary point for it.

The first lemma provides useful relationships between the directional derivatives of the functions ϕ_π and v and their local models, q^π and m.

Lemma 3.1 *Given a point x_k, the directional derivatives of v and ϕ_π along a vector p satisfy*

$$Dv(x_k;p) = Dm_k(0;p) \tag{3.2}$$

and

$$D\phi_\pi(x_k;p) = Dq_k^\pi(0;p). \tag{3.3}$$

Proof. Given x_k and a vector $d \in \mathbb{R}^n$, let us define the sets

$$\begin{aligned} \mathcal{G}_-^k(d) &= \{i \in \mathcal{I} : \nabla g_i(x_k)^T d + g_i(x_k) < 0\}, \\ \mathcal{G}_0^k(d) &= \{i \in \mathcal{I} : \nabla g_i(x_k)^T d + g_i(x_k) = 0\}, \\ \mathcal{G}_+^k(d) &= \{i \in \mathcal{I} : \nabla g_i(x_k)^T d + g_i(x_k) > 0\}, \end{aligned} \tag{3.4}$$

which determine a partition of $\mathcal{I}$. Similarly, we can define a partition $\mathcal{H}_-^k(d), \mathcal{H}_0^k(d)$ and $\mathcal{H}_+^k(d)$ of $\mathcal{E}$ induced by the value of $\nabla h_i(x_k)^T d + h_i(x_k)$.

The directional derivative of $m_k(\cdot)$ at d in the direction p is given by

$$\begin{aligned} Dm_k(d;p) &= \sum_{i \in \mathcal{G}_0^k(d)} [\,\nabla g_i(x_k)^T p\,]^- - \sum_{i \in \mathcal{G}_-^k(d)} \nabla g_i(x_k)^T p \\ &\quad + \sum_{i \in \mathcal{H}_+^k(d)} \nabla h_i(x_k)^T p + \sum_{i \in \mathcal{H}_0^k(d)} |\,\nabla h_i(x_k)^T p\,| - \sum_{i \in \mathcal{H}_-^k(d)} \nabla h_i(x_k)^T p. \end{aligned} \tag{3.5}$$

On the other hand, we have that

$$\begin{aligned} Dv(x_k;p) &= \sum_{i \in \mathcal{G}_0^k(0)} [\,\nabla g_i(x_k)^T p\,]^- - \sum_{i \in \mathcal{G}_-^k(0)} \nabla g_i(x_k)^T p \\ &\quad + \sum_{i \in \mathcal{H}_+^k(0)} \nabla h_i(x_k)^T p + \sum_{i \in \mathcal{H}_0^k(0)} |\,\nabla h_i(x_k)^T p\,| - \sum_{i \in \mathcal{H}_-^k(0)} \nabla h_i(x_k)^T p. \end{aligned}$$

The equality (3.2) follows by comparing this expression with (3.5).

Given a direction p, we have that $D\phi_\pi(x_k;p) = \nabla f(x_k)^T p + Dv(x_k;p)$. Also, for any d we have that $Dq_k^\pi(d;p) = (\nabla f(x_k) + W_k d)^T p + Dm_k(d;p)$. By evaluating $Dq_k^\pi(d;p)$ at $d = 0$ and using (3.2) we obtain (3.3). □

The next result is well known (see e.g. [4, 21]). For a given point x_*, we define the model (2.5) at x_* by q_*^π, and denote its solution by $d_*(\pi)$. Similarly, m_* denotes the model (2.4) at x_*.

Theorem 3.2 *The following three statements are true:*

(a) x_* is a stationary point of the penalty function $\phi_\pi(x)$ if and only if $d_*(\pi) = 0$ solves problem (2.6).

(b) If x_* is a stationary point of $\phi_\pi(x)$ and $v(x_*) = 0$, then x_* is a KKT point of (2.1).

(c) x_* is a stationary point of the infeasibility measure $v(x)$ if and only if, for any $\Delta > 0$, any solution d^{LP} of the linear feasibility problem (2.8) *satisfies*

$$m_*(d^{LP}) = m_*(0). \tag{3.6}$$

Proof. (a) By definition, $d_*(\pi) = 0$ is the minimizer of $q_*^\pi(d)$ if and only if $Dq_*^\pi(0;p) \geq 0$ for any direction p. The result follows from (3.3). (b) See [26, Theorem 17.4].

(c) Given $\Delta > 0$, let d^{LP} be a solution of (2.8). Since $d = 0$ obviously satisfies $\|d\|_\infty \leq \Delta$, we have that $m_*(d^{\text{LP}}) \leq m_*(0)$. Also, from (3.2) we have that x_* is stationary for $v(x)$ if and only if 0 is stationary for $m_*(d)$, which holds (by convexity of m_*) if and only if 0 is an unconstrained global minimizer of $m_*(d)$. Therefore, $m_*(0) \leq m_*(d)$ for any d, and in particular for $d = d^{\text{LP}}$. We conclude that (3.6) holds. □

This theorem justifies the stopping tests in Algorithm I. If Algorithm I stops at Step 1, Theorem 3.2 (a) and (b) imply that the current iterate x_k is a KKT point of the nonlinear program (2.1). If the algorithm stops at Step 3, then Theorem 3.2 (c) and $v(x_k) > 0$ imply that x_k is an infeasible stationary point. If neither stop test is satisfied, we need to show that that Algorithm I will generate a new iterate x_{k+1} and that it is always possible to meet the requirements in Steps 4 and 6. This is done in Lemma 3.5; first we need to establish two auxiliary results.

Lemma 3.3 *Suppose that Assumptions I hold. At any given iterate x_k, and for all $\pi > 0$, the minimizers $d_k(\pi)$ of $q_k^\pi(d)$ are contained in a compact ball*

$$\mathcal{B}^k = \{d : \|d\| \leq r^k\} \quad \textit{with} \quad r^k = \kappa_1 + \kappa_2 \|\bar{d}_k\|, \tag{3.7}$$

for some global constants κ_1 and κ_2 and where $\bar{d}_k$ denotes the minimum norm minimizer of $m_k(d)$.

Proof. Let $\bar{d}_k$ be the minimum norm minimizer of $m_k(d)$; it is well defined because $m_k(d)$ is a piece-wise linear convex function that is bounded below. If $\|d\|$ is large enough that $\mu_{\min}\|d\| \geq 8\|\nabla f(x_k)\|$ and $\mu_{\min}\|d\|^2 \geq 2\mu_{\max}\|\bar{d}\|^2$, then by (3.1)

$$\begin{aligned}
-\nabla f(x_k)^T d + \nabla f(x_k)^T \bar{d} + \tfrac{1}{2}\bar{d}^T W_k \bar{d} \quad &\leq \quad \|\nabla f_k\|\|d\| + \|\nabla f_k\|\|\bar{d}\| + \tfrac{\mu_{\max}}{2}\|\bar{d}\|^2 \\
&\leq \quad \tfrac{\mu_{\min}}{8}\|d\|^2 + \tfrac{\mu_{\min}}{8}\left[\tfrac{\mu_{\min}}{2\mu_{\max}}\right]^{\frac{1}{2}}\|d\|^2 + \tfrac{\mu_{\min}}{4}\|d\|^2 \\
&< \quad \tfrac{\mu_{\min}}{2}\|d\|^2 \\
&\leq \quad \tfrac{1}{2}d^T W d,
\end{aligned}$$

and therefore

$$f(x_k) + \nabla f(x_k)^T d + \tfrac{1}{2}d^T W_k d \quad > \quad f(x_k) + \nabla f(x_k)^T \bar{d} + \tfrac{1}{2}\bar{d}^T W_k \bar{d}.$$

Thus, for all

$$\|d\| > \max\{8\|\nabla f(x_k)\|/\mu_{min}, \sqrt{2\mu_{\max}/\mu_{\min}}\|\bar{d}\|\} \tag{3.8}$$

and all $\pi \geq 0$ we have

$$\begin{aligned}
q_k^\pi(d) \quad &= \quad f(x_k) + \nabla f(x_k)^T d + \tfrac{1}{2}d^T W_k d + \pi m_k(d) \\
&> \quad f(x_k) + \nabla f(x_k)^T \bar{d} + \tfrac{1}{2}\bar{d}^T W_k \bar{d} + \pi m_k(\bar{d}) = q_k^\pi(\bar{d}),
\end{aligned}$$

since $\bar{d}$ is a minimizer of m_k. Therefore, no minimizer $d_k(\pi)$ can be larger in norm than the right hand side of (3.8). To establish (3.7), we let $\tilde{\kappa}_1$ be an upper bound for $\|\nabla f_k\|$, define $\kappa_1 = 8\tilde{\kappa}_1/\mu_{\min}$ and $\kappa_2 = \sqrt{2\mu_{\max}/\mu_{\min}}$. □

The following result shows that by choosing π sufficiently large, the direction $d(\pi)$ can attain any achievable level of linear feasibility.

Lemma 3.4 *At any iterate x_k, for all all π sufficiently large the minimizer $d_k(\pi)$ of q_k^π, also minimizes $m_k(d)$.*

Proof. It is clear from (2.4) that the piecewise linear function $m_k(d)$ may be expressed as

$$m_k(d) = \max_{j\in\mathcal{M}}\{a_j^T d + \beta_j\},$$

where $\mathcal{M}$ is a finite index set, the vectors a_j are in $\mathbb{R}^n$ and the β_j are scalars. It follows (see e.g. [28, p.66]) that for any $d \in \mathbb{R}^n$ the subdifferential $\partial m_k(d)$ is the convex hull of the active support functions at d, i.e.

$$\partial m_k(d) = \text{conv}\{a_j | j \in \mathcal{M} \text{ and } a_j^T d + \beta_j = m_k(d)\}.$$

Since a vector d minimizes the convex function m_k if and only if $\partial m_k(d)$ contains 0, then if d is not a minimizer of m_k it follows that $\partial m_k(d)$ is a closed convex set not containing 0, which implies that

$$\sigma(d) \triangleq \min\{\|a\| | a \in \partial m_k(d)\} > 0.$$

Now since $\partial m_k(d)$ is defined by the active set $\{j \in \mathcal{M} | a_j^T d + \beta_j = m_k(d)\}$ and only a finite number of possible sets $\partial m_k(d)$ exist as d ranges over $\mathbb{R}^n$, the function $\sigma(d)$ takes on only a finite set of values. Therefore

$$\sigma_k \triangleq \min_{d\in\mathbb{R}^n}\{\sigma(d) | 0 \notin \partial m_k(d)\} > 0. \tag{3.9}$$

Now by Lemma 3.3, there is a compact ball $\mathcal{B}^k$ containing the minimizers of q_k^π, for all π. Therefore, for all such minimizers $d_k(\pi)$ we have $\|\nabla f(x_k) + W_k d_k(\pi)\| \le \beta_k$, for some constant β_k. Consider some $\tilde{\pi} > \beta_k/\sigma_k$ and the minimizer $d_k(\tilde{\pi})$ of $q_k^{\tilde{\pi}}$. Any vector $g \in \partial q_k^{\tilde{\pi}}(d_k(\tilde{\pi}))$ may be expressed as

$$g = \nabla f(x_k) + W_k d_k(\tilde{\pi}) + \tilde{\pi} a \quad \text{for some} \quad a \in \partial m_k(d_k(\tilde{\pi})).$$

If $d_k(\tilde{\pi})$ does not minimize m_k, it follows from (3.9) that

$$\|g\| \ge \tilde{\pi}\|a\| - \|\nabla f(x_k) + W_k d_k(\pi)\| \ge \tilde{\pi}\sigma_k - \beta_k > 0.$$

This means $0 \notin \partial q_k^{\tilde{\pi}}$ contradicting the fact that $d_k(\tilde{\pi})$ is a minimizer of $q_k^{\tilde{\pi}}$. Therefore $d_k(\tilde{\pi})$ must minimize m_k. □

For the remainder of the analysis, it is useful to define the model

$$q_k^f(d) = f_k + \nabla f_k^T d + \tfrac{1}{2} d^T W_k d, \tag{3.10}$$

so that

$$q_k^{\pi_k}(d) = q_k^f(d) + \pi_k m_k(d). \tag{3.11}$$

We now prove that Algorithm I is well defined.

Lemma 3.5 *Suppose that x_k is neither a KKT point of nonlinear program* (2.1) *nor a stationary point of the infeasibility measure $v(x)$. Then,*

a) If Algorithm I executes Step 4, it is always possible to find a value π_+ and a corresponding vector $d_k(\pi_+)$ such that condition (2.11) *holds if $m_k(d^{LP}) = 0$, or condition* (2.12) *holds if $m_k(d^{LP}) > 0$.*

b) If Algorithm I executes Step 5, it is always possible to find a value π_+ and a corresponding vector $d_k(\pi_+)$ such that condition (2.13) *holds.*

c) At Step 6, d_k is a descent direction for $\phi_{\pi_+}(x)$ at x_k. Therefore, there exists α_k such that condition (2.14) *is satisfied.*

Proof. (a) By Lemma 3.4, for π_+ sufficient large $d_k(\pi_+)$ is a minimizer of $m_k(d)$ and hence $m_k(d_k(\pi_+)) \le m_k(d_k^{\rm LP})$; this implies (2.11). Moreover, since $\epsilon_1 \le 1$, we have that

$$m_k(0) - m_k(d_k(\pi_+)) \ge m_k(0) - m_k(d_k^{\rm LP}) \ge \epsilon_1[m_k(0) - m_k(d_k^{\rm LP})],$$

so that (2.12) is satisfied.

b) We have already shown in Section 2 that (2.13) is satisfied if the penalty parameter is chosen by (2.15). We now show that if the penalty parameter is increased in Step 5, this new value of π still satisfies (2.11) and (2.12).

Let $\pi_2 > \pi_1$. Then by (3.11) and the fact that $d_k(\pi_k)$ is the minimizer of q_π^k, we have

$$\begin{aligned} q_k^f(d_k(\pi_1)) + \pi_2 m_k(d_k(\pi_1)) &\ge q_k^f(d_k(\pi_2)) + \pi_2 m_k(d_k(\pi_2)) && (3.12)\\ q_k^f(d_k(\pi_1)) + \pi_1 m_k(d_k(\pi_1)) &\le q_k^f(d_k(\pi_2)) + \pi_1 m_k(d_k(\pi_2)). && (3.13) \end{aligned}$$

Hence

$$(\pi_2 - \pi_1) m_k(d_k(\pi_1)) \ge (\pi_2 - \pi_1) m_k(d_k(\pi_2)),$$

which implies that $m_k(d_k(\pi_1)) \ge m_k(d_k(\pi_2))$. We conclude that $m_k(d_k(\pi))$ cannot increase as π is increased.

(c) If x_k is neither stationary for ϕ_{π_k} nor for $v(x)$, then at Step 6 we must have $d_k \ne 0$. This is a consequence of the logic of Algorithm I and of Theorem 3.2, parts (a) and (c). Recall that d_k is defined to be the minimizer of $q_k^{\pi+}$ and hence $q_k^{\pi+}(d_k) \le q_k^{\pi+}(0)$. Furthermore, since $q_k^{\pi+}(d)$ is strictly convex and $d_k \ne 0$, we have that $q_k^{\pi+}(d_k) < q_k^{\pi+}(0)$ and thus d_k is a descent direction for $q_k^{\pi+}$ at 0. By (3.3), we have

$$D\phi_{\pi_+}(x_k; d_k) \quad = \quad Dq_k^{\pi+}(0; d_k) < 0,$$

and therefore d_k is also a descent direction for $\phi_{\pi_+}(x)$ at x_k. Since the constant η is chosen in (0,1), it follows that $\phi_{\pi_+}(x_k + \alpha d_k) < \phi_{\pi_+}(x_k) + \alpha\eta D\phi_{\pi_+}(x_k; d_k)$ for all sufficiently small $\alpha > 0$, or

$$\phi_{\pi_+}(x_k) - \phi_{\pi_+}(x_k + \alpha d_k) > -\alpha\eta D\phi_{\pi_+}(x_k; d_k).$$

From (3.3) and the convexity of $q_{\pi_+}^k(d)$, we have that

$$-\alpha\eta D\phi_{\pi_+}(x_k; d_k) = -\alpha\eta D q_{\pi_+}^k(0; d_k) \quad \ge \quad \alpha\eta[q_k^{\pi+}(0) - q_k^{\pi+}(d_k)].$$

We conclude that there always exists a sufficiently small steplength α_k that satisfies condition (2.14). □

We now establish the first convergence result. It gives conditions under which Algorithm I identifies stationary points of the penalty function.

Theorem 3.6 *Suppose that Algorithm I generates an infinite sequence of iterates $\{x_k\}$ and that Assumptions I hold. Suppose also that $\{\pi_k\}$ is bounded, so that $\pi_k = \bar{\pi}$ for all k large. Then any accumulation point x_* of $\{x_k\}$ is a stationary point of the penalty function $\phi_{\bar{\pi}}(x)$.*

Proof. Note that, whenever π is increased in Algorithm I, it is increased by at least $\rho > 0$. Therefore, if $\{\pi_k\}$ is bounded, there is a value $\bar{\pi}$ such that $\pi_k = \bar{\pi}$ for all sufficiently large k.

Let x_* be a limit point of $\{x_k\}$, which exists because Assumption A1 states that $\{x_k\}$ is bounded. Let $\mathcal{K}$ be an infinite subset of indices such that $\{x_k\}_{k\in\mathcal{K}} \to x_*$. The sequence of matrices $\{W_k\}$ is also bounded, by Assumption A2. We restrict $\mathcal{K}$ if necessary so that $\{W_k\}_{k\in\mathcal{K}} \to W_*$, where W_* is a limit point of $\{W_k\}$. Then, from the continuity of the functions f, g_i, h_i and their gradients, and from the definition (2.5), we have that the sequence of models $q_k^{\bar{\pi}}, k \in \mathcal{K}$ converges (pointwise) to a function $q_*^{\bar{\pi}}$.

Each of the functions $q_k^{\bar{\pi}}$, as well as the limiting function $q_*^{\bar{\pi}}$, are strictly convex and have a unique minimizer. We want to prove that the minimizer of $q_*^{\bar{\pi}}(d)$ is $d_* = 0$, for then Theorem 3.2 (a) implies that x_* is a stationary point of $\phi_{\bar{\pi}}$.

We proceed by contradiction. Assume that $d_* \neq 0$, or equivalently, that $q_*^{\bar{\pi}}(0) - q_*^{\bar{\pi}}(d_*) > 0$. From the pointwise convergence of the functions $q_k^{\bar{\pi}}$, we know that there exists $\epsilon > 0$ such that

$$q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_*) \to q_*^{\bar{\pi}}(0) - q_*^{\bar{\pi}}(d_*) = 2\epsilon > 0.$$

Therefore, there is a number k_0 such that for all $k \geq k_0$, with $k \in \mathcal{K}$, we have that $\pi_k = \bar{\pi}$ and

$$q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k) \geq q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_*) \geq \epsilon. \tag{3.14}$$

It is not difficult to show (see e.g. Lemma 3.4 in [6]) that for any $\alpha \in [0, 1]$,

$$|\phi_{\bar{\pi}}(x_k + \alpha d_k) - q_k^{\bar{\pi}}(\alpha d_k)| \leq c_1 \|\alpha d_k\|^2, \tag{3.15}$$

for some positive constant c_1. Recalling that $q_k^{\bar{\pi}}$ is a convex function, noting that $\phi_{\bar{\pi}}(x_k) = q_k^{\bar{\pi}}(0)$, using (3.14) and (3.15), and assuming that $\alpha_k \leq (1-\eta)\epsilon/(c_1\|d_k\|^2)$, we obtain

$$\begin{aligned}
\phi_{\bar{\pi}}(x_k) - \phi_{\bar{\pi}}(x_k + \alpha_k d_k) &\geq [q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(\alpha_k d_k)] - \phi_{\bar{\pi}}(x_k + \alpha_k d_k) + q_k^{\bar{\pi}}(\alpha_k d_k) \\
&\geq \alpha_k[q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)] + c_1\alpha_k^2\|d_k\|^2 \\
&= \eta\alpha_k[q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)] + (1-\eta)\alpha_k[q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)] - c_1\alpha_k^2\|d_k\|^2 \\
&\geq \eta\alpha_k[q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)] + (1-\eta)\alpha_k\epsilon - c_1\alpha_k^2\|d_k\|^2 \\
&\geq \eta\alpha_k[q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)].
\end{aligned}$$

Thus, for such α_k the sufficient decrease condition (2.14) is satisfied, which implies that Step 6 of Algorithm I will always select α_k satisfying

$$\alpha_k \geq \min\{\tau(1-\eta)\epsilon/(c_1\|d_k\|^2), 1\}. \tag{3.16}$$

Now we argue that optimality of d_k implies it satisfies the bound

$$\|d_k\| \leq \frac{2\|\nabla f_k\|}{\mu_{\min}} + \sqrt{\frac{2\bar{\pi} m_k(0)}{\mu_{\min}}}. \tag{3.17}$$

This is clear since if (3.17) is violated,

$$\begin{aligned}
q_k^{\bar{\pi}}(d) &= f(x_k) + \nabla f_k^T d_k + \frac{1}{2} d^T W_k d + \bar{\pi} m_k(d) & (3.18)\\
&\geq f(x_k) - \|\nabla f_k\| \|d_k\| + \frac{1}{2}\mu_{\min}\|d\|^2 & (3.19)\\
&> f(x_k) + \bar{\pi} m_k(0) & (3.20)\\
&= q_k^{\bar{\pi}}(0), & (3.21)
\end{aligned}$$

which would mean d_k does not minimize $q_k^{\bar{\pi}}$.

Together with (3.16), the bound (3.17) implies there is a constant $c_2 > 0$ such that $\alpha_k > c_2$ for all k. Now, using this bound on α_k together with (3.14), it follows that Step 6 of Algorithm I guarantees that

$$\begin{aligned}
\phi_{\bar{\pi}}(x_k) - \phi_{\bar{\pi}}(x_k + \alpha_k d_k) &\geq \eta \alpha_k [q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k)]\\
&\geq \eta c_2 \epsilon. & (3.22)
\end{aligned}$$

This relation implies that $\phi_{\bar{\pi}}(x_k) \to \infty$, which contradicts Assumption A1. This implies that the hypothesis $q_*^{\bar{\pi}}(0) - q_*^{\bar{\pi}}(d_*) > 0$ is false, and therefore that x_* is a stationary point of $\phi_{\bar{\pi}}$. □

Now that we have established that, when the penalty parameter is bounded the algorithm will locate a stationary point of ϕ, the next result shows that such s stationary point is either a KKT point of the nonlinear program (2.1) or a stationary point of the infeasibility measure.

Theorem 3.7 *Suppose that Algorithm I generates an infinite sequence of iterates $\{x_k\}$, that Assumptions I hold, and that $\{\pi_k\}$ is bounded. Let x_* be any accumulation point of $\{x_k\}$. Then either: (a) $v(x_*) = 0$ and x_* is a KKT point of* (2.1)*; or (b) $v(x_*) > 0$ and x_* is a stationary point of $v(x)$.*

Proof. Let x_* be a limit point of the sequence $\{x_k\}$. Since π_k is bounded there is a scalar $\bar{\pi}$ such that $\pi_k = \bar{\pi}$ for all large k. From Theorem 3.6 we have that x_* is a stationary point of $\phi_{\bar{\pi}}(x)$.

(a) If $v(x_*) = 0$, then by Theorem 3.2 (b), x_* is a KKT point of problem (2.1).

(b) In the case when $v(x_*) > 0$ we want to show that (3.6) holds for some $\Delta > 0$. Let $\mathcal{K}$ be an infinite subset of indices for which $x_k \to x_*$ for $k \in \mathcal{K}$. Let W_* be a limiting matrix of the sequence $\{W_k\}_{k\in\mathcal{K}}$, and let $q_*^{\bar{\pi}}(d)$ be the corresponding piecewise quadratic model. Restricting $\mathcal{K}$ further, if necessary, we obtain pointwise convergence of the models, i.e., $q_k^{\bar{\pi}}(d) \to q_*^{\bar{\pi}}(d)$ for $k \in \mathcal{K}$. As before, let d_k denote the minimizer of $q_k^{\bar{\pi}}(d)$.

Since x_* is a stationary point of $\phi_{\bar{\pi}}(x)$, by Theorem 3.2 (a) we have that the minimizer of $q_*^{\bar{\pi}}(d)$ is $d_* = 0$. From the pointwise convergence of the models, it follows that $d_k \to 0$, which in turn implies that

$$q_k^{\bar{\pi}}(0) - q_k^{\bar{\pi}}(d_k) \to 0. \tag{3.23}$$

This limit together with (2.13) imply that $m_k(0) - m_k(d_k^{\text{LP}}) \to 0$ for $k \in \mathcal{K}$.

We also have pointwise convergence to a limiting piecewise linear model, i.e., $m_k(d) \to m_*(d)$, and hence

$$\begin{aligned}
0 &= \lim_{k\to\infty, k\in\mathcal{K}} [m_k(0) - m_k(d_k^{\text{LP}})] \\
&= \lim_{k\to\infty, k\in\mathcal{K}} [m_k(0) - \min_{\|d\|_\infty \le \Delta_k} m_k(d)] \\
&\ge \lim_{k\to\infty, k\in\mathcal{K}} [m_k(0) - \min_{\|d\|_\infty \le \Delta_{\min}} m_k(d)],
\end{aligned}$$

where the last inequality follows from the fact that Algorithm I requires that $\Delta_k \ge \Delta_{\min} > 0$. Therefore, $0 < v(x_*) = m_*(0) = m_*(d^{\text{LP}}(\Delta_{\min}))$ and by Theorem 3.2 (c) we conclude that x_* is a stationary point of infeasibility for $v(x)$. □

Now we consider the behavior of the algorithm when the penalty parameter increases without bound. The first result focuses on the development of the penalty parameter in a vicinity of an infeasible stationary point.

Lemma 3.8 *Suppose that Algorithm I generates a sequence $\{x_k\}$ that satisfies Assumptions I. Let x_* be a cluster point of this sequence such that $v(x_*) > 0$, and suppose that $m_*(0) - m_*(d_k^{LP}) > 0$. Then, along any subsequence $\{x_k\}_{k\in\mathcal{K}}$ that converges to x_* the penalty parameter is updated only a finite number of times.*

Proof. We will show that for any subsequence that converges to such a point x_*, Step 4(a) cannot be executed infinitely often, and that for x_k sufficiently close to x_*, (2.12) and (2.13) are satisfied for sufficiently large π. This will prove the result because the penalty parameter is only increased in Steps 4 and 5 of Algorithm I.

As a preliminary observation note that, since we assume

$$m_*(0) - m_*(d_k^{\text{LP}}) > 0, \tag{3.24}$$

there exist constants $r > 0$ and $\zeta > 0$ such that

$$m_k(0) - m_k(d_k^{\text{LP}}) > \zeta, \quad \text{for all } x_k \in \mathcal{B}^* \triangleq \{x : \|x_k - x_*\| < r\}. \tag{3.25}$$

Now we study the situations in which the penalty parameter is increased in Algorithm I. This increase can happen in Steps 4(a), 4(b) or 5 of Algorithm I, and we study each case separately.

Case (i) Consider an iterate x_k where Step 4(a) is executed. By (2.13) and (3.25), for any such k

$$q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k(\pi_+)) \geq \epsilon_2 \pi_+ \zeta. \tag{3.26}$$

Now, by the Lipschitz continuity assumptions in A1, one can show [7, Lemma 3.4] that there is a constant c_1 such that for any x_k and any π

$$|\phi_\pi(x_k + \alpha d_k) - q_k^\pi(\alpha d_k)| \leq c_1(1+\pi)\|\alpha d_k\|^2. \tag{3.27}$$

Let us consider the sufficient decrease condition (2.14). From the equality $\phi_{\pi_+}(x_k) = q_k^{\pi_+}(0)$, the convexity of $q_k^{\pi_+}$, (3.27) and (3.26), we have, for $x_k \in \mathcal{B}^*$,

$$\begin{aligned}
\phi_{\pi_+}(x_k) - \phi_{\pi_+}(x_k + \alpha d_k) \quad & - \quad \eta\alpha[q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)] \\
& = \quad [q_k^{\pi_+}(0) - q_k^{\pi_+}(\alpha d_k)] - [\phi_{\pi_+}(x_k + \alpha d_k) - q_k^{\pi_+}(\alpha d_k)] \\
& \quad\ \ -\eta\alpha[q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)] \\
& \geq \quad \alpha[q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)] - c_1(1+\pi_+)\|\alpha d_k\|^2 - \eta\alpha[q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)] \\
& \geq \quad \alpha(1-\eta)[q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k)] - c_1(1+\pi_+)\|\alpha d_k\|^2 \\
& \geq \quad \alpha(1-\eta)\epsilon_2\pi_+\zeta - c_1(1+\pi_+)\alpha^2\|d_k\|^2.
\end{aligned}$$

The right hand side is nonnegative if $\alpha \leq (1-\eta)\epsilon_2\pi_+\zeta / c_1(1+\pi_+)\|d_k\|^2$. Therefore Step 6 of Algorithm I will always choose

$$\alpha_k \geq \min\{1, \tau(1-\eta)\epsilon_2\pi_+\zeta / c_1(1+\pi_+)\|d_k\|^2\}, \tag{3.28}$$

where τ is the contraction factor used in Step 6 of the algorithm.

Since when Step 4(a) is executed $m_k(d_k^{\mathrm{LP}}) = 0$, we have that d_k^{LP} is an unconstrained minimizer of m_k and is thus no smaller in norm than the minimum norm minimizer $\bar{d}_k$ mentioned in Lemma 3.3. By applying Lemma 3.3, we obtain the bound $\|d_k\| \leq \kappa_1 + \kappa_2\|d_k^{\mathrm{LP}}\| \leq \kappa_1 + \kappa_2\Delta_{\max}$, since $\|d_k^{\mathrm{LP}}\| \leq \Delta_{\max}$; see Step 7. Using this bound in (3.28) gives

$$\alpha_k \geq \min\left\{1, \frac{(1-\eta)\tau\epsilon_2\pi_+\zeta}{c_1(1+\pi_+)(\kappa_1 + \kappa_2\Delta_{\max})^2}\right\} \geq c_2,$$

for some constant $c_2 > 0$. This bound, together with (2.14) and (3.26) implies that there is a constant $c_3 > 0$ such that for any $x_k \in \mathcal{B}^*$,

$$\phi_{\pi_{k+1}}(x_{k+1}) \leq \phi_{\pi_{k+1}}(x_k) - c_3\pi_{k+1}. \tag{3.29}$$

Now, consider the scaled penalty function

$$\tfrac{1}{\pi}\phi_\pi(x) = \tfrac{1}{\pi}f(x) + v(x),$$

and note that since $\{f_k\}$ is assumed bounded below and the algorithm is unaffected by adding a constant to f, we may assume without loss of generality that $f(x_k) \geq 0$ for all k. This assumption and the fact that $\{\pi_k\}$ is nondecreasing, imply that

$$\tfrac{1}{\pi_{k+1}}f(x_k) + v(x_k) \leq \tfrac{1}{\pi_k}f(x_k) + v(x_k). \tag{3.30}$$

By the sufficient decrease condition (2.14) we have that, for all k,

$$\tfrac{1}{\pi_{k+1}}\phi(x_{k+1}) \leq \tfrac{1}{\pi_{k+1}}\phi(x_k).$$

By combining this expression with (3.30) we have

$$\tfrac{1}{\pi_{k+1}}\phi(x_{k+1}) \leq \tfrac{1}{\pi_k}\phi(x_k),$$

which shows that the sequence $\{\frac{1}{\pi_k}f(x_k)+v(x_k)\}$ is monotone decreasing for all k. Consider now an iterate $x_k \in \mathcal{B}^*$ at which Step 4(a) is executed. By combining (3.30) with (3.29) we obtain, for such x_k,

$$\tfrac{1}{\pi_{k+1}}f(x_{k+1}) + v(x_{k+1}) \leq \tfrac{1}{\pi_k}f(x_k) + v(x_k) - c_3.$$

This contradicts Assumption A1 that implies that the sequence $\{\frac{1}{\pi_k}f(x_k)+v(x_k)\}$ is bounded below. We conclude that Step 4(a) can only be executed finitely often for $x_k \in \mathcal{B}^*$.

Case (ii) Next, consider Step 4(b). First note that, since W_k is positive definite, the lowest value of the quadratic model $q_k^f(d)$ (defined in (3.10)) is attained at the Newton step, $-W_k^{-1}\nabla f_k$. Thus by (3.1) we have

$$\begin{aligned} q_k^f(d) &\geq f_k - \nabla f_k^T W_k^{-1}\nabla f_k + \tfrac{1}{2}\nabla f_k^T W_k^{-1} W_k W_k^{-1}\nabla f_k \\ &= f_k - \tfrac{1}{2}\nabla f_k^T W_k^{-1}\nabla f_k \\ &\geq f_k - \tfrac{1}{2}\|\nabla f_k\|^2/\mu_{\min}. \end{aligned} \tag{3.31}$$

Also, since the linear program (2.8) is constrained by a trust region whose radius cannot exceed $\Delta_{\max}$, we have from (3.10)

$$q_k^f(d_k^{\text{LP}}) \leq f_k + \|\nabla f_k\|\Delta_{\max} + \tfrac{1}{2}\mu_{\max}\Delta_{\max}^2. \tag{3.32}$$

By combining (3.31) and (3.32) and recalling that $\|\nabla f_k\|$ is bounded (by Assumption A1), we deduce that there is constant ν such that, for all x_k

$$q_k^f(d^{\text{LP}}) - q_k^f(d_k(\pi_k)) \leq \nu. \tag{3.33}$$

Now since $d_k(\pi_k)$ minimizes q_{π_k}, by (3.10) we have that

$$q_k^f(d_k(\pi_k)) + \pi_k m_k(d_k(\pi_k)) \leq q_k^f(d^{\text{LP}}) + \pi_k m_k(d^{\text{LP}}).$$

Combining this relation with (3.33) gives

$$\begin{aligned} \pi_k[m_k(0) - m_k(d_k(\pi_k))] &\geq \pi_k[m_k(0) - m_k(d_k^{\text{LP}})] - q_k^f(d^{\text{LP}}) + q_k^f(d_k(\pi_k)) \\ &\geq \pi_k[m_k(0) - m_k(d_k^{\text{LP}})] - \nu \\ &\geq \pi_k[m_k(0) - m_k(d_k^{\text{LP}})](1 - \tfrac{\nu}{\zeta\pi_k}), \end{aligned}$$

because (3.25) implies that $-\nu \geq -\nu[m_k(0) - m_k(d_k^{\text{LP}})]/\zeta$. If the penalty parameter is large enough that

$$\pi_k \geq \frac{\nu}{\zeta(1-\epsilon_1)} \tag{3.34}$$

then

$$m_k(0) - m_k(d_k(\pi_k)) \geq \epsilon_1[m_k(0) - m_k(d^{\rm LP})], \tag{3.35}$$

and condition (2.12) will be satisfied. Therefore, π_k cannot be increased infinitely often in Step 4(b), for $x_k \in \mathcal{B}^*$.

Case (iii) Finally, consider Step 5 of Algorithm I, which enforces condition (2.13). Suppose that π_+ is chosen so that

$$\pi_+ \geq \frac{\nu}{\zeta(1-\epsilon_1)(1-\epsilon_2/\epsilon_1)} \tag{3.36}$$

(recall that $\epsilon_2 < \epsilon_1$). If we let $\tilde{\pi} = \pi_+(1-\epsilon_2/\epsilon_1)$ then the fact that $d(\tilde{\pi})$ is a minimizer of $q^{\tilde{\pi}}$ implies that

$$q_k^f(0) - q_k^f(d_k(\tilde{\pi})) + \pi_+(1-\epsilon_2/\epsilon_1)(m_k(0) - m_k(d_k(\tilde{\pi})) \geq 0,$$

and therefore

$$q_k^f(0) - q_k^f(d(\tilde{\pi})) + \pi_+[m_k(0) - m_k(d_k(\tilde{\pi}))] \geq \tfrac{\epsilon_2}{\epsilon_1}\pi_+[m_k(0) - m_k(d_k(\tilde{\pi}))].$$

Thus,

$$q_k^{\pi_+}(0) - q_k^{\pi_+}(d_k(\tilde{\pi})) \geq \epsilon_2\pi_+(m_k(0) - m_k(d_k^{\rm LP})),$$

since $\tilde{\pi}$ satisfies (3.34) and thus (2.12). The fact that π_+ satisfies (2.13) follows from noting that $-q_k^{\pi_+}(d_k(\tilde{\pi})) \leq -q_k^{\pi_+}(d_k(\pi_+))$ since $d(\pi_+)$ is a minimizer of q^{π_+}.

Therefore since any π_+ satisfying (3.36) satisfies (2.12) and (2.13) and since π increases by at least ρ, only a finite number of increases can occur for $x_k \in \mathcal{B}^*$. □

We now consider the behavior of the algorithm in the vicinity of a point that satisfies the well-known Mangasarian-Fromovitz constraint qualification (MFCQ) [25].

Lemma 3.9 *Let x_* be a point that satisfies MFCQ and suppose that Assumptions I hold. Then, there is a neighborhood $\mathcal{N}$ of x_* and a constant r_F such that, for any iterate $x_k \in \mathcal{N}$, there is a vector $d_F(x_k)$ with $\|d_F(x_k)\| \leq r_F$ such that $m_k(d_F(x_k)) = 0$. In addition there are constants r and β such that, for $x_k \in \mathcal{N}$, the minimizer $d_k(\pi)$ of q_π^k satisfies*

$$\|d_k\| \leq r \tag{3.37}$$

and such that

$$q_k^f(d_1) - q_k^f(d_2) \leq \beta\|d_1 - d_2\| \tag{3.38}$$

for any vectors d_1, d_2 such that $\|d_1\| \leq r$, $\|d_2\| \leq r$.

Proof. Let $h(x)$ denote the vector with components $h_i(x)$, $i \in \mathcal{E}$, $g(x)$ the vector with components $g_i(x)$, $i \in \mathcal{I}$ and let $\nabla h(x)^T$ and $\nabla g(x)^T$ denote their Jacobians. Since MFCQ holds at x_*, $\nabla h(x_*)$ has full rank and there is a direction $\|d_{\rm M}\| < 1$ such that

$$\nabla h(x_*)^T d_{\rm M} = -h(x_*) \;\text{ and }\; \nabla g(x_*)^T d_{\rm M} + g(x_*) > 0. \tag{3.39}$$

Since $\nabla h(x_*)$ has full rank, for any x sufficiently near x_* the matrix $\nabla h(x)^T \nabla h(x)$ is nonsingular and the vector

$$d_{\rm F}(x) = d_{\rm M} - \nabla h(x)(\nabla h(x)^T \nabla h(x))^{-1}[h(x) + \nabla h(x)^T d_{\rm M}] \tag{3.40}$$

satisfies

$$\nabla h(x)^T d_{\rm F}(x) = -h(x). \tag{3.41}$$

By continuity of $\nabla h(x)$, the vector $d_{\rm F}(x)$ is continuous, and the first condition in (3.39) implies that the term in square brackets in (3.40) is small in norm near x_*. Therefore, from the second condition in (3.39) we have that $\nabla g(x)^T d_{\rm F}(x) + g(x) > 0$ for x near x_*. Thus, for x_k in some neighborhood $\mathcal{N}$ of x_*, we have that $m_k(d_{\rm F}(x_k)) = 0$. Continuity of $d_{\rm F}(x)$ also implies that there is a constant $r_{\rm F}$ such that $\|d_{\rm F}(x_k)\| < r_{\rm F}$ for all x_k in $\mathcal{N}$.

Since $d_{\rm F}(x_k)$ is a minimizer of m_k, we have that $||\bar{d}_k|| \leq ||d_{\rm F}(x_k)||$, where $\bar{d}_k$ is the minimum norm minimizer of m_k mentioned in Lemma 3.3. Thus, by (3.7) we have $\|d_k(\pi)\| \leq r$, with $r = \kappa_1 + \kappa_2 r_{\rm F}$.

From (3.10), $\|\nabla q_k^f(d)\| \leq \|\nabla f(x_k)\| + \|W_k\|\|d\|$. The right hand side of this inequality is bounded for all $x_k \in \mathcal{N}$ due to the bounds on d_1, d_2 and the boundedness of W_k stipulated in Assumptions I . The result (3.38) then follows by a Taylor expansion of q_k^f. □

For the following results, we define $\mathcal{A}^*$ to be the set of active inequality constraints at x_*, i.e., $\mathcal{A}^* = \{i \in \mathcal{I} : g_i(x_*) = 0\}$. The next lemma is a technical result establishing a cone of linearized feasibility with respect to constraints not in $\mathcal{A}^*$.

Lemma 3.10 *Suppose that Assumptions I hold and that x_* is a feasible point with active set $\mathcal{A}^*$. Then, there exists a constant $\gamma > 0$ and a neighborhood of x_*, such that for any x_k in that neighborhood, for any step d_k satisfying*

$$g_i(x_k) + \nabla g_i(x_k)^T d_k \geq 0, \quad \textit{for all} \;\; i \notin \mathcal{A}^* \tag{3.42}$$

and for any direction $\tilde{d}$, we have

$$g_i(x_k) + \nabla g_i(x_k)^T (\alpha d_k + \tau \tilde{d}) \geq 0, \quad \textit{for all} \;\; i \notin \mathcal{A}^*, \tag{3.43}$$

if $\tau > 0$ and $\alpha \in (0,1)$ satisfy

$$\tau \|\tilde{d}\| \leq (1-\alpha)\gamma. \tag{3.44}$$

Proof. Define $\mathcal{N}'$ to be a neighborhood of x_* over which $g_i(x) \geq \frac{1}{2} g_i(x_*) > 0$ for all $i \notin \mathcal{A}^*$. If $\nabla g_i(x_k) = 0$ for all $i \notin \mathcal{A}^*$ then (3.43) holds trivially. Otherwise, the quantity

$$\gamma = \min_{i \notin \mathcal{A}^*, x \in \mathcal{N}'} \frac{g_i(x)}{\|\nabla g_i(x)\|}$$

is positive. Multiplying (3.42) by any $\alpha \in (0,1)$ we obtain

$$g_i(x_k) + \nabla g_i(x_k)^T (\alpha d_k) \geq (1-\alpha) g_i(x_k), \quad \text{for all} \;\; i \notin \mathcal{A}^*. \tag{3.45}$$

Now consider the composite direction $\alpha d_k + \tau \tilde{d}$, with $\tau \geq 0$ and $\tilde{d}$ an arbitrary direction. We have

$$g_i(x_k) + \nabla g_i(x_k)^T(\alpha d_k + \tau \tilde{d}) \geq (1-\alpha) g_i(x_k) + \tau \nabla g_i(x_k)^T \tilde{d}, \quad \text{for all } \; i \notin \mathcal{A}^*. \tag{3.46}$$

If $x_k \in \mathcal{N}'$, then $g_i(x_k) > 0$ for $i \notin \mathcal{A}^*$, so that if $\nabla g_i(x_k)^T \tilde{d} \geq 0$, the right-hand side of (3.46) is nonnegative. If, on the other hand $\nabla g_i(x_k)^T \tilde{d} < 0$ and τ satisfies (3.44), then

$$\tau \leq \frac{(1-\alpha)\gamma}{\|\tilde{d}\|} \leq \frac{(1-\alpha) g_i(x_k)}{\|\nabla g_i(x_k)\| \|\tilde{d}\|} \leq \frac{(1-\alpha) g_i(x_k)}{-\nabla g_i(x_k)^T \tilde{d}}, \qquad i \notin \mathcal{A}^*; \tag{3.47}$$

hence the right-hand side of (3.46) is nonnegative. □

The next result shows that, in a vicinity of a feasible point that satisfies the Mangasarian-Fromowitz constraint qualification and for sufficiently large values of the penalty parameter, the step d_k generated by the algorithm satisfies the linearized constraints (i.e. the vectors r, s, t in (2.7) are all zero).

Lemma 3.11 *Suppose that Algorithm I generates a sequence $\{x_k\}$ that satisfies Assumptions I. Let x_* be a cluster point of this sequence such that $v(x_*) = 0$, and suppose that MFCQ holds at x_*. Then for all x_k sufficiently close to x_* and π_k sufficiently large, the minimizer d_k of $q_k^{\pi_k}$ satisfies $m_k(d_k) = 0$.*

Proof. As in the proof of Lemma 3.9, we denote by $h(x)$ the vector with components $h_i(x)$, $i \in \mathcal{E}$ and denote the Jacobian of h by $\nabla h(x)^T$, and similarly for $g(x)$ and $\nabla g(x)^T$. Since MFCQ is satisfied at x_*, we have that $\nabla h(x_*)$ has full rank and there is a direction $\|d_{\rm MF}\| < 1$ such that

$$h(x_*) + \nabla h(x_*)^T d_{\rm MF} = 0 \quad \text{ and } \quad \nabla g(x_*)^T d_{\rm MF} + g(x_*) > 0. \tag{3.48}$$

Let us define

$$d_{\rm M}(x) \triangleq d_{\rm MF} - \nabla h(x)[\nabla h(x)^T \nabla h(x)]^{-1} \nabla h(x)^T d_{\rm MF}, \tag{3.49}$$

which is well defined for x near x_* because the matrix $\nabla h(x)^T \nabla h(x)$ is nonsingular since it is close to the nonsingular matrix $\nabla h(x_*)^T \nabla h(x_*)$. Clearly,

$$\nabla h(x)^T d_{\rm M}(x) = 0. \tag{3.50}$$

Since x_* is feasible, we have that $h(x_*) = 0$, and thus by the first relation in (3.48) the term $\nabla h(x)^T d_{\rm MF}$ is close to zero for x_k near x_* — and this implies that $d_{\rm M}(x)$ is close to $d_{\rm MF}$. Since ∇h is a continuous function, $d_{\rm M}(x)$ varies continuously with x, and therefore by the second relation in (3.48) there is a constant $\sigma > 0$ such that for all x_k near x_*, the vector $d_{\rm M}^k \triangleq d_{\rm M}(x_k)$ satisfies

$$\nabla g_i(x_k)^T d_{\rm M}^k > \sigma \quad \text{for all } i \in \mathcal{A}^*, \tag{3.51}$$

where, as before, $\mathcal{A}^*$ denotes the set of active inequality constraints at x_*.

Let us define $\underline{g} = \frac{1}{2}\min_{j\notin\mathcal{A}^*}\{g_j(x_*)\}$. Then, for x_k sufficiently near x_*, we have that $v(x_k) < \epsilon$ where $\epsilon > 0$ is a constant that may additionally be chosen sufficiently small to satisfy both

$$g_j(x_k) \geq \underline{g} - \epsilon > 0 \ \text{ for } j \notin \mathcal{A}^*, \quad \text{and } 2\epsilon < \underline{g}. \tag{3.52}$$

We denote by $\mathcal{N}$ a neighborhood of x_* contained in the neighborhoods given by Lemmas 3.9 and 3.10, and such that for all $x_k \in \mathcal{N}$, conditions (3.50), (3.51) and (3.52) hold and $v(x_k) < \epsilon$. Let us re-write (2.4) as

$$m_k(d) = \sum_{i\notin\mathcal{A}^*}[g_i(x_k)+\nabla g_i(x_k)^T d]^- + \sum_{i\in\mathcal{A}^*}[g_i(x_k)+\nabla g_i(x_k)^T d]^- + \sum_{i\in\mathcal{E}}|h_i(x_k)+\nabla h_i(x_k)^T d|. \tag{3.53}$$

The proof proceeds in three stages; each shows that one of the summations is zero for any $x_k \in \mathcal{N}$ and for sufficiently large π_k.

Part (i) Let d_k minimize $q_k^{\pi_k}$, and suppose by way of contradiction that the first summation is nonzero for $x_k \in \mathcal{N}$. Then,

$$g_j(x_k) + \nabla g_j(x_k)^T d_k < 0 \quad \text{for some } j \notin \mathcal{A}^*. \tag{3.54}$$

By (3.52), we have that $g_j(x_k) > 0$ for $x_k \in \mathcal{N}$, and since (3.54) also holds, we know that there exists $\alpha \in (0,1)$ such that

$$g_j(x_k) + \alpha\nabla g_j(x_k)^T d_k = 0. \tag{3.55}$$

It follows that

$$\alpha(g_j(x_k) + \nabla g_j(x_k)^T d) = -(1-\alpha)g_j(x_k),$$

which together with (3.52) and the condition $\alpha < 1$ implies

$$g_j(x_k) + \nabla g_j(x_k)^T d_k \leq -(1-\alpha)(\underline{g} - \epsilon). \tag{3.56}$$

Define the function

$$a_j^k(d) \triangleq m_k(d) - [g_j(x_k) + \nabla g_j(x_k)^T d)]^-, \tag{3.57}$$

which consists of excluding the jth inequality term from (3.53) (and therefore $a_j^k(d_k) \geq 0$). For j satisfying (3.54), we have

$$a_j^k(d_k) = m_k(d_k) + (g_j(x_k) + \nabla g_j(x_k)^T d_k). \tag{3.58}$$

Clearly, a_j^k is a convex function, which implies that for any d_k

$$a_j^k(\alpha d_k) \leq (1-\alpha)a_j^k(0) + \alpha a_j^k(d_k),$$

and thus

$$a_j^k(\alpha d_k) - a_j^k(d_k) \leq (1-\alpha)(a_j^k(0) - a_j^k(d_k)) \leq (1-\alpha)\epsilon, \tag{3.59}$$

since by (3.57) $a_j^k(0) = m_k(0) = v(x_k) < \epsilon$, and $a_j^k(d_k) \geq 0$. Now, we have from (3.57), (3.55), (3.58), (3.59) and (3.56) that

$$\begin{aligned} m(\alpha d_k) - m(d_k) &= a_j^k(\alpha d_k) - a_j^k(d_k) + (g_j(x_k) + \nabla g_j(x_k)^T d_k) \\ &\leq (1-\alpha)\epsilon - (1-\alpha)(\underline{g} - \epsilon) \\ &= (1-\alpha)(2\epsilon - \underline{g}). \end{aligned}$$

Finally, by Lemma 3.9 and since $q_k^{\pi_k}(d_k) = q_k^f(d_k) + \pi_k m_k(d_k)$,

$$q_k^{\pi_k}(\alpha d_k) - q_k^{\pi_k}(d_k) \leq (1-\alpha)\beta r + \pi_k(1-\alpha)(2\epsilon - \underline{g}). \tag{3.60}$$

By (3.52), $2\epsilon - \underline{g} < 0$, and if $\pi_k > \beta r/(\underline{g} - 2\epsilon)$, we have that $q_k^{\pi_k}(\alpha d_k) < q_k^{\pi_k}(d_k)$, which contradicts the fact that d_k is the minimizer of $q_k^{\pi_k}$. Therefore, for $x_k \in \mathcal{N}$ and for π_k sufficiently large, there cannot exist an index j satisfying (3.54), and thus the first summation in (3.53) is zero.

Part (ii) Next, suppose that the step d_k that minimizes $q_k^{\pi_k}$ is such that the second sum in (3.53) is nonzero for $x_k \in \mathcal{N}$, while the first sum is zero. Then

$$g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k < 0 \quad \text{for some } \ell \in \mathcal{A}^*. \tag{3.61}$$

As above, consider the linearized model of the constraints other than ℓ:

$$a_\ell^k(d) = m_k(d) - [g_\ell(x_k) + \nabla g_\ell(x_k)^T d)]^-. \tag{3.62}$$

By (3.50), (3.51) we have that the vector $d_{\mathrm{M}}^k = d_{\mathrm{M}}(x_k)$ satisfies $\nabla h_i(x_k)^T d_{\mathrm{M}}^k = 0$, $i \in \mathcal{E}$ and $\nabla g_i(x_k)^T d_{\mathrm{M}}^k \geq \sigma$, $i \in \mathcal{A}^*$; furthermore, for

$$\tau < (1-\alpha)\gamma, \tag{3.63}$$

we have by Lemma 3.10 that (3.43) is satisfied for $\tilde{d} = d_{\mathrm{M}}^k$ and for $i \notin \mathcal{A}^*$, (recall that $\|d_{\mathrm{M}}\| < 1$). These observations show that each of the terms in (3.53) is not larger for $d = \alpha d_k + \tau d_{\mathrm{M}}^k$ than for $d = \alpha d_k$, and the same is true for a_ℓ^k since a_ℓ^k consists of all but one of the terms in m_k. Thus,

$$a_\ell^k(\alpha d_k + \tau d_{\mathrm{M}}^k) \leq a_\ell^k(\alpha d_k) \leq (1-\alpha)a_\ell^k(0) + a_\ell^k(d_k), \tag{3.64}$$

where the second inequality follows from the convexity of a_ℓ^k and the condition $\alpha \in (0,1)$. Since $\ell \in \mathcal{A}^*$, we also have from (3.51) that

$$g_\ell(x_k) + \nabla g_\ell(x_k)^T(\alpha d_k + \tau d_{\mathrm{M}}^k) \quad \geq \quad g_\ell(x_k) + \alpha \nabla g_\ell(x_k)^T d_k + \tau\sigma. \tag{3.65}$$

If we choose $\tau > 0$ small enough so that

$$\tau\sigma \leq -(g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k) \tag{3.66}$$

then by (3.61)

$$\begin{aligned} [g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k + \tau\sigma]^- &= -(g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k) - \tau\sigma \\ &= [g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k]^- - \tau\sigma. \end{aligned} \tag{3.67}$$

By making use of (3.65), the fact that $[\,\cdot\,]^-$ is a non-increasing convex function, the condition $\alpha < 1$ and (3.67) we have

$$
\begin{aligned}
[g_\ell(x_k) + \nabla g_\ell(x_k)^T(\alpha d_k + \tau d_{\mathrm{M}}^k)]^- &\leq [g_\ell(x_k) + \alpha \nabla g_\ell(x_k)^T d_k + \tau\sigma]^- \\
&\leq (1-\alpha)[g_\ell(x_k) + \tau\sigma]^- + \alpha[g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k + \tau\sigma]^- \\
&\leq (1-\alpha)[g_\ell(x_k)]^- + \alpha[g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k + \tau\sigma]^- \\
&\leq (1-\alpha)[g_\ell(x_k)]^- + [g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k + \tau\sigma]^- \\
&\leq (1-\alpha)[g_\ell(x_k)]^- + [g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k]^- - \tau\sigma. \quad (3.68)
\end{aligned}
$$

Now, using (3.62) to decompose m_k and then applying (3.64) and (3.68), we obtain

$$
\begin{aligned}
m_k(\alpha d_k + \tau d_{\mathrm{M}}^k) &= a_\ell^k(\alpha d_k + \tau d_{\mathrm{M}}^k) + [g_\ell(x_k) + \nabla g_\ell(x_k)^T(\alpha d_k + \tau d_{\mathrm{M}}^k)]^- \\
&\leq (1-\alpha)a_\ell^k(0) + a_\ell^k(d_k) + (1-\alpha)[g_\ell(x_k)]^- + [g_\ell(x_k) + \nabla g_\ell(x_k)^T d_k]^- - \tau\sigma \\
&\leq (1-\alpha)m_k(0) + m_k(d_k) - \tau\sigma \\
&\leq (1-\alpha)\epsilon + m_k(d_k) - \tau\sigma, \qquad (3.69)
\end{aligned}
$$

for any $\alpha \in [0,1]$, and $\tau > 0$ satisfying (3.63) and (3.66). Since, by Lemma 3.9, $\|d_k\| \leq r$, we may also choose τ small enough that $\|\alpha d_k + \tau d_{\mathrm{M}}^k\| \leq 2r$. Then, if we additionally require that (3.63) holds with equality, we have from (3.11), Lemma 3.9 and (3.69) that

$$
\begin{aligned}
q_\pi^k(\alpha d_k + \tau d_{\mathrm{M}}^k) - q_\pi^k(d_k) &\leq q_f^k(\alpha d_k + \tau d_{\mathrm{M}}^k) - q_k^f(d_k) - \pi(\tau\sigma - (1-\alpha)\epsilon) \\
&\leq \beta\|(\alpha - 1)d_k + \tau d_{\mathrm{M}}^k\| - \pi(\tau\sigma - (1-\alpha)\epsilon) \\
&\leq \left(\frac{r}{\gamma} + \|d_{\mathrm{M}}^k\|\right)\beta\tau - \pi\tau\left(\sigma - \frac{\epsilon}{\gamma}\right),
\end{aligned}
$$

where we have applied Lemma 3.10 with the condition that τ is small enough that $\alpha \in (0,1)$. If we choose $\epsilon < \gamma\sigma/2$, then for $\pi > 2\beta(r/\gamma + \|d_{\mathrm{M}}^k\|)/\sigma$ the right hand side is negative, which is not possible because d_k is the minimizer of q_π^k. Therefore the inequality (3.54) cannot hold for $x_k \in \mathcal{N}$ and π_k large enough.

Part (iii) Last, suppose that the step d_k that minimizes $q_k^{\pi_k}$ satisfies all the linearized inequalities (so that the first two summations in (3.53) are zero), but is such that $h(x_k) + \nabla h(x_k)^T d_k \neq 0$. Then

$$
m(d_k) = \|h(x_k) + \nabla h(x_k)^T d_k\|. \qquad (3.70)
$$

Consider taking a step from d_k in the direction

$$
p = -\nabla h(x_k)[\nabla h(x_k)^T \nabla h(x_k)]^{-1}(h(x_k) + \nabla h(x_k)^T d_k) + \theta d_{\mathrm{M}}^k, \qquad (3.71)
$$

for some $\theta > 0$. We have from (3.50) that, for any α, τ, such that $\tau < \alpha < 1$,

$$
\begin{aligned}
h(x_k) + \nabla h(x_k)^T(\alpha d_k + \tau p) &= h(x_k) + \nabla h(x_k)^T \alpha d_k - \tau[h(x_k) + \nabla h(x_k)^T d_k], \\
&= (\alpha - \tau)[h(x_k) + \nabla h(x_k)^T d_k] + (1-\alpha)h(x_k). \quad (3.72)
\end{aligned}
$$

Since for $x_k \in \mathcal{N}$ we have $\|h(x_k)\| \leq \epsilon$, and since $\alpha_k \leq 1$, we obtain

$$\|h(x_k) + \nabla h(x_k)^T(\alpha d_k + \tau p)\| \leq (1-\tau)\|h(x_k) + \nabla h(x_k)^T d_k\| + (1-\alpha)\epsilon. \tag{3.73}$$

By Assumption A1, the fact that $\nabla h(x)$ has full rank near x_* and (3.51), we have that for $i \in \mathcal{A}^*$, there is a constant C_1 such that

$$\begin{aligned}
\nabla g_i(x_k)^T p &\geq -\nabla g_i(x_k)^T \nabla h(x_k) \left[\nabla h(x_k)^T \nabla h(x_k)\right]^{-1} (h(x_k) + \nabla h(x_k)^T d_k) + \theta\sigma \\
&\geq -C_1\|h(x_k) + \nabla h(x_k)^T d_k\| + \theta\sigma \\
&= \tfrac{2}{3}\theta\sigma > 0,
\end{aligned}$$

provided we choose $\theta = 3C_1\|h(x_k) + \nabla h(x_k)^T d_k\|/\sigma$. This bound and the fact that $[\;\cdot\;]^-$ is a non-increasing convex function, imply that for all $i \in \mathcal{A}^*$

$$\begin{aligned}
[g_i(x_k) + \nabla g_i(x_k)^T(\alpha d_k + \tau p)]^- &\leq [g_i(x_k) + \nabla g_i(x_k)^T \alpha d_k]^- \\
&\leq (1-\alpha)[g_i(x_k)]^- + \alpha[g_i(x_k) + \nabla g_i(x_k)^T d_k]^- \\
&\leq (1-\alpha)[g_i(x_k)]^-,
\end{aligned} \tag{3.74}$$

where the last inequality follows from the assumption that d_k satisfies all linearized inequalities. Our choice of θ implies that the length of vector p is bounded as follows,

$$\begin{aligned}
\|p\| &\leq C_2\|h(x_k) + \nabla h(x_k)^T d_k\| + \theta\|d_{\mathrm{M}}^k\| \\
&= \|h(x_k) + \nabla h(x_k)^T d_k\|(C_2 + 3C_1\|d_{\mathrm{M}}^k\|/\sigma) \\
&\leq C_3\|h(x_k) + \nabla h(x_k)^T d_k\|,
\end{aligned} \tag{3.75}$$

for suitable constants C_2 and C_3. If we choose τ and $\alpha \in (0,1)$ to satisfy

$$\tau C_3\|h(x_k) + \nabla h(x_k)^T d_k\| = (1-\alpha)\gamma, \tag{3.76}$$

then by Lemma 3.10 we have that condition (3.43) is satisfied for $\tilde{d} = p$. This observation together with (3.73), (3.74) and the convexity of $[\;\cdot\;]^-$, yield

$$\begin{aligned}
m_k(\alpha d_k + \tau p) &= \|h(x_k) + \nabla h(x_k)^T(\alpha d_k + \tau p)\| + \sum_{i \notin \mathcal{A}^*} [g_i(x_k) + \nabla g_i(x_k)^T(\alpha d_k + \tau p)]^- \\
&\quad + \sum_{i \in \mathcal{A}^*} [g_i(x_k) + \nabla g_i(x_k)^T(\alpha d_k + \tau p)]^- \\
&\leq \|h(x_k) + \nabla h(x_k)^T(\alpha d_k + \tau p)\| + \sum_{i \in \mathcal{A}^*} [g_i(x_k) + \nabla g_i(x_k)^T(\alpha d_k + \tau p)]^- \\
&\leq (1-\tau)\|h(x_k) + \nabla h(x_k)^T d_k\| + (1-\alpha)\epsilon + (1-\alpha)\sum_{i \in \mathcal{A}^*} [g_i(x_k)]^- \\
&\leq m(d_k) - \tau\|h(x_k) + \nabla h(x_k)^T d_k\| + 2(1-\alpha)\epsilon,
\end{aligned}$$

where the last inequality follows from (3.70) and the condition $m_k(0) < \epsilon$. It follows from this inequality, the Lipschitz condition (3.38) of Lemma 3.9, (3.37) and (3.75), that

$$\begin{aligned}
q_\pi^k(\alpha d_k + \tau p) - q_\pi^k(d_k) &\leq q_k^f(\alpha d_k + \tau p) - q_k^f(d_k) + \pi[-\tau\|h(x_k) + \nabla h(x_k)^T d_k\| + 2(1-\alpha)\epsilon] \\
&\leq \beta(\|\tau p\| + (1-\alpha)\|d_k\|) + \pi[-\tau\|h(x_k) + \nabla h(x_k)^T d_k\| + 2(1-\alpha)\epsilon] \\
&\leq \beta(1-\alpha)r + (\beta C_3 - \pi)\tau\|h(x_k) + \nabla h(x_k)^T d_k\| + 2\pi(1-\alpha)\epsilon] \\
&\leq (\beta C_3 - \pi)\tau\|h(x_k) + \nabla h(x_k)^T d_k\| + (\beta r + 2\pi\epsilon)(1-\alpha).
\end{aligned}$$

Since we have chosen τ and α to satisfy (3.76), we have

$$\begin{aligned} q_\pi^k(\alpha d_k + \tau p) - q_\pi^k(d_k) &\leq [(\beta C_3 - \pi) + (\beta r + 2\pi\epsilon)C_3/\gamma]\tau\|h(x_k) + \nabla h(x_k)^T d_k\| \\ &\leq [(\beta C_3 + \beta r C_3/\gamma) + \pi(-1 + 2C_3\epsilon/\gamma)]\tau\|h(x_k) + \nabla h(x_k)^T d_k\|. \end{aligned}$$

If the neighborhood of x_* is chosen small enough that $\epsilon C_3/\gamma < 1/4$, then for $\pi > 2\beta C_3(1 + r/\gamma)$, we have that $q_\pi^k(\alpha d_k + \tau p) - q_\pi^k(d_k) < 0$, which contradicts the fact that d_k is the minimizer of q_{π_k}. Therefore, we must have that $h(x_k) + \nabla h(x_k)^T d_k = 0$, and this concludes the proof. □

We can now prove the main convergence result of this paper.

Theorem 3.12 *Suppose that Algorithm I generates an infinite sequence of iterates $\{x_k\}$ and that Assumptions I hold. Then,*

(a) If $\{\pi_k\}$ is bounded, any limit point of $\{x_k\}$ is either a KKT point of the nonlinear program (2.1) *or is an infeasible stationary point;*

(b) If $\{\pi_k\} \to \infty$, then either there is a limit point x_* that is an infeasible stationary point or there is a feasible limit point x_* where MFCQ fails.

Proof. Part (a) follows directly from Theorem 3.7.

To prove part (b), when $\{\pi_k\} \to \infty$, consider an infinite subsequence $x_k, k \in \mathcal{K}$ over which π_k is increased without bound. Since by Assumption A1 this sequence is bounded, it has at least one limit point, say x_*.

Suppose that $v(x_*) > 0$. Then by Lemma 3.8, if $m_*(0) - m_*(d^{\rm LP}) > 0$, the penalty parameter π can be increased only finitely often in a neighborhood of x_*. So the fact that x_* is a limit point of the sequence $x_k, k \in \mathcal{K}$ defined above, implies that $m_*(0) - m_*(d^{\rm LP}) = 0$, i.e. that x_* is an infeasible stationary point (see Theorem 3.2 (c)).

Suppose on the other hand that $v(x_*) = 0$. If x_* satisfies MFCQ, then by Lemma 3.11 we have that, for π_k sufficiently large, $m_k(d_k) = 0$, for all x_k in a neighborhood of x_*. By Step 2 of Algorithm I, this implies that once π_k is large enough it will no longer be increased in this neighborhood of x_*. This contradicts our assumption that x_* is the limit point of a subsequence over which the penalty parameter is increased without bound. Therefore, MFCQ must fail at x_*.

□

4 Numerical Experiments

We developed a MATLAB implementation of Algorithm I and tested its performance on several difficult situations. We present results on five small-dimensional examples that exhibit inconsistent constraint linearizations at some iterate or that fail to satisfy the linear independence constraint qualification at the solution. One of the test problems is infeasible. The analysis in this paper indicates that the algorithm should be very robust, and these examples are chosen to test that robustness in cases where the theory applies and in cases that go beyond the theory. Important issues related to the efficient sparse implementation of Algorithm I are not addressed here as they lie outside the scope of this paper.

To solve the subproblems in Algorithm I, we employed the codes provided by the MATLAB OPTIMIZATION TOOLBOX. The linear program (2.9) was solved using `linprog` and the quadratic program (2.7), using `quadprog`.

We mentioned in Section 2 that the trust region radius Δ_k used in the linear program (2.9) is not crucial; in fact the convergence properties established in the previous section hold even if this radius is kept constant. In practice, however, it may be advantageous to choose Δ_k based on local information of the problem, and in our implementation this choice is based on the most recently accepted step. Given the search direction d_k and the step length α_k at the end of iteration k of Algorithm I, we compute

$$a^k_{red} = \phi_{\pi_+}(x_k) - \phi_{\pi_+}(x_k + \alpha_k d_k), \qquad p^k_{red} = q_k^{\pi_+}(0) - q_k^{\pi_+}(\alpha_k d_k)$$

and update the LP trust region radius as follows

Procedure for Updating Δ_k.
Initial data: $\eta_1 < \eta_2 \in (0,1)$, $\Delta_{min}, \Delta_{\max} > 0$.
If $a^k_{red} < \eta_1 p^k_{red}$
    set $\Delta_{k+1} = \frac{1}{2}\|\alpha_k d_k\|$;
else if $a^k_{red} > \eta_2 p^k_{red}$
    set $\Delta_{k+1} = 2\|\alpha_k d_k\|$;
else
    set $\Delta_{k+1} = \|\alpha_k d_k\|$;
Set $\Delta_{k+1} = \text{mid}(\Delta_{\min}, \Delta_{k+1}, \Delta_{\max})$.

The initial penalty π_1 is set to 1 in all tests. Algorithm I stops in Step 1 and reports optimality if the infinity norm of the KKT error is less than 10^{-6}. Convergence to an infeasible stationary point is reported if Algorithm I executes Step 3 and $m_k(0) - m_k(d^{\text{LP}}) < 10^{-15}$. We multiply π by 10 whenever it is increased and set $\Delta_{\min}, \Delta_1, \Delta_{\max}$ to $10^{-3}, 1, 10^3$, respectively. The rest of the parameters are chosen as $\tau = 0.5$, $\eta = 10^{-4}$, $\epsilon_1 = \epsilon_2 = 0.1$ and $\eta_1 = 0.25, \eta_2 = 0.75$.

The first example illustrates the behavior of Algorithm I when the linearizations of the constraints are inconsistent. In this situation, some SQP methods trigger a switch and revert either to a feasibility restoration phase in which the objective function is ignored, as in FILTERSQP [15], or to an elastic mode (ℓ_1 minimization) phase, as in SNOPT [18]. There is no switch in Algorithm I, which always takes steps based on the penalty function ϕ_π. An important difference between Algorithm I and the penalty update strategy in SNOPT is that the latter follows a traditional approach in which the penalty parameter is held fixed during the course of the minimization, and is only increased when a stationary point of the penalty function is approximated. Algorithm I, on the other hand, employs the *steering rules* described in §2 for updating π.

Example 1. The problem

$$\begin{array}{rrcl} \text{minimize} & x_1 & & \\ \text{subject to} & x_1^2 + 1 - x_2 & = & 0, \\ & x_1 - 1 - x_3 & = & 0, \\ & x_2 \geq 0,\ x_3 & \geq & 0 \end{array} \tag{4.1}$$

was introduced by Wächter and Biegler [29] to show that a class of line search interior-point methods may converge to a non-stationary point. We use the starting point $(-3, 1, 1)$; the solution is $x_* = (1, 2, 0)$. The output is summarized in Table 1, which reports the iteration number (it) the value of the penalty parameter π_k, the trust region Δ_{k-1} used to generate the latest iterate, the number of quadratic programs (QPs) solved at the current iterate, and the number of linear programs (LPs) solved (0 or 1). The table also prints the values of the first two components of x, the KKT and feasibility errors, as well as the value of the objective function f and the penalty function ϕ.

Table 1: Output for Example 1.

it	π_k	Δ_{k-1}	QPs	LPs	x_1	x_2	KKT(x)	feas(x)	$f(x)$	$\phi_{\pi_+}(x)$
0	1				-3	1	9.00E+00	1.40E+01	-3.00E+00	1.10E+01
1	1	1.00E+00	1	1	-1.5405	1.2432	2.54E+00	4.67E+00	-1.54E+00	3.13E+00
2	1	2.92E+00	1	1	-0.9428	1.5316	1.94E+00	2.30E+00	-9.43E-01	1.36E+00
3	1	1.20E+00	1	1	-0.8115	1.6414	1.81E+00	1.83E+00	-8.12E-01	1.02E+00
4	1	2.63E-01	1	1	-0.8043	1.6468	1.80E+00	1.80E+00	-8.04E-01	1.00E+00
5	1	1.45E-02	2	1	-0.2924	0.8234	6.10E+00	1.55E+00	-2.92E-01	1.53E+01
6	10	8.23E-01	1	0	0.3538	0.5766	8.25E+00	1.19E+00	3.54E-01	1.23E+01
7	10	6.46E-01	1	1	1	1.4265	1.23E+01	5.73E-01	1.00E+00	6.73E+00
8	10	1.70E+00	1	0	1	2	5.73E-01	2.69E-16	1.00E+00	1.00E+00
9	10	1.15E+00	1	0	1	2	2.22E-15	2.69E-16	1.00E+00	1.00E+00

The linearized constraints are not satisfied in the first few iterations, i.e., if $m_k(d_k) > 0$ and Algorithm I therefore solves the linear feasibility LP subproblem. Note that progress toward the solution is made during these initial iterations. The penalty parameter is increased only once, at iteration 5, meaning that the initial penalty $\pi = 1$ adequately relaxed the constraints at the earlier iterations. At iteration 5, the search direction has to be recomputed and hence two QPs are solved. Accurate optimal values for the primal variables are found at iteration 8, but Algorithm I performs one extra iteration to determine the correct final multipliers.

Example 2. The problem

$$\begin{array}{rl} \text{minimize} & (x_2 - 1)^2 \\ \text{subject to} & x_1^2 = 0, \\ & x_1^3 = 0 \end{array} \tag{4.2}$$

is presented in Fletcher et al. [17] and is also discussed by Chen and Goldfarb [10]. MFCQ is violated at the solution $x_* = (0, 1)$, and the linearized constraints are inconsistent at every infeasible point.

Fletcher et al. [17] mention that, starting from the infeasible point $(1, 0)$, a feasibility restoration phase is likely to converge to $(0, 0)$, which is not the solution of the problem. We ran the FILTERSQP solver [14] and observed that it did indeed converge to $(0, 0)$. Algorithm I does not exhibit such behavior. The sequence of iterates moves toward the solution from the very first step, and is not attracted to the origin because the objective function influences the choice of search direction. As shown in Table 2, the linearized constraints are never satisfied (an LP is solved at every iteration) but the algorithm finds that the penalty $\pi = 1$ is adequate to enforce progress. (No LP is solved in the last iteration, because the optimality stopping test is satisfied at that point). Thus, although the linear feasibility subproblem (2.8) of Algorithm I has some of the flavor of a feasibility restoration phase, it is only used to determine the penalty parameter and not to compute iterates, which is beneficial in this example. □

Table 2: Output for Example 2.

it	π_k	Δ_{k-1}	QPs	LPs	x_1	x_2	KKT(x)	feas(x)	$f(x)$	$\phi_{\pi_+}(x)$
0	1				1	0	2.00E+00	2.00E+00	1.00E+00	3.00E+00
1	1	1.00E+00	1	1	0.6667	0.6667	6.67E-01	7.41E-01	1.11E-01	8.52E-01
2	1	1.33E+00	1	1	0.4444	0.8736	3.95E-01	2.85E-01	1.60E-02	3.01E-01
3	1	4.44E-01	1	1	0.2222	0.952	2.59E-01	6.04E-02	2.30E-03	6.27E-02
4	1	4.44E-01	1	1	0.1111	1	6.48E-02	1.37E-02	0.00E+00	1.37E-02
5	1	2.22E-01	1	1	0.0556	1	1.62E-02	3.26E-03	4.93E-32	3.26E-03
6	1	1.11E-01	1	1	0.0278	1	4.05E-03	7.93E-04	4.93E-32	7.93E-04
7	1	5.56E-02	1	1	0.0139	1	1.01E-03	1.96E-04	1.23E-32	1.96E-04
8	1	2.78E-02	1	1	0.0069	1	2.53E-04	4.86E-05	0.00E+00	4.86E-05
9	1	1.39E-02	1	1	0.0035	1	6.33E-05	1.21E-05	4.93E-32	1.21E-05
10	1	6.94E-03	1	1	0.0017	1	1.58E-05	3.02E-06	4.93E-32	3.02E-06
11	1	3.47E-03	1	1	0.0009	1	3.96E-06	7.54E-07	1.23E-32	7.54E-07
12	1	1.74E-03	1	0	0.0009	1	8.48E-07	7.54E-07	1.23E-32	7.54E-07

Example 3. The following problem belongs to the class of mathematical programs with complementarity constraints (MPCCs). These problems have received much attention in recent years because of their many practical applications [12]; they can be challenging to solve because MFCQ is violated at every feasible point. The problem is given by

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 && && (4.3)\\ \text{subject to} \quad & x_2^2 - 1 &&\geq 0, \\ & x_1 x_2 &&\leq 0, \\ & x_1 \geq 0,\ x_2 &&\geq 0. \end{aligned}$$

The solution is $x_* = (0, 1)$ and is a *strongly stationary point*, which in the context of this paper means that there is a finite value of the penalty parameter π_* such that x_* is a stationary point for the penalty function $\phi_\pi(x)$, for all $\pi \geq \pi_*$. Equivalently, there exist multipliers at x_* that satisfy the KKT conditions for (2.1) (although these multipliers are not unique; in fact the set of multipliers is unbounded). Fletcher et al. [16] show that the linearization of the constraints of this problem is inconsistent for any point of the form $(\epsilon, 1 - \delta)$, with $\epsilon, \delta > 0$.

In the results reported in Table 3, the starting point was chosen as $(0.1, 0.9)$. At the first iteration, the linearized constraints are not satisfied; the search direction computed with the initial penalty parameter satisfies condition (2.12), and the penalty parameter is not increased. At the second iteration, the search direction violates the linearized constraints, and the LP subproblem indicates that the linearized constraints can be satisfied. The

Table 3: Output for Example 3.

it	π_k	Δ_{k-1}	QPs	LPs	x_1	x_2	KKT(x)	feas(x)	$f(x)$	$\phi_{\pi_+}(x)$
0	1				1.00E-01	9.00E-01	1.70E+00	2.80E-01	1.00E+00	1.28E+00
1	1	1.00E+00	1	1	-1.17E-02	1.01E+00	3.08E-01	1.17E-02	9.94E-01	1.01E+00
2	1	2.23E-01	3	1	-6.42E-17	1.01E+00	1.00E+00	6.42E-17	1.01E+00	1.01E+00
3	100	2.35E-02	1	0	-1.28E-16	1.00E+00	4.82E-01	1.28E-16	1.00E+00	1.00E+00
4	100	1.11E-02	1	0	-2.57E-16	1.00E+00	3.07E-05	2.57E-16	1.00E+00	1.00E+00
5	100	1.00E-03	1	0	-2.57E-16	1.00E+00	2.36E-10	2.57E-16	1.00E+00	1.00E+00

penalty parameter needs to be increased twice so that the solution of the QP satisfies the linearized constraints. The new iterate is feasible and from that point on, the iterates converge quadratically to the solution.

Several specialized methods have been developed in recent years that exploit the structure of MPCCs (see e.g. [2, 3, 11, 16, 22, 23, 24, 27]. In these methods, the complementarity constraints must be singled out and relaxed (or penalized). Algorithm I is, in contrast, a general-purpose nonlinear programming solver that treats MPCCs as any other problem. □

Example 4. MFCQ is also violated in a subset of the feasible region in the class of *switch-off* problems, also known as problems with *vanishing constraints*; see Achtziger and Kanzow [1]. An instance of such problems is

$$\begin{aligned} \text{minimize} \quad & 2(x_1 + x_2) \\ \text{subject to} \quad x_1 & \geq 0, \\ x_1 x_2 & \geq 0, \\ x_2 & \geq -1, \end{aligned} \tag{4.4}$$

which has a unique solution at $x_* = (0, -1)$. The feasible region is the union of the first quadrant, where the constraints are regular (except at the origin), and a portion of the negative x_2–axis, where MFCQ is violated. The performance of Algorithm I, starting at the origin, is summarized in Table 4.

Table 4: Output for Example 4.

it	π_k	Δ_{k-1}	QPs	LPs	x_1	x_2	KKT(x)	feas(x)	$f(x)$	$\phi_{\pi_+}(x)$
0	1				0	0	1.00E+00	0.00E+00	0.00E+00	0.00E+00
1	1	1.00E+00	2	1	0	-1	9.00E+00	0.00E+00	-2.00E+00	-2.00E+00
2	10	2.00E+00	1	0	0	-1	2.66E-15	0.00E+00	-2.00E+00	-2.00E+00

At the starting point, the direction obtained by solving subproblem (2.7) is given by $d = (-1, -1)$ and leads away from the feasible region. Algorithm I, however, discards this direction because it does not satisfy the linearization of the constraints, which are satisfiable. The penalty parameter is increased from 1 to 10, and the new search direction $d = (0, -1)$ not only satisfies the linearized constraints but leads straight to the solution of problem (4.4). The second iteration is required simply to compute the optimal multipliers.

This example shows how a prompt identification of an inadequate penalty parameter can save a great deal of computational work. Classical penalty methods would not increase the penalty parameter at the first iteration and would follow the initial direction $(-1,-1)$. The penalty function ϕ_π is unbounded below along this direction and a classical algorithm would generate a series of iterates with decreasing values of x_1 until the algorithm detects that the iterates appear to be diverging. Not only would those iterations be wasted, but extra effort would be required to return to the vicinity of the solution.

□

Example 5. The following infeasible problem has been studied by Burke and Han [5] and Chen and Goldfarb [10]:

$$\begin{aligned} \text{minimize} \quad & x \\ \text{subject to} \quad & x^2 + 1 \leq 0, \\ & x \leq 0. \end{aligned} \tag{4.5}$$

Chen and Goldfarb report that, starting from $x_1 = 10$, their method converges to the infeasible stationary point $x_* = 0$ after 50 iterations, with a final penalty parameter of $\pi \simeq 10^6$. Algorithm I converges to that infeasible stationary point in 3 iterations; see Table 5.

It may seem surprising that the final penalty parameter reported in the Table 5 is only 10, given that our analysis suggests that the penalty parameter will tend to infinity in the infeasible case. We note, however, that $\pi = 10$ is the penalty at the beginning of iteration 3 and that Algorithm I drives the penalty parameter to infinity in Step 4. It does so, while staying at the current iterate, and once it detects that the problem is locally infeasible, it stops. □

Table 5: Output for Example 5.

it	π_k	Δ_{k-1}	QPs	LPs	x	KKT(x)	feas(x)	$f(x)$	$\phi_{\pi_+}(x)$
0	1				10	1.01E+02	1.11E+02	1.00E+01	1.21E+02
1	1	1.00E+00	1	1	4.95	2.55E+01	3.05E+01	4.95E+00	3.54E+01
2	1	1.01E+01	2	1	0	4.01E+00	1.00E+00	8.88E-16	1.00E+01
3	10	9.90E+00	1	1	0	1.00E+00	1.00E+00	8.88E-16	1.00E+01

More results obtained with our MATLAB implementation of Algorithm I are reported in [24]. The test set used in those experiments includes both regular problems and problems that are infeasible or do not satisfy constraint qualifications. The results in [24] indicate that Algorithm I is efficient on problems that do not require regularization because condition (2.11) guarantees that a pure SQP step is used whenever the linearized constraints can be satisfied in a neighborhood of the current iterate. Therefore, for regular problems, Algorithm I performs very similarly to a classical SQP method, except that a few extra QPs and LPs are solved when the initial penalty parameter is too small. On the other hand, Algorithm I is more robust and efficient than a classical SQP method on problems (such as those in the Examples 1-5) that require regularization. We believe that by using warm starts, the cost of solving the additional QPs is not significant, but a careful sparse implementation of Algorithm I is needed to measure the computational cost of various components of the iteration.

5 Final Remarks

In this paper we have proposed a line search SQP penalty method for nonlinear programming. The method updates the penalty parameter dynamically using an extension of the *steering rules* described in [8] to the line search setting. The resulting algorithm is robust and its global convergence properties are as strong as those of trust region methods. Specifically, we have proved that under common assumptions all limit points of the sequence of iterates are either KKT points, infeasible stationary points, or points of MFCQ failure. This fact shows that use of exact penalties, together with a positive definite Hessian approximation, has a regularizing effect similar to a trust region.

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