Using Common Random Numbers for Indifference-zone Selection and Multiple Comparisons in Simulation

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We present a general recipe for constructing experiment design and analysis procedures that simultaneously provide indifference-zone selection and multiple-comparison inference for choosing the best among k simulated systems. We then exhibit two such procedures that exploit the variance-reduction technique of common random numbers to reduce the sample size required to attain a fixed precision. One procedure is based on the Bonferroni inequality and is guaranteed to be statistically conservative. The other procedure is exact under a specific dependence structure, but may be slightly liberal otherwise. Both are easy to apply, requiring only simple calculations and tabled constants. We illustrate the procedures with a numerical example.

(*Common Random Numbers; Variance Reduction; Multiple Comparisons; Output Analysis; Simulation*)

1. Introduction

In this paper we consider the problem of comparing a small number of systems, say 2 to 20, in terms of the expected value of some given stochastic performance measure. We assume that the expected performance will be estimated via a simulation experiment. At an aggregate level we are interested in determining which system is best, where "best" is defined as having the maximum or minimum expected performance. At a more refined level we may also be interested in how much better the best is relative to each alternative, since secondary criteria that are not reflected in the performance measure (such as ease of installation, cost to maintain, etc.) may tempt us to choose an inferior system if it is not deficient by much.

Because we are estimating expected performance we can neither select the best system nor bound the differences between systems with certainty. Instead, we present procedures that simultaneously control the error in selecting the best and bounding the differences. These procedures unify standard indifference-zone-selection procedures—that control the error when choosing the best—and standard multiple-comparison procedures—that control the error in making simultaneous comparisons. The procedures depend upon having normally distributed data, but they do not require known or equal variances across systems and they exploit the use of common random numbers (CRN) to reduce the computational effort. When the simulation outputs are sample averages, the normality assumption is typically not a serious restriction.

In an earlier paper (Matejcik and Nelson 1994) we exhibited one procedure that achieves indifference-zone selection and multiple-comparison inference simultaneously. This combined procedure, denoted Procedure $\mathcal{R}$, is based on Rinott's (1978) indifference-zone-selection procedure. Unfortunately, Rinott's procedure assumes independent samples across systems, proscribing the use of common random numbers, or at least preventing us from realizing the full benefit of using

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them. One contribution of the present paper is to establish a general recipe for constructing combined procedures. We then apply the recipe to a selection procedure due to Clark and Yang (1986) that does permit CRN. The Clark and Yang procedure, denoted Procedure $\mathcal{CY}$, is based on the Bonferroni inequality and is therefore statistically conservative, again preventing us from realizing the full benefits of CRN. Therefore, we also present a new combined procedure, Procedure $\mathcal{NM}$, that may sometimes be slightly liberal. We show that $\mathcal{NM}$ can have much smaller expected sample size than Rinott's procedure, Clark and Yang's procedure, and another popular procedure due to Dudewicz and Dalal (1975), which we denote as Procedure $\mathcal{DD}$.

The following motivating example will be used throughout the paper to illustrate the procedures: Consider an (s, S) inventory system in which some discrete item is periodically reviewed. If the inventory level is found to be below s units, then an order is issued to bring the inventory level up to S units; otherwise no additional items are ordered. Different (s, S) inventory policies result in different inventory systems. Koenig and Law (1985) used this example to illustrate a subset selection procedure; see their paper for a detailed description of the model. The only stochastic input process in the simulation is the demand for inventory in each period.

Suppose that five (s, S) inventory policies have been identified for study and we are interested in determining which policy has the minimum expected cost per period for 30 periods, where cost is measured in thousands of dollars. Differences of less than one thousand dollars are considered practically insignificant, so while we want to choose the best system we also want to know which policies are nearly equivalent to the best policy. This is an ideal setting for a procedure that provides both indifference-zone selection and multiple-comparison inference.

The paper is organized as follows: We first provide background necessary to understand the inference that the new procedures provide and give our main result for forming combined procedures. Section 3 contains the procedures themselves along with a numerical example. The properties of Procedure $\mathcal{NM}$ are explored in §4. We close with some conclusions in §5. A preliminary report of this work appeared in Matejcik and Nelson (1993).

2. Background

Let Y_{ij} represent the output from the jth replication (or batch mean in a steady-state simulation) of system i, for $i = 1, 2, \ldots, k$, so that $\mathbf{Y}_j = (Y_{1j}, Y_{2j}, \ldots, Y_{kj})'$ is the $k \times 1$ vector of outputs across all systems on replication j. We assume throughout that $\mathbf{Y}_1, \mathbf{Y}_2, \cdots$ are i.i.d., and that $\mathbf{Y}_j \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the multivariate normal distribution with unknown mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)'$ and unknown variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ & & \ddots & \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{pmatrix}.$$

We are interested in comparing the k systems in terms of their expected performance, μ_i. In the inventory example there are $k = 5$ policies, Y_{ij} is the average cost for 30 periods observed on the jth replication of the ith inventory policy, and μ_i is the expected cost per period of the ith inventory policy.

If we simulate the systems independently—meaning we use different random numbers to drive the simulation of each system—then

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \sigma_{kk} \end{pmatrix}.$$

However, since using common random numbers across systems often reduces the variance of comparisons, we are also interested in the case when CRN forces the covariances $\sigma_{ij} > 0$, for $i \neq j$. In the inventory example we can use CRN to force each inventory policy to be subjected to the same sequence of demands, providing a statistically fair comparison of policy performance.

Output-analysis methods that exploit CRN and furnish appropriate statistical inference have long been of interest to the simulation community. Yang and Nelson (1991) provide additional references and some solutions for multiple comparisons in conjunction with CRN. This paper provides methods for simultaneously selecting the best system and providing confidence intervals for certain differences under CRN. Moreover, all of our methods allow for unequal and unknown variances across the systems.

The following sections review the confidence-interval procedure, multiple comparisons with the best, and the decision-theory procedure, indifference-zone selection. We then show that both types of inference can be attained simultaneously from a single experiment.

2.1. Multiple Comparisons with the Best

Suppose that larger μ_i is better. *Multiple Comparisons with the Best* (MCB) provides simultaneous confidence intervals for the parameters $\mu_i - \max_{j \neq i} \mu_j$ for $i = 1, 2, \ldots, k$. These confidence intervals bound the difference between the performance of each system and the best of the others with a prespecified confidence level. For minimization problems, such as the inventory example, we consider $\mu_i - \min_{j \neq i} \mu_j$, for $i = 1, 2, \ldots, k$.

Most MCB procedures assume the variances across systems are equal. See Hochberg and Tamhane (1987) for a general discussion of MCB procedures.

2.2. Indifference-zone Selection

Let $\mu_{(1)} \leq \mu_{(2)} \leq \cdots \leq \mu_{(k)}$ be the (unknown) ordered means. Two-stage, indifference-zone-selection procedures yield estimators $\hat{\mu}_i, i = 1, 2, \ldots, k$, that guarantee

$$\Pr\{\hat{\mu}_{(k)} > \hat{\mu}_{(i)}, \forall i \neq k\} \geq 1 - \alpha$$

whenever $\mu_{(k)} - \mu_{(i)} \geq \delta, \forall i \neq k$, where $\hat{\mu}_{(i)}$ is the estimator associated with the (unknown) system having the ith smallest expectation. The user-specified value δ is called the *indifference zone*. The implication is that if we use the procedure and then select the system with the largest performance estimate $\hat{\mu}_i$ as the best system we will be correct with probability greater than or equal to $1 - \alpha$ when the best is at least δ better than the others. In the inventory example we could set δ = one thousand dollars since we are indifferent to policies with expected costs that differ by less than one thousand dollars.

Indifference-zone-selection procedures typically do not exploit CRN, and do not provide inference about systems other than the best. They do, however, allow for unequal variances across systems. Goldsman (1983) and Bechhofer et al. (1995) provide expositions of indifference-zone selection and related topics.

2.3. Simultaneous Ranking, Selection and Multiple Comparisons

In this section we establish that MCB intervals and indifference-zone selection can be derived simultaneously from the same experiment. We begin with a lemma that establishes sufficient conditions under which MCB intervals can be formed.

LEMMA 1. *If*

$$\Pr\{\hat{\mu}_{(k)} - \hat{\mu}_{(i)} - (\mu_{(k)} - \mu_{(i)}) > -w, \forall i \neq k\} \geq 1 - \alpha$$

then with probability greater than or equal to $1 - \alpha$

$$\mu_i - \max_{j \neq i} \mu_j \in [-(\hat{\mu}_i - \max_{j \neq i} \hat{\mu}_j - w)^-, \; (\hat{\mu}_i - \max_{j \neq i} \hat{\mu}_j + w)^+]$$

for $i = 1, 2, \ldots, k$, *where* $-x^- = \min\{0, x\}$ *and* $x^+ = \max\{0, x\}$.

PROOF. The proof of this lemma can be extracted from various papers by Hsu, including Hsu (1984); it is also the same as the proof of balanced MCB in Hochberg and Tamhane (1987, p. 151). □

The quantity w is the *whisker length* of the MCB intervals, and it is analogous to the half width of symmetric confidence intervals. In standard MCB procedures w is a random variable, but our Theorem 1 will establish that the estimators formed by an indifference-zone-selection procedure with indifference zone δ satisfy the condition in Lemma 1 with $w = \delta$. Therefore, both types of inference can be derived simultaneously from the same experiment, and the whisker length of the MCB intervals can be specified in advance.

In order to state the theorem precisely, let $\pi_1, \pi_2, \ldots, \pi_k$ represent the k systems with expected performance parameters $\mu_1, \mu_2, \ldots, \mu_k$, and let $\mathcal{S}$ be the indifference-zone-selection procedure that guarantees $\Pr\{\hat{\mu}_{(k)} > \hat{\mu}_{(i)}, \forall i \neq k\} \geq 1 - \alpha$, whenever $\mu_{(k)} - \mu_{(i)} \geq \delta, \forall i \neq k$. Let $\zeta_1, \zeta_2, \ldots, \zeta_k$ represent k systems whose simulation outputs are identical in distribution to $\pi_1, \pi_2, \ldots, \pi_k$ except that their expected performance parameters are $\theta_1, \theta_2, \ldots, \theta_k$ with

$$\theta_{(k)} = \mu_{(k)}, \quad \theta_{(i)} = \mu_{(k)} - \delta, \quad i \neq k.$$

Finally, let $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ be the estimators obtained by applying Procedure $\mathcal{S}$ to $\zeta_1, \zeta_2, \ldots, \zeta_k$.

THEOREM 1. *If*

$$\begin{pmatrix} \hat{\mu}_{(k)} \\ \hat{\mu}_{(k-1)} + (\mu_{(k)} - \mu_{(k-1)} - \delta) \\ \vdots \\ \hat{\mu}_{(1)} + (\mu_{(k)} - \mu_{(1)} - \delta) \end{pmatrix} \stackrel{\mathcal{D}}{=} \begin{pmatrix} \hat{\theta}_{(k)} \\ \hat{\theta}_{(k-1)} \\ \vdots \\ \hat{\theta}_{(1)} \end{pmatrix} \quad (1)$$

then the statement

$$\Pr\{\hat{\mu}_{(k)} > \hat{\mu}_{(i)}, \forall i \neq k\} \geq 1 - \alpha$$

whenever $\mu_{(k)} - \mu_{(i)} \geq \delta, \forall i \neq k$, *implies that*

$$\Pr\{\hat{\mu}_{(k)} - \hat{\mu}_{(i)} - (\mu_{(k)} - \mu_{(i)}) > -\delta, \forall i \neq k\} \geq 1 - \alpha$$

for any *values of the true means*.

REMARK. Condition (1) insures that when indifference-zone-selection procedure $\mathcal{S}$ is applied to two collections of k systems for which the probability distributions of their outputs differ only in their location parameters, then the point estimators for each collection of systems also differ only in their location parameters. For example, condition (1) will be satisfied when $\mu_i = E[Y_{ij}]$ and the procedure $\mathcal{S}$ implies estimators of the form

$$\hat{\mu}_i = \sum_{j=1}^{N} a_j Y_{ij}$$

where $\sum_j a_j = 1$ and N is independent of μ.

PROOF. The result follows by noticing that condition (1) implies that

$$\Pr\{\hat{\mu}_{(k)} - \hat{\mu}_{(i)} - (\mu_{(k)} - \mu_{(i)}) > -\delta, \forall i \neq k\}$$
$$= \Pr\{\hat{\theta}_{(k)} - \hat{\theta}_{(i)} > 0, \forall i \neq k\} \geq 1 - \alpha$$

where the last inequality follows from the properties of $\mathcal{S}$. □

The primary consequence of Theorem 1 is that we can use the outcome of many two-stage, indifference-zone-selection procedures to form MCB intervals with whisker length w equal to the indifference zone δ, and simultaneously guarantee both the correct selection and the coverage of the MCB differences with overall confidence level $1 - \alpha$. Condition (1) will be satisfied, for instance, when the point estimators of the selection procedure are sample means or the generalized sample means of Dudewicz and Dalal (1975).

To illustrate how easy it is to apply Theorem 1 we show how it can be used to extend Rinott's (1978) indifference-zone-selection procedure, a well known procedure that requires independently simulated systems (this combined procedure was first introduced in Matejcik and Nelson 1995). In the procedure we use the convention that a "$\cdot$" subscript indicates averaging with respect to that subscript. For example, $\bar{Y}_{i\cdot}$ is the sample average of $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$.

Procedure $\mathcal{R}$

1. Specify w, α and n_0. Let h solve Rinott's integral for n_0, k and α (see the tables in Wilcox 1984).
2. Take i.i.d. sample $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from each of the k systems *simulated independently*.
3. Compute the marginal sample variances

$$S_i^2 = \frac{\sum_{j=1}^{n_0} (Y_{ij} - \bar{Y}_{i\cdot})^2}{n_0 - 1} \quad \text{for } i = 1, 2, \ldots, k.$$

4. Compute the final sample sizes

$$N_i = \max\{n_0, \lceil (hS_i/w)^2 \rceil\} \quad \text{for } i = 1, 2, \ldots, k.$$

5. Take $N_i - n_0$ additional i.i.d. observations from system i, independently of the first-stage sample and the other systems.
6. Compute the overall sample means

$$\bar{\bar{Y}}_{i\cdot} = \frac{1}{N_i} \sum_{j=1}^{N_i} Y_{ij} \quad \text{for } i = 1, 2, \ldots, k.$$

7. Select the system with the largest $\bar{\bar{Y}}_{i\cdot}$ as best.
8. Simultaneously form the MCB confidence intervals

$$\mu_i - \max_{j \neq i} \mu_j \in [-(\bar{\bar{Y}}_{i\cdot} - \max_{j \neq i} \bar{\bar{Y}}_{j\cdot} - w)^-,$$
$$(\bar{\bar{Y}}_{i\cdot} - \max_{j \neq i} \bar{\bar{Y}}_{j\cdot} + w)^+] \quad \text{for } i = 1, 2, \ldots, k.$$

Steps 1–7 are simply Rinott's procedure. Theorem 1 allows us to add the MCB confidence intervals in Step 8 and simultaneously guarantee the probability of correct selection *and* the coverage probability.

As a more concrete illustration we simulated the $k = 5$ inventory systems (described in §1) independently with first-stage number of replications $n_0 = 10$, indifference zone $w = 1$ (thousand dollars), and confidence level $1 - \alpha = 0.95$. From the tables in Wilcox (1984) we obtain $h = 3.692$. The procedure selected inventory policy 2 as the best (that is, policy 2 had the smallest estimated cost per period), and provided the MCB intervals in Table 1. Recall that these are confidence intervals for $\mu_i - \min_{j \neq i} \mu_j$, where μ_i is the unknown expected cost per period of inventory policy i.

The point estimate for $\mu_2 - \min_{j \neq 2} \mu_j$ is -1.4, indicating that policy 2 appears to be 1.4 thousand dollars less expensive than the best of the other policies. The intervals tell us that, with confidence level 0.95, policy 2 is no worse than any of the others (the upper endpoint of the confidence interval is 0), and it may be as much

Table 1 **MCB Results for Procedure $\mathscr{R}$ Applied to the Inventory Example**

Policy i	Sample Size N_i	Lower MCB Limit	$\bar{\bar{Y}}_{i\cdot} - \min_{j\neq i} \bar{\bar{Y}}_{j\cdot}$	Upper MCB Limit
1	220	0	1.4	2.4
2	211	−2.4	−1.4	0
3	270	0	17.1	18.1
4	241	0	17.6	18.6
5	130	0	34.3	35.3

as 2.4 thousand dollars less expensive (the lower endpoint is −2.4). Notice that the whisker length is $w = -1.4 - (-2.4) = 1$, precisely as specified.

The intervals also indicate that the other four policies are inferior to policy 2 (the lower endpoints of their intervals are all 0), and may be as much as 2.4, 18.1, 18.6 and 35.3 thousand dollars more expensive for policies 1, 3, 4 and 5, respectively. These constrained intervals—which either contain 0 or have 0 as one endpoint—are a characteristic of MCB. Technically, the most MCB can declare is that a system is *no worse* than the best, it cannot declare that the system is better.

Also presented in Table 1 are the total sample sizes (number of replications) for each policy, N_i. They range from 130 replications for policy 5 to 270 for policy 3, for a total of 1072 replications. The different sizes are a function of the variances of the systems; the larger the variance the greater the number of replications. Procedures that exploit CRN should reduce the number of replications required to attain the same confidence level and whisker length.

Another popular indifference-zone-selection procedure that also assumes independence across systems is due to Dudewicz and Dalal (1975). Their procedure generally requires fewer replications than Rinott's procedure, but is more complicated to implement because it employs specially-weighted sample means. We extended their procedure in the same manner as Procedure $\mathscr{R}$ to create Procedure $\mathscr{DD}$. For completeness, and to provide a fair comparison with the new procedures presented later, we also applied $\mathscr{DD}$ to the inventory problem and obtained the results in Table 2. The constant in Procedure $\mathscr{DD}$, denoted f, corresponding to h in Procedure $\mathscr{R}$ is $f = 3.531$. Notice that the total number of replications is 981, fewer than the 1072 required by Procedure $\mathscr{R}$.

3. CRN Procedures

We now present two combined indifference-zone-selection and MCB procedures that exploit CRN, and illustrate them using the inventory example described in §1. As is traditional, the procedures are stated in a maximization context, but the inventory example used to illustrate the procedures is a minimization problem. The first procedure is a direct application of Theorem 1 to an existing indifference-zone-selection procedure. The second procedure is new and, we think, better than the first; its properties will be explored more fully in §4.

3.1. Clark and Yang's Procedure

We first extend Clark and Yang's (1986) indifference-zone-selection procedure. This procedure exploits the Bonferroni inequality to account for the dependence induced by CRN. Thus, it is a conservative procedure that typically prescribes more replications than actually necessary to make a correct selection under CRN. In the procedure, $t = t_{1-[\alpha/(k-1)],n_0-1}$ is the $(1 - [\alpha/(k-1)])$-quantile of the t distribution with $n_0 - 1$ degrees of freedom.

Procedure $\mathscr{CY}$

1. Specify w, α and n_0. Let $t = t_{1-[\alpha/(k-1)],n_0-1}$.
2. Take i.i.d. sample $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from each of the k systems *using CRN across systems*.
3. Compute the sample variances of the differences

$$S_{ij}^2 = \frac{1}{n_0 - 1} \sum_{l=1}^{n_0} (Y_{il} - Y_{jl} - (\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot}))^2 \quad \text{for all } i \neq j.$$

4. Compute the final sample size

$$N = \max\{n_0, \lceil \max_{j\neq i} (tS_{ij}/w)^2 \rceil\}.$$

Table 2 **MCB Results for Procedure $\mathscr{DD}$ Applied to the Inventory Example**

Policy i	Sample Size N_i	Lower MCB Limit	$\bar{\bar{Y}}_{i\cdot} - \min_{j\neq i} \bar{\bar{Y}}_{j\cdot}$	Upper MCB Limit
1	202	0	1.5	2.5
2	193	−2.5	−1.5	0
3	247	0	17.2	18.2
4	220	0	17.9	18.9
5	119	0	34.7	35.7

5. Take $N - n_0$ additional i.i.d. observations from each system, using CRN across systems.
6. Compute the overall sample means

$$\bar{\bar{Y}}_i. = \frac{1}{N}\sum_{j=1}^{N} Y_{ij} \quad \text{for } i = 1, 2, \ldots, k.$$

7. Select the system with the largest $\bar{\bar{Y}}_i.$ as best.
8. Simultaneously form the MCB confidence intervals

$$\mu_i - \max_{j\neq i}\mu_j \in [-(\bar{\bar{Y}}_i. - \max_{j\neq i}\bar{\bar{Y}}_j. - w)^-, (\bar{\bar{Y}}_i. - \max_{j\neq i}\bar{\bar{Y}}_j. + w)^+] \quad \text{for } i = 1, 2, \ldots, k.$$

We performed the same experiment for the inventory example, but this time using CRN across systems. The value of $t = t_{1-(0.05/4),9} = 2.685$. Procedure $\mathcal{CY}$ also selected inventory policy 2 as the best, but it did so with many fewer total replications (875 for $\mathcal{CY}$ versus 981 for $\mathcal{DD}$ versus 1072 for $\mathcal{R}$). The MCB results are displayed in Table 3.

To illustrate the impact of CRN, we estimated the correlation matrix of $(Y_{1j}, Y_{2j}, \ldots, Y_{kj})'$ to be

$$\begin{pmatrix} 1 & 0.44 & 0.86 & 0.27 & 0.68 \\ 0.44 & 1 & 0.41 & 0.92 & 0.41 \\ 0.86 & 0.41 & 1 & 0.23 & 0.63 \\ 0.27 & 0.92 & 0.23 & 1 & 0.35 \\ 0.68 & 0.41 & 0.63 & 0.35 & 1 \end{pmatrix}.$$

Therefore, CRN is effective, inducing positive correlations across systems ranging from 0.23 to 0.92. Procedure $\mathcal{CY}$ exploits this dependence and is clearly superior to $\mathcal{R}$ and $\mathcal{DD}$ in this example. Unfortunately, procedures based on the Bonferroni inequality become more conservative as the number of systems, k, increases. At some point this conservatism overwhelms the benefit from CRN; avoiding this problem is the motivation for the procedure presented in the next section.

Table 3 **MCB Results for Procedure $\mathcal{CY}$ Applied to the Inventory Example**

Policy i	Sample Size N_i	Lower MCB Limit	$\bar{\bar{Y}}_i. - \min_{j\neq i}\bar{\bar{Y}}_j.$	Upper MCB Limit
1	175	0	1.4	2.4
2	175	−2.4	−1.4	0
3	175	0	17.9	18.9
4	175	0	18.1	19.1
5	175	0	34.4	35.4

3.2. Nelson and Matejcik's Procedure

We now present a new procedure that was motivated by Nelson's (1993) robust MCB procedure. This procedure assumes that $\boldsymbol{\Sigma}$ has a particular structure known as *sphericity*, specifically

$$\boldsymbol{\Sigma} = \begin{pmatrix} 2\psi_1 + \tau^2 & \psi_1 + \psi_2 & \cdots & \psi_1 + \psi_r \\ \psi_2 + \psi_1 & 2\psi_2 + \tau^2 & \cdots & \psi_2 + \psi_r \\ & & \ddots & \\ \psi_r + \psi_1 & \psi_r + \psi_2 & \cdots & 2\psi_r + \tau^2 \end{pmatrix}$$

where $\tau^2 > \sqrt{k\sum_{i=1}^{k}\psi_i^2} - \sum_{i=1}^{k}\psi_i$ to guarantee that $\boldsymbol{\Sigma}$ is positive definite. Sphericity implies that

$$\mathrm{Var}[Y_{ij} - Y_{lj}] = 2\tau^2$$

for all $i \neq l$. In other words, the variances of all pairwise differences across systems are equal, even though the marginal variances and covariance may be unequal. Sphericity generalizes *compound symmetry*, which is

$$\boldsymbol{\Sigma} = \sigma^2\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ & & \ddots & \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$$

Compound symmetry has been assumed by many researchers to account for the effect of CRN (e.g., Schruben and Margolin (1978), Nozari, Arnold and Pegden (1987), and Tew and Wilson (1994)).

The procedure below is valid when $\boldsymbol{\Sigma}$ satisfies sphericity, as we prove in the next section. We also show that the procedure is robust to departures from sphericity provided that the covariances are nonnegative ($\sigma_{ij} \geq 0$ is the assumed effect of CRN). This property is consistent with the performance of the robust MCB procedure in Nelson (1993).

In the procedure, $g = T^{(1-\alpha)}_{k-1,(k-1)(n_0-1),1/2}$ is the $(1-\alpha)$-quantile of the maximum of a multivariate t random variable of dimension $k-1$ with $(k-1)(n_0-1)$ degrees of freedom and common correlation $\frac{1}{2}$; see, for instance, Table 4 in Hochberg and Tamhane (1987).

Procedure $\mathcal{NM}$

1. Specify w, α and n_0. Let $g = T^{(1-\alpha)}_{k-1,(k-1)(n_0-1),1/2}$.
2. Take i.i.d. sample $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from each of the k systems *using CRN across systems*.

3. Compute the sample variance of the difference under the condition of sphericity

$$S^2 = \frac{2\sum_{i=1}^{k}\sum_{j=1}^{n_0}(Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot})^2}{(k-1)(n_0-1)}.$$

4. Compute the final sample size

$$N = \max\{n_0, \lceil (gS/w)^2 \rceil\}.$$

5. Take $N - n_0$ additional i.i.d. observations from each system, using CRN across systems.

6. Compute the overall sample means

$$\bar{\bar{Y}}_{i\cdot} = \frac{1}{N}\sum_{j=1}^{N} Y_{ij} \quad \text{for } i = 1, 2, \ldots, k.$$

7. Select the system with the largest $\bar{\bar{Y}}_{i\cdot}$ as best.

8. Simultaneously form the MCB confidence intervals

$$\mu_i - \max_{j\neq i}\mu_j \in [-(\bar{\bar{Y}}_{i\cdot} - \max_{j\neq i}\bar{\bar{Y}}_{j\cdot} - w)^-,$$
$$(\bar{\bar{Y}}_{i\cdot} - \max_{j\neq i}\bar{\bar{Y}}_{j\cdot} + w)^+] \quad \text{for } i = 1, 2, \ldots, k.$$

We performed the same experiment for the inventory example using CRN across systems. The value of $g = T^{(0.95)}_{4,36,1/2} = 2.238$. Procedure $\mathcal{NM}$ also selected inventory policy 2 as the best, but it did so with significantly fewer total replications (330 for $\mathcal{NM}$ versus 875 for $\mathcal{CY}$ versus 981 for $\mathcal{DD}$ versus 1072 for $\mathcal{R}$). The MCB results are displayed in Table 4, and are nearly identical to the results obtained by the other procedures.

Table 4 **MCB Results for Procedure $\mathcal{NM}$ Applied to the Inventory Example**

Policy i	Sample Size N_i	Lower MCB Limit	$\bar{\bar{Y}}_{i\cdot} - \min_{j\neq i}\bar{\bar{Y}}_{j\cdot}$	Upper MCB Limit
1	66	0	1.5	2.5
2	66	−2.5	−1.5	0
3	66	0	18.2	19.2
4	66	0	18.2	19.2
5	66	0	34.7	35.7

4. Properties of Procedure $\mathcal{NM}$

In this section we study the properties of Procedure $\mathcal{NM}$. We begin by proving that $\mathcal{NM}$ is valid when Σ satisfies sphericity. Then we investigate the robustness of Procedure $\mathcal{NM}$ to departures from sphericity. Finally, we show that we can expect a smaller total sample size using Procedure $\mathcal{NM}$ relative to Procedures $\mathcal{R}$, $\mathcal{DD}$ and $\mathcal{CY}$.

4.1. Validity Under Sphericity

We will exploit the following lemmas. The proof of Lemma 2 can be found in Hochberg and Tamhane (1987, pp. 210–211), while the proof of Lemma 3 is a straightforward exercise in mathematical statistics.

LEMMA 2. *If $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ are distributed i.i.d. $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\boldsymbol{\Sigma}$ has the property of sphericity, then*

$$S^2 = \frac{2\sum_{i=1}^{k}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot})^2}{(k-1)(n-1)}$$

is distributed as $2\tau^2\chi^2_{(k-1)(n-1)}/((k-1)(n-1))$ and is independent of $\bar{Y}_{1\cdot}, \bar{Y}_{2\cdot}, \ldots, \bar{Y}_{k\cdot}$, where χ^2_{df} denotes a chi-squared random variable with df degrees of freedom.

LEMMA 3. *If $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ are i.i.d. with marginal variance-covariance matrix $\boldsymbol{\Sigma}$, and $\boldsymbol{\Sigma}$ has the property of sphericity, then $\mathrm{Var}[\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot}] = 2\tau^2/n$ for all $i \neq j$, and $\mathrm{Corr}[\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot}, \bar{Y}_{l\cdot} - \bar{Y}_{j\cdot}] = \frac{1}{2}$ for all $i \neq j \neq l$.*

We now establish the key result.

THEOREM 2. *If $\mathbf{Y}_1, \mathbf{Y}_2, \cdots$ are distributed i.i.d. $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\boldsymbol{\Sigma}$ has the property of sphericity, then after applying Procedure $\mathcal{NM}$*

$$\Pr\{\mathrm{CS}\} \equiv \Pr\{\bar{\bar{Y}}_{(i)} < \bar{\bar{Y}}_{(k)}, \forall i \neq k\} \geq 1 - \alpha$$

whenever $\mu_{(k)} - \mu_{(i)} \geq w, \forall i \neq k$.

PROOF. When $\boldsymbol{\Sigma}$ has the property of sphericity, the probability of correct selection (CS) is minimized over the space of all $\{\mu_{(k)} - \mu_{(i)} \geq w, \forall i \neq k\}$ at the *least favorable configuration* (LFC): $\mu_{(k)} = \mu_{(k-1)} + w$ and $\mu_{(k-1)} = \mu_{(k-2)} = \cdots = \mu_{(1)}$. Therefore we assume that the LFC holds, and we write

$$\Pr\{\mathrm{CS}\} \equiv \Pr\{\bar{\bar{Y}}_{(i)} < \bar{\bar{Y}}_{(k)}, \forall i \neq k\}$$
$$= \Pr\left\{\frac{\bar{\bar{Y}}_{(i)} - (\bar{\bar{Y}}_{(k)} - w)}{\sqrt{2\tau^2/N}} < \frac{w}{\sqrt{2\tau^2/N}}, \forall i \neq k\right\}.$$

For convenience, let

$$Z_{(i)} \equiv \frac{\bar{\bar{Y}}_{(i)} - (\bar{\bar{Y}}_{(k)} - w)}{\sqrt{2\tau^2/N}}.$$

Then since $N \geq (gS/w)^2$ we have

$$\Pr\{\text{CS}\} = \Pr\left\{Z_{(i)} < \frac{w}{\sqrt{2\tau^2/N}}, \forall i \neq k\right\}$$

$$\geq \Pr\left\{Z_{(i)} < \frac{gS}{\sqrt{2}\tau}, \forall i \neq k\right\}$$

$$= E\left[\Pr\left\{\max_{i\neq k} Z_{(i)} < \frac{gS}{\sqrt{2}\tau}\middle|\frac{S}{\sqrt{2}\tau}\right\}\right].$$

Notice that, conditional on $S/(\sqrt{2}\tau)$, N is a constant; and by Lemma 2, $Z_{(1)}, Z_{(2)}, \ldots, Z_{(k-1)}$ are independent of $S/(\sqrt{2}\tau)$. We can therefore apply Lemma 3 to obtain

$$\begin{pmatrix} Z_{(1)} \\ Z_{(2)} \\ \vdots \\ Z_{(k-1)} \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ & & \ddots & \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{bmatrix}\right)$$

conditional on $S/(\sqrt{2}\tau)$. That is, the conditional distribution of $Z_{(1)}, Z_{(2)}, \ldots, Z_{(k-1)}$ is multivariate normal with zero mean vector, unit variance, and common covariance $\frac{1}{2}$. Therefore,

$$\Pr\{\text{CS}\} \geq E\left[\Pr\left\{\max_{i\neq k} Z_{(i)} < \frac{gS}{\sqrt{2}\tau}\middle|\frac{S}{\sqrt{2}\tau}\right\}\right]$$

$$= E\left[\int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{\frac{1}{\sqrt{2}}x + gS/(\sqrt{2}\tau)}{\sqrt{1-\frac{1}{2}}}\right) d\Phi(x)\right]$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{\frac{1}{\sqrt{2}}x + gy}{\sqrt{1-\frac{1}{2}}}\right) d\Phi(x)\,d\Gamma(y) = 1-\alpha$$

where Φ is the univariate standard-normal cumulative distribution function (cdf) and Γ is the cdf of $S/(\sqrt{2}\tau)$. The first equality follows from Equation (1.1a) in Hochberg and Tamhane (1987, p. 374), while the last equality follows from the definition of the critical value g and Equation (1.2a) in Hochberg and Tamhane (1987, p. 375). □

REMARK. When $k = 2$, any covariance matrix satisfies sphericity, so Procedure $\mathcal{NM}$ is always valid when there are only two systems (and the data $\mathbf{Y}_j$ are normally distributed).

4.2. Robustness

The assumption of sphericity will not be exactly or even approximately satisfied in many situations. For instance, we estimated $\boldsymbol{\Sigma}$ for the inventory example from 5000 replications, then calculated Grieve and Ag's (1984) ϵ measure; ϵ takes values between 0 and 1, with 1 indicating perfect conformance to sphericity. The value of ϵ was 0.53, which indicates a significant departure from sphericity in this example.

To evaluate the robustness of Procedure $\mathcal{NM}$ to departures from sphericity we performed an empirical study. Since it is not possible to control the extent to which system-simulation examples depart from sphericity, we focused instead on the space of normally-distributed output vectors with nonnegative correlations (the assumed effect of CRN). We estimated the probability of correct selection over this space, but did not estimate MCB coverage separately since it is implied by the correct-selection guarantee.

We considered only the LFC because the minimum probability of correct selection occurs at the LFC. Without loss of generality we set $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$ and $\mu_k = w = \delta$, implying that system k is the best. We fixed $w = \frac{1}{2}$, 1 and 2 in units of the standard error of the first-stage sample means; specifically,

$$w = \frac{1}{2\sqrt{n_0}}, \frac{1}{\sqrt{n_0}} \quad \text{and} \quad \frac{2}{\sqrt{n_0}}.$$

When $w = \frac{1}{2}$ there will be a large second-stage sample; $w = 1$ implies that there will usually be a modest second-stage sample; while $w = 2$ implies that second-stage sampling is rarely required.

The experiments were conducted as follows:

1. Fix the number of systems, k, number of first-stage replications from each system, n_0, and confidence level $1-\alpha$. We considered $k = 3$, 5 and 10 systems, $n_0 = 10$ and 30 replications, and $1-\alpha = 0.95$.

2. Generate a random k-dimensional correlation matrix $\boldsymbol{\Xi}$ using the method of Marsaglia and Olkin (1984). This method transforms a randomly generated point on the k-dimensional unit sphere into a correlation matrix. We modified the method to generate a point on the unit sphere with all nonnegative coordinates, which leads to a correlation matrix with all nonnegative elements.

3. Generate n_0 i.i.d. random vectors $\mathbf{Y}_j \sim N(\mathbf{0}, \boldsymbol{\Xi})$, for $j = 1, 2, \ldots, n_0$.

4. Compute the total sample size $N = \max\{n_0, \lceil (gS/w)^2 \rceil\}$.

5. Generate $N - n_0$ i.i.d. random vectors $\mathbf{Y}_j \sim N(\mathbf{0}, \boldsymbol{\Xi})$, for $j = n_0 + 1, n_0 + 2, \ldots, N$.

6. Score a correct selection if $\{\bar{\bar{Y}}_{k\cdot} + w > \bar{\bar{Y}}_{i\cdot}, \forall i \neq k\}$.

7. Repeat steps 3–6 a total of 2,000 times to obtain an estimate of $\Pr\{\text{CS}\}$ for the correlation matrix Ξ (2,000 replications gives two significant digits of precision).

8. Repeat steps 2–7 a total of 5,000 times to estimate the distribution of $\Pr\{\text{CS}\}$ over the space of correlation matrices, Ξ.

The experiments bypass two problems that affect all parametric multiple-comparison procedures—nonnormal data and heteroscedastic data—and instead focus on the effect of positive correlation. The results are therefore optimistic in the same way that any parametric multiple-comparison procedure is optimistic with regard to these assumptions. The results are pessimistic in the sense that we seldom encounter the LFC in practice, and that the probability of MCB coverage is typically larger than the probability of correct selection (Nelson 1993).

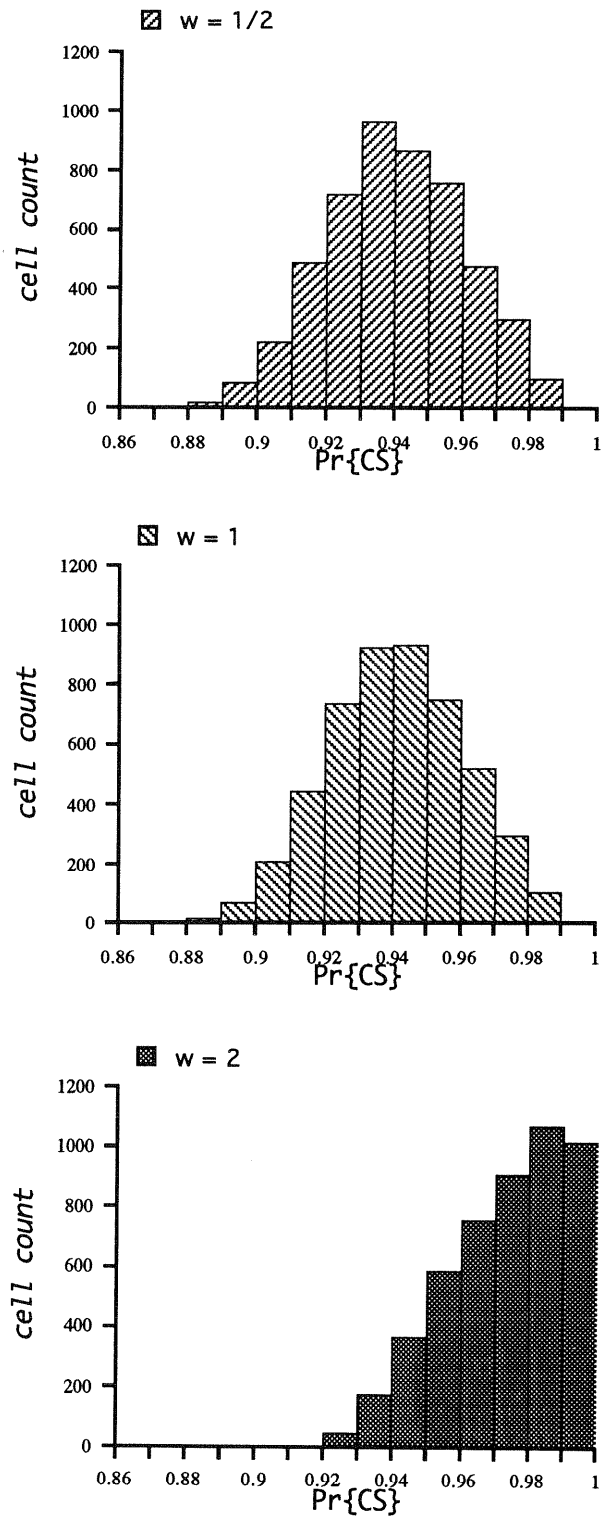
Since the results were nearly identical for all cases of k and n_0, we only present the single case with $k = 5$ and $n_0 = 30$. Over the 5000 generated correlation matrices and the three values of w, the minimum and maximum estimated values of $\Pr\{\text{CS}\}$ were 0.88 and 1.00, respectively; the average was about 0.94. Histograms for each value of w are given in Figure 1. Notice that a $\Pr\{\text{CS}\}$ less than 0.9 was rare.

Our experience with other system-simulation examples—including several simple queueing models—indicates that coverage as low as 0.88 when the nominal level is 0.95 is rather pathological, provided the normal-theory assumptions are not significantly violated. And we know that typically we do not have the LFC and that the coverage of the associated MCB intervals will be higher than the probability of correct selection. Therefore, Figure 1 encourages us to believe that the procedure is robust enough to be used in practice. Since the performance of the procedure was not affected by the number of systems, k, one could inflate the nominal coverage probability somewhat (say 0.97 when we want 0.95) and still do better than Procedure $\mathcal{CY}$, where the necessary inflation is an increasing function of k.

4.3. Sample-Size Comparisons

In this section we compare Procedure $\mathcal{NM}$ to Procedures $\mathcal{R}$, $\mathcal{DD}$ and $\mathcal{CY}$ in terms of total sample size. To do so

Figure 1 **Estimated Probability of Correct Selection, $\Pr\{\text{CS}\}$, for $k = 5$ Systems and $n_0 = 30$ Replications over the Space of Randomly Generated Ξ**



we will need the following lemma, which is a portion of Theorem 6.1 in Nelson (1993). In the lemma, S^2 is the variance estimator used in Procedure $\mathcal{NM}$.

LEMMA 4. *Let* $s = k^{-1} \sum_{i=1}^{k} \sigma_{ii}$, *the average marginal variance of the observations across systems, and let* $\varrho = (k(k-1))^{-1} \sum_{i \neq j} \sigma_{ij}$, *the average marginal covariance across systems. Then for any* $\boldsymbol{\Sigma}$,

$$E[S^2] = 2s(1 - \varrho/s).$$

Assuming that the indifference zone w is small enough that second-stage sampling is effectively certain, and using Lemma 4, it is easy to derive the expected total sample sizes for each procedure; they are displayed in Table 5.

Rinott (1978) showed that $f \leq h$ for fixed k and n_0, with strict inequality when $k > 2$. This establishes that Procedure $\mathcal{DD}$ has a smaller expected sample size than Procedure $\mathcal{R}$. Direct comparisons against the other procedures are not possible unless we assume that the first-stage sample size is large enough to be effectively infinite. Then it follows directly from the definitions of the critical values that $\sqrt{2} g_\infty = f_\infty \leq h_\infty$, where the subscript "$\infty$" indicates an infinite first-stage sample. So for large first-stage samples $\mathcal{NM}$ has a smaller expected total sample size than $\mathcal{DD}$ and $\mathcal{R}$ provided

$$ks\left(1 - \frac{\varrho}{s}\right) < ks.$$

If CRN induces positive covariances, as it is designed to do, then $\varrho > 0$ and the inequality holds. Therefore, we can conclude that Procedure $\mathcal{NM}$ will require a smaller total sample size whenever CRN is effective.

Finally we compare $\mathcal{NM}$ to $\mathcal{CY}$. When the first-stage sample size is effectively infinite, the total sample size for $\mathcal{CY}$ is

$$\frac{t_\infty^2}{w^2} k \max_{i \neq j} (\sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}).$$

To make the comparison to $\mathcal{NM}$, notice that

$$2s\left(1 - \frac{\varrho}{s}\right) = 2(s - \varrho) = \frac{1}{k(k-1)} \sum_{i \neq j} (\sigma_{ii} + \sigma_{jj} - 2\sigma_{ij})$$

the average of the $\sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}$ terms. Since the average cannot be greater than the maximum term, it follows that

$$2ks\left(1 - \frac{\varrho}{s}\right) \leq k \max_{i \neq j} (\sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}).$$

We can also show that $g_\infty < t_\infty$, in general. For instance, when $k = 3$ we have $g_\infty = 1.916$ and $t_\infty = 1.960$; when $k = 5$ we have $g_\infty = 2.160$ and $t_\infty = 2.241$; and when $k = 10$ we have $g_\infty = 2.417$ and $t_\infty = 2.539$. Therefore, we can conclude that Procedure $\mathcal{NM}$ will require a smaller total sample size than Procedure $\mathcal{CY}$, also.

Table 5 **Expected Total Sample Sizes When Second-Stage Sampling is Effectively Certain**

Procedure	Total Sample Size
$\mathcal{R}$	$\frac{h^2}{w^2} ks$
$\mathcal{DD}$	$\frac{f^2}{w^2} ks$
$\mathcal{CY}$	$\frac{t^2}{w^2} kE[\max_{i \neq j} S_{ij}]$
$\mathcal{NM}$	$\frac{g^2}{w^2} 2ks\left(1 - \frac{\varrho}{s}\right)$

5. Summary and Conclusions

In this paper we presented two procedures that simultaneously control the error in selecting the best of k systems and comparing the best system to each of the other competitors. These procedures are based on a new theorem that allows MCB confidence intervals to be appended to indifference-zone-selection procedures, thereby unifying an inference approach and a decision-theory approach. This inference facilitates making selections based on secondary criteria that are not reflected in the primary performance parameter.

The procedures allow CRN to be used to reduce the sample size required to attain a fixed precision. This is in contrast to most applications of variance reduction that increase the precision of an estimator for a fixed sample size.

Each of the procedures employs a two-stage-sampling approach, which is more natural in simulation than in most other sampling experiments. However, Procedures $\mathcal{CY}$ and $\mathcal{NM}$ are a bit more difficult to apply than Procedures $\mathcal{R}$ and $\mathcal{DD}$ (which do not exploit CRN) because

the first-stage data from all k systems must be available before the second-stage sample size can be computed.*

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