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The $Ph_t/Ph_t/\infty$ Queueing System: Part I—The Single Node

Barry L. Nelson
Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, Illinois 60208, nelsonb@northwestern.edu

Michael R. Taaffe
Department of Industrial and Systems Engineering, Virginia Tech, Blacksburg, Virginia 24061, taaffe@vt.edu

We develop a numerically exact method for evaluating the time-dependent mean, variance, and higher-order moments of the number of entities in a $Ph_t/Ph_t/\infty$ queueing system. We also develop a numerically exact method for evaluating the distribution function and moments of the virtual sojourn time for any time t; in our setting, the virtual sojourn time is equivalent to the service time for virtual entities arriving to the system at that time t. We include several examples using software that we have developed and have put in downloadable form in the Online Supplement to this paper on the journal's website.

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1. Introduction

Most real-world queueing systems exhibit some sort of time-dependent behavior, including but not limited to time-varying arrival and service processes. However, analysis of the time-dependent behavior of even the simple $M_t/M_t/\infty$ queueing system, having general initial conditions, requires numerical integration of an infinite number of differential-difference equations for general, real-valued and integrable arrival/service-rate functions. Although we know that the time-dependent number of entities in the system for the $M_t/M_t/\infty$ system, with empty and idle initial conditions, has a time-dependent Poisson distribution, we do not know, in general, the value of its mean without at least integrating the differential equations for the mean (Massey and Whitt 1993).

Consider the *moment differential equations* (MDEs) approach for analysis of the $M_t/M_t/\infty$ system. The MDEs are the derivatives of the moments of $N(t)$—the number of entities in the system at time t—with respect to time. The MDEs for the $M_t/M_t/\infty$ system are *closed*, meaning the right-hand side (RHS) of the pth MDE contains *no* state probabilities and no moments of order greater than p, just system parameters (the time-dependent arrival and service rates) and system moments of order p or less. As a result, we can obtain the time-dependent expected number in the system, $\mathrm{E}[N(t)]$, by solving a single differential equation, the first MDE. If the initial conditions are indeed empty and idle, then for any particular time t we can numerically evaluate just one differential equation, the first MDE, from time 0 to time t and compute individual state probabilities via

$$\mathrm{P}(N(t)=i)=\frac{(\mathrm{E}[N(t)])^i}{i!}e^{-\mathrm{E}[N(t)]}$$

for $i=0,1,\ldots$. For more general initial conditions we could approximate the time-dependent distribution of the number of entities in the system at time t via numerical integration of the first two MDEs, for instance, and then match those two moments to some approximate distribution; see Ong and Taaffe (1987, 1988, 1989).

In this paper we develop similar results for the number of entities in a $Ph_t/Ph_t/\infty$ system, where Ph_t denotes a time-dependent generalization of a phase-type renewal process (Neuts 1981; Taaffe and Ong 1984; Ong and Taaffe 1987, 1988, 1989). Stated differently, the interarrival-time and the service-time random variables are each represented by the time until absorption in a finite-state, nonstationary Markov process having exactly one absorbing state.

Specifically, we develop the *partial-moment differential equations* (PMDEs) for the number of entities currently in phase l of their service and the arrival process in phase i. We show that the first PMDEs are closed; i.e., there are no state probabilities on the RHSs of the PMDEs. The corresponding MDEs are *quasi-closed*, because their RHSs contain only system parameters and system partial moments, which are themselves closed. Therefore, the time-dependent values of, say, the mean and variance of the number of entities in the system at time t can be computed by numerically integrating a small number of differential equations whose size is a function of the number of arrival-process and service-process phases. The standard solution methods for the state probabilities of this system require evaluation of an infinite number of differential equations, or a very large number of differential equations if we truncate the state space.

The paper is organized as follows. In §2 we briefly review major results for infinite-server, time-dependent Markovian queueing models. In §3 we define the $Ph_t/Ph_t/\infty$ system. Section 4 develops the MDEs and PMDEs and establishes their closure properties. We present several numerical examples in §5 and provide downloadable software for evaluating $Ph_t/Ph_t/\infty$ models. The virtual sojourn time cdf and moment calculations are described in §6 along with a numerical example. Finally, §7 offers a brief set of conclusions and an indication of some extensions of the $Ph_t/Ph_t/\infty$ queue to be presented in the companion paper to this one by Nelson and Taaffe (2004).

2. Background

Infinite-server queueing models have been the subject of research in recent years, partially because of their central role in approximating systems with many servers. Time-dependent arrival and service processes for a variety of queueing models have also received increasing attention from the applied probability research community because few real-world systems are truly time homogeneous.

Applications for models of time-dependent, infinite-server queueing models include population processes in biology, migration processes, and immigration processes. Perhaps the most interesting application for such models is found in the wireless telecommunication industry. Time-dependent, infinite-server networks of queues have become a standard model for analysis of mobile cellular telecommunication system design and management problems (Lee 1989).

The intersection of the two topics—time-dependent arrival/service processes and infinite-server (network) queueing models—is the focus of some classic papers as well as a wealth of recent applied probability research (see, for instance, Brown and Ross 1967, Collings and Stoneman 1976, Eick et al. 1993a, Foley 1984, Glynn and Whitt 1991, Harrison and Lemoine 1981, Massey and Whitt 1993, Mirasol 1963, Newell 1966, Whitt 1982). Massey and Whitt (1993) contains a thorough review of what is known about time-dependent, Poisson-arrival, infinite-server queues and networks of time-dependent, exogeneous Poisson-arrival, infinite-server queues.

A fundamental result for the $M_t/G_t/\infty$ queue, and thus the $M_t/Ph_t/\infty$, is that the distribution of the number of entities in the system at any time t, given that the initial conditions are empty and idle, is the time-dependent Poisson distribution. Thus, an ordinary differential equation for the time-dependent mean is sufficient to characterize the time-dependent distribution fully (Massey and Whitt 1994). There are no similar results for time-dependent, infinite-server queueing models where the arrival process is *not* (time-dependent) Poisson. In this paper, we extend results in the literature to the more general case of time-dependent infinite-server models having general (Ph_t) arrival. *Ph*-type processes are general because the family of phase distributions is dense over the space of probability distributions with support on $[0, \infty)$. We develop numerically exact solutions and demonstrate that the time-dependent distribution of the number of entities in the system at time t is *not* Poisson, in general.

3. The $Ph_t/Ph_t/\infty$ System

In this section we define the $Ph_t/Ph_t/\infty$ queueing system and present the Kolmogorov forward equations for the time-dependent system state.

3.1. The Arrival Process

We represent the time-dependent phase-type arrival process by its underlying Markov chain, the vector of time-dependent Poisson rates associated with each (nonabsorbing) state, and the vector of initial state probabilities.

Let

$$\boldsymbol{\mathcal{A}}(t) = \begin{pmatrix} \boldsymbol{\mathcal{A}}_1(t) & \boldsymbol{\mathcal{A}}_2(t) \\ \boldsymbol{\alpha}(t)^T & 0 \end{pmatrix}$$

be the one-step transition matrix of the Markov chain underlying the m_A-phase arrival process, where

$$\boldsymbol{\mathcal{A}}_1(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1m_A}(t) \\ \vdots & \ddots & \vdots \\ a_{m_A 1}(t) & \cdots & a_{m_A m_A}(t) \end{pmatrix}$$

is the matrix of transient-to-transient state transition probabilities, and

$$\boldsymbol{\mathcal{A}}_2(t) = \begin{pmatrix} a_{1,\, m_A+1}(t) \\ \vdots \\ a_{m_A,\, m_A+1}(t) \end{pmatrix}$$

is the vector of transition probabilities from transient states to state $m_A + 1$, the instantaneous absorbing state representing an entity arriving to a server. The vector $\boldsymbol{\alpha}(t) = [\alpha_1(t), \ldots, \alpha_{m_A}(t)]^T$ contains the initial arrival-phase probabilities for the next entity to start through the arrival process.

Let $\boldsymbol{\lambda}(t) = [\lambda_1(t), \ldots, \lambda_{m_A}(t)]^T$ be the vector of real-valued integrable rate functions for transient states of the arrival process. Thus, the $(m_A + 1)$-dimensional rate vector for the entire phase arrival process is $[\boldsymbol{\lambda}(t)^T, \infty]$. The infinite rate corresponds to the instantaneous absorbing state.

Finally, let $\{A(t);\ t \geq 0\}$ be the random process representing the arrival phase of the next arrival to the system at time t, where $A(t) \in \{1, \ldots, m_A\}$. The instantaneous absorbing state need not be explicitly represented.

3.2. The Service Process

We define the (m_B+1)-dimensional phase-type service process in a manner similar to the definition of the arrival process. Let

$$\mathcal{B}(t) = \begin{pmatrix} \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \boldsymbol{\beta}(t)^T & 0 \end{pmatrix},$$

where

$$\mathcal{B}_1(t) = \begin{pmatrix} b_{11}(t) & \cdots & b_{1m_B}(t) \\ \vdots & \ddots & \vdots \\ b_{m_B 1}(t) & \cdots & b_{m_B m_B}(t) \end{pmatrix}$$

is the underlying Markov chain one-step transition matrix for transient-to-transient state transitions, and

$$\mathcal{B}_2(t) = \begin{pmatrix} b_{1,\, m_B+1}(t) \\ \vdots \\ b_{m_B,\, m_B+1}(t) \end{pmatrix}$$

is the matrix of transition probabilities from transient states to the instantaneous absorbing state $m_B + 1$, which represents a service completion (a departure from the queue). The vector $\boldsymbol{\beta}(t) = [\beta_1(t), \ldots, \beta_{m_B}(t)]^T$ contains the initial service-phase probabilities for an entity completing its arrival process (and hence beginning its service process).

Let $\boldsymbol{\mu}(t) = [\mu_1(t), \ldots, \mu_{m_B}(t)]^T$ be the vector of real-valued integrable rate functions for the transient states of the service process, so that $[\boldsymbol{\mu}(t)^T, \infty]$ is the $(m_B + 1)$-dimensional rate vector for the entire phase service process.

3.3. The $Ph_t/Ph_t/\infty$ Kolmogorov Forward Equations

The state of the $Ph_t/Ph_t/\infty$ process at time t is given by

$$[\mathbf{N}(t), A(t)] = [\{N_1(t), \ldots, N_{m_B}(t)\}, A(t)],$$

where $\{N_i(t);\ t \geq 0\}$ for $i = 1, \ldots, m_B$ is the random process representing the number of entities who, at time t, are in the ith phase of their service. Therefore, the total number of entities in service at time t is $N(t) = \sum_{j=1}^{m_B} N_j(t)$. The instantaneous absorbing state in the service process need not be explicitly represented.

Let

$$\mathrm{P}(t; n_1, \ldots, n_{m_B}, k) \equiv \mathrm{P}(N_1(t) = n_1, \ldots, N_{m_B}(t) = n_{m_B},\ A(t) = k),$$

and

$$\mathrm{P}(t; n_1, \ldots, n_{m_B}, k)' \equiv \frac{d}{dt}\mathrm{P}(N_1(t) = n_1, \ldots, N_{m_B}(t) = n_{m_B},\ A(t) = k).$$

The infinite number of Kolmogorov forward equations for the model are tedious but straightforward to derive and are as follows:

$$\begin{aligned}
&\mathrm{P}(t; n_1, \ldots, n_{m_B}, k)' \\
&\quad = -\lambda_k(t)\mathrm{P}(t; n_1, \ldots, n_{m_B}, k) \\
&\qquad - \sum_{l=1}^{m_B} n_l \mu_l(t)[1 - b_{ll}(t)]\mathrm{P}(t; n_1, \ldots, n_{m_B}, k) \\
&\qquad + \sum_{l=1}^{m_A} a_{l,\, m_A+1}(t)\alpha_k(t)\lambda_l(t) \\
&\qquad \cdot \left\{ \sum_{h=1}^{m_B} \mathrm{I}_{[n_h>0]}\beta_h(t)\mathrm{P}(t; n_1, \ldots, n_h - 1, \ldots, n_{m_B}, l) \right\} \\
&\qquad + \sum_{l=1}^{m_A} a_{lk}(t)\lambda_l(t)\mathrm{P}(t; n_1, \ldots, n_{m_B}, l) \\
&\qquad + \sum_{l=1}^{m_B} b_{l,\, m_B+1}(t)[n_l + 1]\mu_l(t) \\
&\qquad \cdot \mathrm{P}(t; n_1, \ldots, n_l + 1, \ldots, n_{m_B}, k) \\
&\qquad + \sum_{l=1}^{m_B} \mathrm{I}_{[n_l>0]} \left\{ \sum_{\substack{h=1 \\ h \neq l}}^{m_B} b_{hl}(t)[n_h + 1]\mu_h(t) \right. \\
&\qquad \left. \cdot \mathrm{P}(t; n_1, \ldots, n_l - 1, \ldots, n_h + 1, \ldots, n_{m_B}, k) \right\}, \qquad (1)
\end{aligned}$$

where

$$\mathrm{I}_{[a>0]} \equiv \begin{pmatrix} 0, & a \leq 0 \\ 1, & a > 0 \end{pmatrix}$$

for $k = 1, \ldots, m_A$, $n_h = 0, 1, \ldots, \infty$, $h = 1, \ldots, m_B$, and $t \geq 0$. Of course we also know that for all t,

$$\sum_{k,\, n_1, \ldots, n_{m_B}} \mathrm{P}(t; n_1, \ldots, n_{m_B}, k) = 1.$$

4. The MDEs, PMDEs, and Their Closure Properties

In this section we derive the MDEs and PMDEs for the $Ph_t/Ph_t/\infty$ queueing system. For the purpose of intuition and motivation, we start by presenting the first MDEs for the $M_t/M_t/\infty$ and $M_t/M_t/1$ systems and observe their similar structure. We also observe that their MDEs are closed or *almost* closed. In §4.2 we develop a similar structure and set of closure properties for the $Ph_t/Ph_t/\infty$ system.

4.1. The $M_t/M_t/\infty$ and $M_t/M_t/1$ First MDEs

For most queueing systems the MDEs are *not closed*; i.e., the RHSs of the MDEs contain individual state probabilities. Thus, for most time-dependent queueing systems we must numerically integrate the entire set of Kolmogorov forward equations for the state probabilities to evaluate the set of time-dependent moments of the number of entities in the system, or develop an approximation algorithm to *close* the set of differential equations.

The $M_t/M_t/\infty$ system is an exception to this rule. For that system the set of MDEs is closed. The first MDE for the $M_t/M_t/\infty$ system is:

$$\frac{d}{dt}\mathrm{E}[N(t)] = \lambda(t) - \mu(t)\mathrm{E}[N(t)]. \tag{2}$$

Thus, for the $M_t/M_t/\infty$ we need only numerically integrate one MDE to obtain the set of time-dependent first moments. This result can be obtained by multiplying both sides of the ith Kolmogorov forward equation for the $M_t/M_t/\infty$ by i and summing. When this is done all of the state probabilities on the RHS cancel—thus, the MDE is closed. Massey and Whitt (1993) show similar closed MDEs for the $M_t/G/\infty$ system.

The form of the first MDE for the $M_t/M_t/\infty$ is intuitive because the positive flux is the system arrival rate, and the negative flux is the system departure rate. Because the first MDE is closed, an infinite amount of numerical work can be avoided in computing the trajectory of the time-dependent first moment. This is also true for trajectories of all higher-order moments, as will be shown later.

Notice that for the stationary case of this model—when the arrival and service rates are constant with respect to time—we can easily confirm that the steady-state value of the first moment of $N \equiv \lim_{t\to\infty} N(t)$ is

$$\mathrm{E}[N] = \frac{\lambda}{\mu}.$$

This result is obtained by setting $d\mathrm{E}[N]/dt = 0$ in (2) and solving for $\mathrm{E}[N]$.

Now consider the $M_t/M_t/1$ system. If we multiply both sides of the ith Kolmogorov forward equation by i and sum we get the following first MDE:

$$\frac{d}{dt}\mathrm{E}[N(t)] = \lambda(t) - \mu(t)[1 - P_0(t)]; \tag{3}$$

see Ong and Taaffe (1987). Observe the similarity in the structure of the first MDEs for these two models. Interpreting $\mathrm{E}[N(t)]$ on the RHS of Equation (2) as the expected number of busy servers at time t for the $M_t/M_t/\infty$, and $1 - P_0(t)$ on the RHS of Equation (3) as the expected number of busy servers at time t for the $M_t/M_t/1$, we see that the two MDEs have exactly the same form.

The truncated versions of these models—the $M_t/M_t/c/c$ and $M_t/M_t/1/k$ systems—have the following first MDEs:

$$\frac{d}{dt}\mathrm{E}[N(t)] = \lambda(t)[1 - P_c(t)] - \mu(t)\mathrm{E}[N(t)], \quad \text{and}$$
$$\frac{d}{dt}\mathrm{E}[N(t)] = \lambda(t)[1 - P_k(t)] - \mu(t)[1 - P_0(t)],$$

respectively. We can interpret $1 - P_c(t)$ and $1 - P_k(t)$ as being the probabilities that the system is in a state that would allow for entities arriving at time t to be admitted to the system (for the $M_t/M_t/\infty$ and the $M_t/M_t/1$ this probability is 1). Thus, for all four of these systems the positive flux in the first-moment differential equation is the actual or effective *system* arrival rate, and the negative flux is the actual or effective *system* departure rate. We will demonstrate similar results and interpretations for the $Ph_t/Ph_t/\infty$ system.

For the $M_t/M_t/1$ system the existence of one probability (or two probabilities for the $M_t/M_t/1/k$) on the RHS implies that the MDEs are not closed. Taaffe and Ong (1984) develop approximations for $P_0(t)$ and $P_k(t)$ that are functions of the first two moments which then close the set of the first and second MDEs. We call a set of MDEs that require a closure approximation *pseudo-closed*. Ong and Taaffe (1987, 1988, 1989) develop closure approximations for a variety of nonstationary phase-type queueing models each having sets of pseudo-closed MDEs. In the cited papers the closure approximations are termed *surrogate distribution approximations*, or SDAs, but we now use the term *closure approximations*, as in Rothkopf and Oren (1979).

4.2. The MDEs and PMDEs for the $Ph_t/Ph_t/\infty$ System

Here we establish our key results for the $Ph_t/Ph_t/\infty$ queue. All of the proofs involve tedious summing and index-shifting operations, and are omitted. The details are given in the Online Supplement to this paper, available on the website of this journal.

Theorem 1. *The $Ph_t/Ph_t/\infty$ first MDE is*:

$$\frac{d}{dt}\mathrm{E}[N(t)] = \sum_{l=1}^{m_A} \lambda_l(t) a_{l,\,m_A+1}(t)\mathrm{P}(t;\cdot,l) - \sum_{j=1}^{m_B} \mu_j(t) b_{j,\,m_B+1}(t)\mathrm{E}[N_j(t)], \tag{4}$$

where $\mathrm{P}(t;\cdot,l)$ *is the marginal probability that the arrival process is in state* l *at time* t.

If we interpret each of the $\mathrm{E}[N_j(t)]$ terms as the expected number of servers busy in phase j of service, and interpret $\lambda_l(t)a_{l,m_A+1}(t)\mathrm{P}(t;\cdot,l)$ as the system arrival rate via arrival phase l, then the first MDEs positive flux is the effective system arrival rate and the negative flux is the effective system departure rate, analogous to the simple models of the previous section.

Notice that the RHS of the first MDE for $Ph_t/Ph_t/\infty$ is not closed because it contains expressions other than simply the system parameters (rates) and the current value of the first moment. Specifically, it includes the marginal moments of the number of entities in a particular phase of service at time t, $\mathrm{E}[N_j(t)]$, and marginal arrival-process state probabilities, $\mathrm{P}(t;\cdot,l)$.

In this section we develop the *arrival-process state-probability differential equations* (ADEs) and show that they *are* closed. We also develop the *marginal-moment differential equations* (MMDEs) and show that they are not closed. We then show that the set consisting of the ADEs and the MMDEs together *is* closed. Therefore, the first moment of the overall systems size can be computed by summing the first marginal moments, and we can evaluate the time-dependent behavior of the first moment of the overall system size without numerically integrating any of the individual system-state probabilities. Thus, no closure approximation is needed for this system. We call MDEs with this property *quasi-closed*; i.e., the system first MDE is *not* closed, but the first moment can be computed as a function of a set of other variables whose associated set of differential equations *is* closed.

However, we can do more than the first moment. We also show that we can evaluate the RHSs of the system MMDEs for moments greater than 1 by evaluating the marginal moments, arrival-state probabilities, and a set of *partial moments*. An example of a partial moment is:

$$\mathrm{E}[N_j(t),A(t)=k]\equiv\sum_{n_j=0}^{\infty}n_j\mathrm{P}(N_j(t)=n_j,\,A(t)=k)$$

for $j=1,\ldots,m_B$, $k=1,\ldots,m_A$. The *conditional* first moment $\mathrm{E}[N_j(t)\mid A(t)=k]$ is obtained by dividing the partial moment by $\mathrm{P}(A(t)=k)$ when $\mathrm{P}(A(t)=k)\neq 0$.

We show that the set of time-dependent moments for the $Ph_t/Ph_t/\infty$ system can be evaluated via the set of PMDEs, which is closed. The fact that the small set of PMDEs is closed eliminates the need for evaluation of the infinite set of state probabilities. We give specific formulas for the MDEs of orders one and two, allowing us to obtain the time-dependent mean, $\mathrm{E}[N(t)]$, and variance, $\mathrm{Var}[N(t)]$, of number in the system.

Define the following notation:

- $\mathrm{E}_j(t)\equiv\mathrm{E}[N_j(t)]$
  First Marginal Moment
- $\mathrm{P}(t;\cdot,k)\equiv\sum_{n_1,n_2,\ldots,n_{m_B}}\mathrm{P}(t;n_1,\ldots,n_{m_B},k)$
  Arrival-Process State Probability
- $\mathrm{E}^p_{j,k}(t)\equiv\mathrm{E}[N^p_j(t),A(t)=k]$
  pth Partial Moment
- $\mathrm{E}^q_{ij,k}(t)\equiv\mathrm{E}[N^q_i(t)N_j(t),A(t)=k]$
  $(q,1)$th-order Cross-Product Partial Moment

Theorem 2. *The $Ph_t/Ph_t/\infty$ marginal first-moment differential equations (MMDEs) are*:

$$\mathrm{E}_j(t)'\equiv\frac{d}{dt}\mathrm{E}_j(t)=\beta_j(t)\sum_{l=1}^{m_A}a_{l,m_A+1}(t)\lambda_l(t)\mathrm{P}(t;\cdot,l)+\sum_{l=1}^{m_B}b_{lj}(t)\mu_l(t)\mathrm{E}_l(t)-\mu_j(t)\mathrm{E}_j(t)$$

for $j=1,\ldots,m_B$.

Notice that the RHSs of the MMDEs contain some arrival-process state probabilities; thus, the set of MMDEs is not closed. To evaluate the RHSs of the MMDEs we need to develop the ADEs.

Theorem 3. *The $Ph_t/Ph_t/\infty$ arrival-process state-probability differential equations (ADE's) are*:

$$\mathrm{P}(t;\cdot,k)'=\alpha_k(t)\sum_{l=1}^{m_A}a_{l,m_A+1}(t)\lambda_l(t)\mathrm{P}(t;\cdot,l)+\sum_{l=1}^{m_A}a_{lk}(t)\lambda_l(t)\mathrm{P}(t;\cdot,l)-\lambda_k(t)\mathrm{P}(t;\cdot,k)$$

for $k=1,\ldots,m_A$.

Clearly, the set of ADEs is closed. The theorem is quite intuitive, and detailed proofs can be found in the Online Supplement to this paper on the journal's website.

Notice that the sum of $\mathrm{E}_j(t)'$ over all j results in the first MDE, $\mathrm{E}(t)'$, and that neither the set $\mathrm{E}_j(t)'$, $j=1,\ldots,m_B$, nor $\mathrm{E}(t)'$, is closed. However, collectively, the set of first MMDEs given in Theorem 2 and the ADEs given in Theorem 3 *are* closed. Because the set consisting of the MMDEs and ADEs is closed, the time-dependent first moment can be evaluated by numerically integrating the set of MMDEs and ADEs and combining the results as $\mathrm{E}[N(t)]=\sum_{j=1}^{m_B}\mathrm{E}_j(t)$.

We can represent this system of simultaneous, linear differential equations in the following compact matrix form:

$$\mathbf{E}(t)'=\boldsymbol{\beta}(t)\big[\boldsymbol{\mathcal{A}}_2(t)^T\mathrm{Diag}[\mathbf{P}(t)]\boldsymbol{\lambda}(\mathbf{t})\big]+\big[\boldsymbol{\mathcal{B}}_1(t)^T-\mathbf{I}\big]\mathrm{Diag}[\mathbf{E}(t)]\boldsymbol{\mu}(\mathbf{t}),\quad\text{and}$$

$$\mathbf{P}(t)'=\boldsymbol{\alpha}(t)\big[\boldsymbol{\mathcal{A}}_2(t)^T\mathrm{Diag}[\mathbf{P}(t)]\boldsymbol{\lambda}(\mathbf{t})\big]+\big[\boldsymbol{\mathcal{A}}_1(t)^T-\mathbf{I}\big]\mathrm{Diag}[\mathbf{P}(t)]\boldsymbol{\lambda}(\mathbf{t}),$$

where

$$\mathbf{E}(t) = [\mathrm{E}_1(t), \ldots, \mathrm{E}_{m_B}(t)]^T,$$
$$\mathbf{P}(t) = [\mathrm{P}(t;\cdot,1), \ldots, \mathrm{P}(t;\cdot,m_A)]^T,$$
$$\mathbf{E}(t)' = [\mathrm{E}_1(t)', \ldots, \mathrm{E}_{m_B}(t)']^T,$$
$$\mathbf{P}(t)' = [\mathrm{P}(t;\cdot,1)', \ldots, \mathrm{P}(t;\cdot,m_A)']^T.$$

Diag[$\mathbf{X}$] is defined to be a matrix whose diagonal is composed of the elements of the vector $\mathbf{X}$ and 0s elsewhere, and $\mathbf{I}$ is the identity matrix of the appropriate dimension.

In this form, it is easy to obtain steady-state results for the stationary case where $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}$, $\boldsymbol{\mu}(t) = \boldsymbol{\mu}$, $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}$, and $\boldsymbol{\beta}(t) = \boldsymbol{\beta}$. Let $\mathbf{E} \equiv \lim_{t\to\infty} \mathbf{E}(t)$ and $\mathbf{P} \equiv \lim_{t\to\infty} \mathbf{P}(t)$. Then

$$\mathbf{E} = -\mathrm{Diag}[\boldsymbol{\mu}]^{-1}\left(\boldsymbol{\mathcal{B}}_1^T - \mathbf{I}\right)^{-1}\boldsymbol{\beta}\,\boldsymbol{\mathcal{A}}_2^T\mathrm{Diag}[\mathbf{P}]\boldsymbol{\lambda}, \quad \text{and} \quad (5)$$

$$\mathbf{P} = -\mathrm{Diag}[\boldsymbol{\lambda}]^{-1}\left(\boldsymbol{\mathcal{A}}_1^T - \mathbf{I}\right)^{-1}\boldsymbol{\alpha}\,\boldsymbol{\mathcal{A}}_2^T\mathrm{Diag}[\mathbf{P}]\boldsymbol{\lambda}. \quad (6)$$

Thus, the steady-state expected number in the system for the stationary case (5) can be obtained by inverting a matrix of size m_B by m_B. Although the steady-state arrival-state probabilities are required to solve (5), they are easily obtained via any standard method for calculating the steady-state distribution of a small Markov process, including solving a set of simultaneous linear equations, uniformization-based methods (Tijms 1995), or the so-called GTH algorithm (Kao 1997). Iterative methods to solve (6) directly should also work well here because the state space is relatively small and we have the condition that the sum of the probabilities is always equal to one.

If the expected number of entities in a $Ph_t/Ph_t/\infty$ system had a time-dependent Poisson distribution, then computing E[$N(t)$] would be sufficient to characterize the entire time-dependent state distribution. Unfortunately, as we demonstrate in §5.2, the distribution of $N(t)$ (even assuming empty and idle initial conditions) and $\lim_{t\to\infty} N(t)$ need *not* be Poisson. Further, none of the $N_j(t)$ are Poisson in general, and neither are any of the $N_j(t)$ or $N(t)$ when conditioned on the arrival-process state. This is in contrast to the $M_t/G_t/\infty$ system where the distribution of $N(t)$ is Poisson when the initial condition is empty and idle. Because $N(t)$ for the $Ph_t/Ph_t/\infty$ system is not, in general, Poisson, $N(t)$ will not necessarily have equal mean and variance. Thus, we develop higher-order MDEs, MMDEs, and PMDEs to evaluate the higher-order moments of system size, concentrating on Var[$N(t)$] in particular.

Theorem 4. *The $Ph_t/Ph_t/\infty$ pth partial-moment differential equations (pth PMDE's) are*:

$$\begin{aligned}\mathrm{E}^p_{j,k}(t)' &\equiv \frac{d}{dt}\mathrm{E}^p_{j,k}(t)\\ &= -\lambda_k(t)\mathrm{E}^p_{j,k}(t)\\ &\quad + \mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}\\ &\quad + \alpha_k(t)\sum_{l=1}^{m_A}\lambda_l(t)a_{l,m_A+1}(t)\mathrm{E}^p_{j,l}(t)\\ &\quad + \alpha_k(t)\beta_j(t)\sum_{l=1}^{m_A}\lambda_l(t)a_{l,m_A+1}(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,l}(t)\\ &\quad + \sum_{l=1}^{m_A}\lambda_l(t)a_{lk}(t)\mathrm{E}^p_{j,l}(t)\\ &\quad + \sum_{\substack{l=1\\ l\neq j}}^{m_B}\mu_l(t)b_{lj}(t)\cdot\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{jl,k}(t)\end{aligned}$$

for $i = 1, \ldots, m_A$, $j = 1, \ldots, m_B$, and $p = 0, 1, \ldots$.

First observe that if $p = 0$ we retrieve the ADEs. Observe that these pth PMDEs are not closed because of the presence of $(q,1)$th-order cross-product terms of $\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{jl,k}(t)$ on the RHS; thus, PMDEs for these $(q,1)$th-order cross-product terms would need to be developed to evaluate higher-order moments numerically. However, if one is interested in only the mean and variance of the number of entities in the system at time t, as we are in this paper, then the $(q,1)$th-order cross-product terms simplify considerably. Observe that for $q = 1$ the $(q,1)$th-order cross-product terms simplify to $\mathrm{E}_{l,k}(t)$; thus the first PMDE is indeed closed. Also observe that for $q = 2$ the $(q,1)$th-order cross-product terms simplify to a sum of a first partial-moment and simple first-order cross-product partial-moment terms, $(\mathrm{E}_{h,k}(t) + 2\mathrm{E}_{jh,k}(t))$. Although the second PMDE is not closed, the only cross-product terms present are simple first-order cross-product partial-moment terms. We next present the first-order cross-product PMDE so that we can evaluate both the 2nd PMDEs and Var[$N(t)$].

Theorem 5. *The $Ph_t/Ph_t/\infty$ first-order cross-product partial-moment differential equations are*:

$$\begin{aligned}\mathrm{E}_{ij,k}(t)' &\equiv \frac{d}{dt}\mathrm{E}_{ij,k}(t)\\ &= \sum_{l=1}^{m_B}b_{li}(t)\mu_l\mathrm{E}_{lj,k}(t) + \sum_{l=1}^{m_B}b_{lj}(t)\mu_l\mathrm{E}_{li,k}(t)\\ &\quad - b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t) - b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t)\\ &\quad - [\mu_i(t)+\mu_j(t)]\mathrm{E}_{ij,k}(t) - \lambda_k(t)\mathrm{E}_{ij,k}(t)\\ &\quad + \sum_{l=1}^{m_A}\lambda_l(t)a_{lk}(t)\mathrm{E}_{ij,l}(t)\\ &\quad + \alpha_k(t)\sum_{l=1}^{m_A}\lambda_l(t)a_{l,m_A+1}(t)\mathrm{E}_{ij,l}(t)\end{aligned}$$

$$+\beta_i(t)\alpha_k(t)\sum_{l=1}^{m_A}\lambda_l(t)a_{l,m_A+1}(t)\mathrm{E}_{j,l}(t)$$

$$+\beta_j(t)\alpha_k(t)\sum_{l=1}^{m_A}\lambda_l(t)a_{l,m_A+1}(t)\mathrm{E}_{i,l}(t)$$

for $k=1,\ldots,m_A$, $i=1,\ldots,m_B$, $j=1,\ldots,m_B$, *and* $i\neq j$.

The entire set of differential equations described by Theorems 1, 2, 3, and 5 are collectively closed; thus $\mathrm{Var}[N(t)]$ can be evaluated in a small number of differential equations rather than the infinite set of differential equations described in Equation (1). Specifically,

$$\mathrm{Var}[N(t)]=\sum_{j=1}^{m_B}\sum_{k=1}^{m_A}\mathrm{E}_{j,k}^2(t)-\left[\sum_{j=1}^{m_B}\mathrm{E}_j(t)\right]^2.$$

The number of differential equations required to evaluate $\mathrm{E}[N(t)]$ is m_A+m_B-1, and the number of additional differential equations required to evaluate $\mathrm{Var}[N(t)]$ numerically (by evaluating $\mathrm{E}[N(t)^2]$) is $m_A m_B(m_B+1)$.

5. Examples

In this section we present a series of examples that illustrate some of the types of models that can be represented within our general framework. The MAPLE software used to do the numerical calculations and an illustrative MAPLE session can be found in the Online Supplement to this paper on the journal's website.

5.1. $M_t/Ph_t/\infty$

Consider an infinite-server queue in which arrivals occur according to a nonstationary Poisson process, and the service time is represented by a nonstationary phase distribution. Specifically, let

$$\boldsymbol{\mathcal{A}}(t)=\left[\begin{array}{c|c}0&1\\\hline 1&0\end{array}\right],$$

$\boldsymbol{\lambda}(t)=[1+\sin(t)]$ be the phase representation of the nonstationary Poisson arrival process. Let

$$\boldsymbol{\mathcal{B}}(t)=\left[\begin{array}{cc|c}0.0&0.6&0.4\\0.3&0.0&0.7\\\hline 0.1&0.9&0.0\end{array}\right],$$

and $\boldsymbol{\mu}(t)=[0.75+0.1\sin(2t),1+0.5\sin(3t)]^T$ represent the nonstationary phase service process. For reference, $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\mu}(t)$ are plotted in Figure 1. Assuming that the queue is empty and idle at time 0, the mean and variance of the total number of entities in the queue are shown in Figure 2. Because the number in queue is a nonstationary Poisson process for this system, the mean and variance are identical.

5.2. $E_{10}/H_2/\infty$

Consider an infinite-server queue in which arrivals occur according to a stationary renewal process with the interarrival times modeled as an Erlang order 10 random variable, where each phase has mean 1/50; thus the coefficient of variation of the time between arrivals is $1/\sqrt{10}\approx 0.32$. Suppose that the service time is represented by a stationary hyper-exponential distribution where the actual service time is equally likely to be exponential with mean 2 or mean 0.5 time units.

Assuming that the queue is empty and idle at time 0, the mean and variance of the total number of entities in the queue is shown in Figure 3. Notice that the curves do not coincide, so the number of entities in the queue is not Poisson for this system.

5.3. $Ph/Ph/\infty$

Consider an infinite-server queue in which arrivals occur according to a stationary renewal process with renewal times and service times represented by phase distributions. Specifically, let

$$\boldsymbol{\mathcal{A}}(t)=\left[\begin{array}{cc|c}0.2&0.1&0.7\\0.4&0.2&0.4\\\hline 0.8&0.2&0.0\end{array}\right],$$

$\boldsymbol{\lambda}(t)=[7,9]^T$,

$$\boldsymbol{\mathcal{B}}(t)=\left[\begin{array}{cc|c}0.3&0.4&0.3\\0.4&0.2&0.4\\\hline 0.1&0.9&0.0\end{array}\right],$$

and let $\boldsymbol{\mu}(t)=[1,2]^T$.

Assuming that the queue is empty and idle at time 0, the mean and variance of the total number of

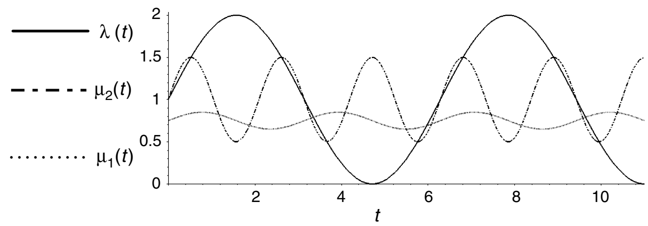

Figure 1 **Arrival Rate and Service-Process Rates for the $M_t/Ph_t/\infty$ Example**

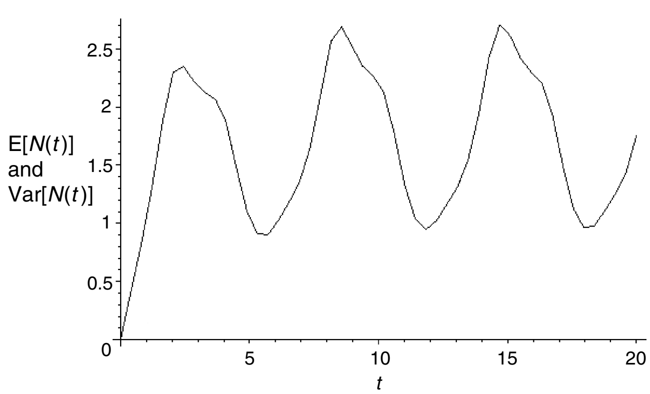

Figure 2 **Mean and Variance of the Number in Queue for the $M_t/Ph_t/\infty$ Example**

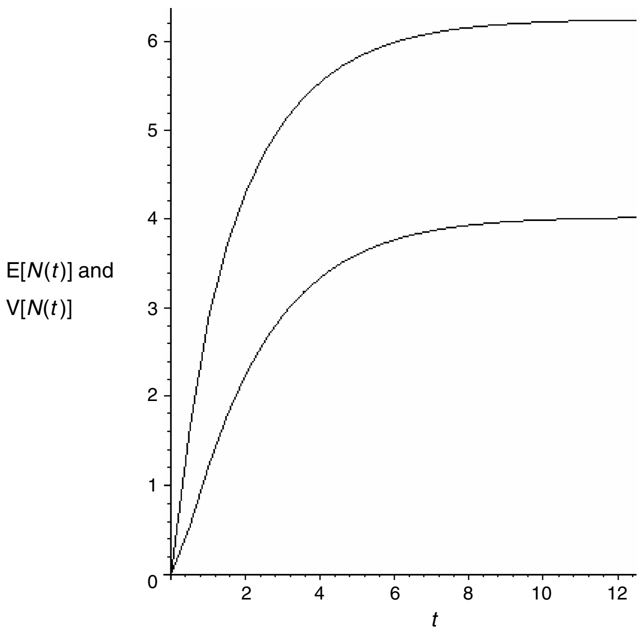

Figure 3 **Mean and Variance of the Number in Queue for the $E_{10}/H_2/\infty$ Example**

entities in the queue are close in value but differ. For example, at time $t = 0.5$, we have $E[N(t)] = 1.98045$ and $\text{Var}[N(t)] = 1.96084$; this demonstrates that the distribution of the number of entities in the system is not Poisson.

6. Virtual Sojourn Time

Let W_t be the sojourn time of a virtual entity arriving at time t. Because there are an infinite number of servers, the sojourn time is just the entity's service time, which is the absorption time in a time-dependent finite-state Markov process having exactly one absorbing state. In this section we show how to obtain the cdf and moments of virtual sojourn time from the first-moment differential equations for number in each phase of service.

Let $N(t) = \sum_{j=1}^{m_B} N_j(t)$ be the total number of entities in the queue at time t. Suppose that no entities are permitted to arrive before time t, a single entity arrives at time t, and no further arrivals are permitted after t. Therefore $N(t+x)$ for all $x \geq 0$, is a Bernoulli random variable because under this scenario $N(t+x)$ is either 0 (if the entity has departed) or 1 (if the entity is still in the queue). Thus $E[N(t+x)] = P(N(t+x) = 1)$.

Let
$$W_t = \inf\{\tau\colon \tau \geq 0,\, N(t+\tau) = 0\},$$
and therefore
$$\begin{aligned}1 - F_{W_t}(x) &\equiv P(W_t > x)\\ F_{W_t}(x) &= 1 - P(W_t > x)\\ &= 1 - P(N(t+x) > 0) = 1 - P(N(t+x) = 1)\\ &= 1 - E[N(t+x)]\\ &= 1 - \sum_{j=1}^{m_B} E[N_j(t+x)] = 1 - \sum_{j=1}^{m_B} E_j(t+x). \quad (7)\end{aligned}$$

To evaluate (7) numerically, we set $\lambda_k(t+x) = 0$ for all $x \geq 0$ and $k = 1, \ldots, m_A$, and initialize the number in the queue at time t to be one. The first-moment differential equations for number in service phase j simplify to
$$E_j(t)' = \sum_{l=1}^{m_B} b_{lj}(t)\mu_l(t)E_l(t) - \mu_j(t)E_j(t) \quad (8)$$
for $j = 1, \ldots, m_B$. Because the single arrival at time t could start in any initial phase of service, the initial conditions become $E_j(t) = \beta_j(t)$, $j = 1, \ldots, m_B$.

Of course, given the cdf of sojourn time we can evaluate the pth moment of sojourn time (for positive integer p) as
$$E[W_t^p] = \int_{0^+}^{\infty} p x^{p-1}(1 - F_{W_t}(x))\, dx. \quad (9)$$

We can in fact evaluate the pth moment of sojourn time simply by adding another differential equation to our system of equations (8), and then numerically integrate this differential equation along with those for $E_j(t)$. Let
$$E[W_t^p(\tau)] = \int_{0^+}^{\tau} p x^{p-1}(1 - F_{W_t}(x))\, dx,$$
which is the pth moment of sojourn time if we let $\tau \to \infty$. Thus,
$$\begin{aligned}\frac{\partial}{\partial\tau} E[W_t^p(\tau)] &= p\tau^{p-1}(1 - F_{W_t}(\tau))\\ &= p\tau^{p-1}[E_1(t+\tau) + \cdots + E_{m_B}(t+\tau)]. \quad (10)\end{aligned}$$
By numerically integrating this differential equation until τ is large we obtain the pth moment of sojourn time for an arrival at time t.

For example, consider the $M_t/Ph_t/\infty$ queue in §5.1. Using this approach, we can calculate that for an arrival at time $t = 10$, $E[W_{10}] \approx 1.80$ and $\text{Var}[W_{10}] \approx 3.65$. The cdf of sojourn time for this entity can be plotted using our software as can be seen in the MAPLE examples found in the Online Supplement for this paper on the journal's website.

7. Conclusions

In this paper we have presented efficient computational procedures for analyzing the time-dependent behavior of $Ph_t/Ph_t/\infty$ queueing systems in which no truncation of the state space is required. We did this by developing finite and relatively small sets of differential equations that can be integrated numerically to calculate the time-dependent behavior of this queue.

Downloadable copies of MAPLE code used to do the evaluations described in this paper and an illustrative MAPLE session can be found in the Online Supplement for this paper on the journal's website. We verified the code by evaluating many cases and comparing the answers provided by the MAPLE code to solutions for the few cases where analytic answers are known, and to Monte Carlo simulation experiments for more general cases. In all cases the MAPLE code provided accurate solutions. The details of the proofs of the key results in the paper can be found in the Online Supplement for this paper on the website for this journal.

If we are interested in individual state probabilities, then the results of this paper can be used to match two systems-size moments (or conditional moments) to a two-parameter approximating distribution, such as the Polya Eggenberger distribution (Johnson and Kotz 1977). The state probabilities can then be approximated by the Polya Eggenberger probabilities. This is the approach that provided many accurate approximations to a variety of time-dependent queueing models in Ong and Taaffe (1987, 1988, 1989).

In the companion paper, Nelson and Taaffe (2004), we present a network generalization of the $Ph_t/Ph_t/\infty$ system.

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References

Brown, M., S. H. Ross. 1967. Some results for infinite-server Poisson queues. *J. Appl. Probab.* **6** 453–458.

Collings, T., C. Stoneman. 1976. The $M/M/\infty$ queue with varying arrival and service rates. *Oper. Res.* **24** 760–773.

Eick, S. G., W. A. Massey, W. Whitt. 1993a. $M_t/G/\infty$ queues with sinusoidal arrival rates. *Management Sci.* **39** 241–252.

Eick, S. G., W. A. Massey, W. Whitt. 1993b. The physics of the $M_t/G/\infty$ queue. *Oper. Res.* **41** 731–742.

Foley, R. D. 1984. The nonhomogeneous $M/G/\infty$ queue. *Oper. Res.* **19** 40–48.

Glynn, P. W., W. Whitt. 1991. A new view of the heavy-traffic limit theorem for infinite-server queues. *Adv. Appl. Probab.* **23** 188–209.

Harrison, J. M., A. J. Lemoine. 1981. A note on networks of infinite-server queues. *J. Appl. Probab.* **18** 561–567.

Johnson, N. L., S. Kotz. 1977. *Urn Models and Their Applications*. John Wiley & Sons, New York.

Kao, E. 1997. *An Introduction to Stochastic Processes*. Duxbury Press, Belmont, CA.

Lee, W. C. Y. 1989. *Mobile Cellular Telecommunications Systems*. McGraw-Hill, New York.

Massey, W. A., W. Whitt. 1993. Networks of infinite-server queues with nonstationary Poisson input. *Queueing Systems: Theory Appl.* **13** 183–250.

Massey, W. A., W. Whitt. 1994. An analysis of the modified offered-load approximation for the nonstationary Erlang loss model. *Ann. Appl. Probab.* **4** 1145–1160.

Mirasol, N. M. 1963. The output of an $M/G/\infty$ queueing system is Poisson. *Oper. Res.* **11** 282–284.

Nelson, B. L., M. R. Taaffe. 2004. The $[Ph_t/Ph_t/\infty]^K$ queueing system: Part II—The multiclass network. *INFORMS J. Comput.* **16**(3) 275–283.

Neuts, M. F. 1981. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. The Johns Hopkins University Press, Baltimore, MD.

Newell, G. F. 1966. The $M/G/\infty$ queue. *SIAM J. Appl. Math.* **14** 86–88.

Ong, K. L., M. R. Taaffe. 1987. Approximating nonstationary $Ph(t)/M(t)/S/C$ queueing systems. *Ann. Oper. Res.* **8** 103–116.

Ong, K. L., M. R. Taaffe. 1988. Approximating $Ph(t)/Ph(t)/1/c$ nonstationary queueing systems. *Math. Comput. Simulation* **30** 441–452.

Ong, K. L., M. R. Taaffe. 1989. Nonstationary queues with interrupted Poisson arrivals and unreliable/repairable servers. *Queueing Systems: Theory Appl.* **4** 27–46.

Rothkopf, M. H., S. S. Oren. 1979. A closure approximation for the nonstationary $M/M/s$ queue. *Management Sci.* **25** 522–534.

Taaffe, M. R., K. L. Ong. 1984. Approximating time-dependent nonexponential queues. S. Sheppard, U. Pooch, C. D. Pegden, eds. *Proc. 1984 Winter Simulation Conf.* Institute of Electrical and Electronic Engineers, Piscataway, NJ, 175–180.

Tijms, H. C. 1995. *Stochastic Models: An Algorithmic Approach*. John Wiley & Sons, Chichester, U.K.

Whitt, W. 1982. On the heavy-traffic limit theorem for the $GI/G/\infty$ queues. *Adv. Appl. Probab.* **14** 171–190.