IIE Transactions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713772245

Selecting a good system: procedures and inference

Barry L. Nelson[a]; Souvik Banerjee[a]
[a] Department of Industrial Engineering & Management Sciences, Northwestern University, Evanston, IL, USA

To cite this Article Nelson, Barry L. and Banerjee, Souvik(2001) 'Selecting a good system: procedures and inference', IIE Transactions, 33: 3, 149 — 166
To link to this Article: DOI: 10.1080/07408170108936819
URL: http://dx.doi.org/10.1080/07408170108936819

IIE Transactions (2001) **33**, 149–166

Selecting a good system: procedures and inference

BARRY L. NELSON* and SOUVIK BANERJEE

Department of Industrial Engineering & Management Sciences, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208-3119, USA
E-mail: nelsonb@northwestern.edu

Received May 1999 and accepted April 2000

We present two-stage experiment designs for use in simulation experiments that compare systems in terms of their expected (long-run average) performance. These procedures simultaneously achieve the following with a prespecified probability of being correct: (i) find the best system or a near-best system; (ii) identify a subset of systems that are more than a practically insignificant difference from the best; and (iii) provide a lower confidence bound on the probability that the best or near-best system will be selected. All of the procedures assume normally distributed data, but versions allow unequal variances and common random numbers.

1. Introduction

In this paper we address problems that arise in the *design, reporting and interpretation of simulation experiments performed to identify the best system*, where best means maximum or minimum expected (long-run average) performance. The procedures we derive allow the simulation analyst to achieve the following goals, all with a prespecified probability of being correct:

1. Design their experiment so as to find the best system, or one within a practically insignificant difference from the best system (we refer to this as a "good selection").
2. Bound the difference between each system and the best system, and thereby eliminate all systems that are more than the practically insignificant difference from the best.
3. Report a lower confidence bound on the true, but unknown, probability of selecting a good system, and on the probability of selecting the unique best system, for this combination of procedure and systems.

Many indifference-zone ranking and selection procedures exist that achieve goal 1 (see, for instance, Bechhofer *et al.* (1995) or Goldsman and Nelson (1998)). Multiple comparison procedures, specifically Multiple Comparisons with the Best (MCB), can satisfy goal 2 (Hsu, 1996). However, the bounds provided by standard MCB procedures are difficult to interpret because they are *constrained* confidence intervals: each interval either contains zero or has zero as one endpoint. A zero endpoint means that a system can be declared either "no better than the best" or "no different from the best," depending on which endpoint it is. This subtlety is confusing to many analysts. We solve the problem by providing fixed-width, unconstrained MCB intervals, a small extension to existing theory. An important use of such intervals is to eliminate from further consideration all systems that are clearly inferior to the best system.

A more fundamental contribution is made by addressing goal 3. For a given procedure, the Probability of a Good Selection (PGS) is an unknown property of the procedure that depends on characteristics of the simulated systems, most critically their true means. In this paper we provide a Lower Confidence Bound (LCB) for this property. Our LCB for PGS will never be smaller than $1 - \alpha$, the guaranteed PGS of the procedure, but can be considerably higher, indicating a favorable setting for finding a system near the best. We also provide a LCB for the probability of selecting the unique best system, which we refer to as the Probability of a Correct Selection (PCS). The LCB for PCS can be less than $1 - \alpha$, since our procedures only guarantee to select the unique best system under certain configurations of the means (specifically, when the best mean is at least δ superior to any other).

The concept of a lower confidence bound on the probability of a correct selection is similar in spirit to Hsu's (1984) *S*-value, which is the smallest confidence level at which the sample best system would be declared

*Corresponding author

to be the true best system. In our case we design the experiment to achieve a given, nominal probability of a *good* selection that we may exceed. For a philosophical argument against computing lower confidence bounds on the probability of selecting the *unique* best system, see Bofinger (1994).

The paper is organized as follows: We first describe how lower confidence bounds on PCS and PGS can be obtained in general. These LCBs depend on LCBs for the difference between the best system and each inferior system; we obtain those bounds in Section 3. Section 4 introduces several specific procedures for computing LCBs on PCS and PGS, and we compare them in Section 5. Some illustrative examples are provided in Section 6.

2. The probability of a good selection

Throughout this paper we assume that larger expected performance implies a better system. Let $\mu_1, \mu_2, \ldots, \mu_k$ denote the unknown means of the k systems to be compared, and let $\hat{\mu}_i$ denote our point estimator of μ_i. In this section we assume only that the distribution of $\hat{\mu}_i - \hat{\mu}_\ell - (\mu_i - \mu_\ell)$, for all $i \neq \ell$, does not depend on $\mu_1, \mu_2, \ldots, \mu_k$.

Denote the ordered means by $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[k-1]} < \mu_{[k]}$, and for the moment suppose that our goal is to find the unique best system $[k]$; later we address the problem of finding either system $[k]$ or a system $[i]$ whose true mean is close enough to $\mu_{[k]}$.

Our rule will be to select the system with the largest performance estimate, $\hat{\mu}_i$. Therefore, the probability of a (unique) correct selection is

$$\begin{aligned} \text{PCS} &= \Pr\left\{\hat{\mu}_{[i]} < \hat{\mu}_{[k]}, \quad i = 1, 2, \ldots, k-1\right\}, \\ &= \Pr\left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \mu_{[k]} - \mu_{[i]}, \right. \\ &\qquad \left. i = 1, 2, \ldots, k-1\right\}, \\ &= \Pr\left\{D_i < \mu_{[k]} - \mu_{[i]}, \quad i = 1, 2, \ldots, k-1\right\}, \quad (1) \end{aligned}$$

where $D_i, i = 1, 2, \ldots, k-1$ has the same joint distribution as $\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}), i = 1, 2, \ldots, k-1$. If the values of the differences $\mu_{[k]} - \mu_{[i]}$ were known, as well as the joint distribution of $D_1, D_2, \ldots, D_{k-1}$, then (1) might be evaluated exactly. Since this is impossible in practice, Kim (1986) suggested replacing $\mu_{[k]} - \mu_{[i]}$ in (1) with $(1-\alpha)100\%$ LCBs on these differences, thereby providing a $(1-\alpha)100\%$ LCB on the probability of a correct selection (see also Anderson *et al.* (1977) and Jeong *et al.* (1989)). Kim (1986) was only able to provide a LCB on the single difference $\mu_{[k]} - \mu_{[k-1]}$, whereas we will provide bounds on all $k-1$ differences leading to a much tighter LCB on PCS (however, Kim's bound for the *single* difference $\mu_{[k]} - \mu_{[k-1]}$ could be tighter than ours for that single difference). Related bounds are considered by Jeon *et al.* (1988) and Kim (1988).

One shortcoming of our proposal is that the LCB on PCS can be small when there are one or more systems whose performance is very close to the best, making a unique correct selection unlikely. Thus, it makes sense to provide a LCB on the probability of choosing the best system or a system whose mean is within a practically insignificant difference δ of the best. That is, we want to select a system i such that $\mu_{[k]} - \mu_i \leq \delta$. We call this event a "good" selection, and let PGS denote the probability of a good selection.

We will show, following Section 3 below, that for the procedures we derive

$$\begin{aligned} \text{PGS} &= \Pr\left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) \right. \\ &\qquad \left. < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1\right\}, \\ &= \Pr\left\{D_i < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1\right\}, \end{aligned} \quad (2)$$

no matter what the configuration of the true means. Our approach will be to substitute LCBs for $\mu_{[k]} - \mu_{[i]}$ into (2) and then evaluate this probability numerically to obtain our LCB on PGS.

3. Two-sided, fixed-width MCB

The following lemma will be useful for deriving two-sided, fixed-width MCB confidence intervals:

Lemma 1. (Hsu, 1996, Section 4.2.1) *If*

$$\Pr\{\hat{\mu}_i - \hat{\mu}_\ell - (\mu_i - \mu_\ell) < \delta, \quad \forall i : i \neq \ell\} \geq 1 - \alpha, \quad (3)$$

then with probability greater than or equal to $1-\alpha$

$$\mu_i - \max_{\ell \neq i} \mu_\ell \in \left[\hat{\mu}_i - \max_{\ell \neq i} \hat{\mu}_\ell \pm \delta\right],$$

for $i = 1, 2, \ldots, k$.

Remark. Notice that (3) will hold when we can form simultaneous two-sided confidence intervals for all-pairwise differences $\mu_i - \mu_\ell$ that take the form $\hat{\mu}_i - \hat{\mu}_\ell \pm \delta$. For instance, in the usual one-way analysis of variance model with normally distributed data and equal variances, setting $\delta = q^{(1-\alpha)}_{k,k(n-1)} S_p / \sqrt{n}$ – where $q^{(1-\alpha)}_{k,k(n-1)}$ is the $1-\alpha$ quantile of the studentized range distribution of dimension k and degrees of freedom $k(n-1)$, and S_p^2 is the usual pooled variance estimator (see Equation (7)) – achieves (3). This is the procedure given by Hsu (1996,

pp. 103–104). Below we propose two-stage procedures that allow unequal variances across systems, and allow the value of δ to be specified in advance.

Let $T_1, T_2, \ldots, T_k$ be independent standard t random variables, each with ν degrees of freedom, and let $\bar{T} = k^{-1}\sum_{i=1}^{k} T_i$. Define the random variables

$$V = \left\{\sum_{i=1}^{k}(T_i - \bar{T})^2\right\}^{1/2},$$

and $R = \max_i T_i - \min_i T_i$. Let $v_\nu^{(1-\alpha)}$ and $r_\nu^{(1-\alpha)}$ be the $1-\alpha$ quantiles of V and R, respectively. The quantities v and r will be the critical constants in our two-stage procedures.

3.1. *Procedures*

Consider the following algorithm for producing fixed-width confidence intervals for all pairwise comparisons when the data are normally distributed and independent. This single algorithm contains two procedures that differ as to whether sample means or generalized sample means are used as the point estimators for $\mu_1, \mu_2, \ldots, \mu_k$.

Fixed-width, all-pairwise comparisons

Step 1. Specify confidence level $1-\alpha$, halfwidth $\delta > 0$, and initial sample size $n_0 \geq 2$.

Step 2. Sample i.i.d. observations $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from all systems $i = 1, 2, \ldots, k$, and compute the sample variances

$$S_i^2 = \frac{1}{n_0 - 1}\sum_{j=1}^{n_0}(Y_{ij} - \bar{Y}_i)^2,$$

for $i = 1, 2, \ldots, k$.

Step 3. Determine the total sample size needed:

(a) For a sample-mean-based procedure (Procedure S), let

$$N_i = \max\left\{n_0, \left\lceil\left(\frac{\sqrt{2}v_{n_0-1}^{(1-\alpha)}S_i}{\delta}\right)^2\right\rceil\right\}. \tag{4}$$

(b) For a generalized-sample-mean-based procedure (Procedure G), let

$$N_i = \max\left\{n_0 + 1, \left\lceil\left(\frac{r_{n_0-1}^{(1-\alpha)}S_i}{\delta}\right)^2\right\rceil\right\}. \tag{5}$$

Step 4. If $N_i > n_0$ take additional samples $Y_{i,n_0+1}, Y_{i,n_0+2}, \ldots, Y_{i,N_i}$ from each system $i = 1, 2, \ldots, k$.

Step 5. Compute $\hat{\mu}_i, i = 1, 2, \ldots, k$:

(a) For a sample-mean-based procedure, let

$$\hat{\mu}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} Y_{ij}.$$

(b) For a generalized-sample-mean-based procedure, let

$$\hat{\mu}_i = \sum_{j=1}^{N_i}\beta_{ij}Y_{ij},$$

where for each i the β_{ij} are chosen such that $\beta_{i1} = \beta_{i2} = \cdots = \beta_{in_0}$, $\sum_{j=1}^{N_i}\beta_{ij} = 1$, and

$$S_i^2\sum_{j=1}^{N_i}\beta_{ij}^2 = \left(\frac{\delta}{r_{n_0-1}^{(1-\alpha)}}\right)^2.$$

Step 6. Report the $k(k-1)/2$ simultaneous confidence intervals

$$\mu_i - \mu_\ell \in [\hat{\mu}_i - \hat{\mu}_\ell \pm \delta],$$

for all $i \neq \ell$.

A proof that the intervals in Step 6 are indeed simultaneous $(1-\alpha)100\%$ confidence intervals when the simulation output data are normally distributed and Procedure S is employed can be found in Hochberg (1975). The corresponding result for the generalized-sample-mean-based Procedure G is given in Hochberg and Tamhane (1987, pp. 200–201). In fact, both of these procedures produce simultaneous confidence intervals for all contrasts involving $\mu_1, \mu_2, \ldots, \mu_k$, but the pairwise differences are all that we require.

Application of Lemma 1 immediately yields the desired two-sided MCB confidence intervals

$$\mu_i - \max_{\ell\neq i}\mu_\ell \in \left[\hat{\mu}_i - \max_{\ell\neq i}\hat{\mu}_\ell \pm \delta\right], \tag{6}$$

for $i = 1, 2, \ldots, k$.

Remark. The weights β_{ij} that are used to form the generalized sample means are chosen so that $(\hat{\mu}_i - \mu_i)/(\delta/r_{n_0-1}^{(1-\alpha)})$ has a standard t distribution with $n_0 - 1$ degrees of freedom. See Hochberg and Tamhane (1987, pp. 196–197).

Remark. If we are willing to believe that the variances across systems are equal, then it is possible to derive a sample-mean-based procedure whose expected sample size is smaller than either version of the procedure proposed above. To do so, replace Steps 2 and 3 with the following:

Step 2. Sample i.i.d. observations $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from all systems $i = 1, 2, \ldots, k$, and compute the pooled sample variance

$$S_p^2 = \frac{1}{k(n_0 - 1)}\sum_{i=1}^{k}\sum_{j=1}^{n_0}(Y_{ij} - \bar{Y}_i)^2. \tag{7}$$

Step 3. Determine the additional sample size needed:
(a) For an equal-variance, sample-mean-based procedure (Procedure E), let

$$N_i = \max\left\{n_0, \left\lceil \left(\frac{q^{(1-\alpha)}_{k,k(n_0-1)} S_p}{\delta}\right)^2 \right\rceil\right\}. \quad (8)$$

A proof that this modification leads to valid all-pairwise confidence intervals, and therefore valid MCB intervals, can be found in Hochberg and Tamhane (1987, pp. 172–173).

Remark. If we are willing to believe that the joint distribution of $(Y_{1j}, Y_{2j}, \ldots, Y_{kj})$ can be modeled as multivariate normal with a variance-covariance matrix that satisfies the *sphericity* condition, then it is possible to derive a sample-mean-based procedure that allows for dependence across systems. Such dependence might occur if Common Random Numbers (CRN) were employed (for a discussion of sphericity and its use to approximate the effect of CRN, see Nelson and Matejcik (1995)). To do so, replace Steps 2–4 with the following:

Step 2. Sample i.i.d. random vectors $(Y_{1j}, Y_{2j}, \ldots, Y_{kj})$, $j = 1, 2, \ldots, n_0$ across all systems, and compute the sample variance of the difference as

$$S_d^2 = \frac{1}{(k-1)(n_0-1)} \sum_{i=1}^{k} \sum_{j=1}^{n_0} (Y_{ij} - \bar{Y}_i - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot})^2, \quad (9)$$

where $\bar{Y}_{\cdot j} = k^{-1} \sum_{i=1}^{k} Y_{ij}$ and $\bar{Y}_{\cdot\cdot} = k^{-1} n_0^{-1} \times \sum_{i=1}^{k} \sum_{j=1}^{n_0} Y_{ij}$.

Step 3. Determine the additional sample size needed:
(a) For a sample-mean-based procedure that allows CRN (Procedure C), let

$$N_i = N = \max\left\{n_0, \left\lceil \left(\frac{q^{(1-\alpha)}_{k,(k-1)(n_0-1)} S_d}{\delta}\right)^2 \right\rceil\right\}.$$

Step 4. If $N > n_0$, sample additional i.i.d. random vectors $(Y_{1j}, Y_{2j}, \ldots, Y_{kj})$, $j = n_0 + 1, n_0 + 2, \ldots, N$ across all systems.

A proof that this modification leads to valid all-pairwise confidence intervals, and therefore valid MCB intervals, can be based on Hochberg and Tamhane (1987, pp. 210–211) and Stein (1945).

3.2. *Inference*

What inference is possible at the end of the procedures defined above? The MCB intervals (6) imply that $\mu_i - \max_{\ell \neq i} \mu_\ell \leq \hat{\mu}_i - \max_{\ell \neq i} \hat{\mu}_\ell + \delta$ with confidence level at least $1 - \alpha$. Therefore, if this upper bound is less than zero we can infer that system i is inferior to the best; if this upper bound is less than $-\delta$, and we have chosen δ so that differences greater than δ are practically significant, then we can claim that system i is inferior to the best by more than δ.

Let $B = \text{argmax}\, \hat{\mu}_i$; that is, B is the index of the system selected as best. Notice that we can also claim, with probability $\geq 1 - \alpha$, that

$$\mu_B - \max_{\ell \neq B} \mu_\ell \geq \hat{\mu}_B - \max_{\ell \neq B} \hat{\mu}_\ell - \delta \geq -\delta.$$

Thus, with confidence level $1 - \alpha$ we are assured that we have made a good selection in that the mean of the selected system is within δ of the true best mean. Stated differently, the event

$$\mathcal{A} = \{\hat{\mu}_i - \hat{\mu}_\ell - (\mu_i - \mu_\ell) < \delta, \quad \forall i, \ell : i \neq \ell\},$$

implies a good selection will be made. In fact, even less is required. Nelson and Goldsman (1998) show that

$$\mathcal{G} = \left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \delta, \quad i = 1, 2, \ldots, k-1\right\},$$

is sufficient to guarantee a good selection. Clearly $\mathcal{A} \subseteq \mathcal{G}$.

Now consider the event

$$\mathcal{G}' = \left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1\right\}.$$

Clearly $\mathcal{G} \subseteq \mathcal{G}'$, so $\Pr\{\mathcal{G}'\} \geq \Pr\{\mathcal{G}\}$. It is also the case the $\mathcal{G}'$ implies a good selection will be made:

- If $\mu_{[i]} < \mu_{[k]} - \delta$, so that $[i]$ is not a good selection, then $\mathcal{G}'$ implies that $\hat{\mu}_{[i]} < \hat{\mu}_{[k]}$ and system $[i]$ will not be selected.
- If $\mu_{[i]} \geq \mu_{[k]} - \delta$, so that $[i]$ is a good selection, then $\mathcal{G}'$ implies that $\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) \leq \delta$ for all such $[i]$. This is precisely the event $\mathcal{G}$ that guarantees a good selection will be made.

Thus, PGS $\geq \Pr\{\mathcal{G}'\}$ and we will obtain a LCB on PGS by obtaining a LCB on $\Pr\{\mathcal{G}'\}$. To achieve this we replace the unknown differences $\mu_{[k]} - \mu_{[i]}$ by $(1 - \alpha)100\%$ simultaneous LCBs of the form

$$\mu_{[k]} - \mu_i \geq \hat{\mu}_B - \hat{\mu}_i - \delta,$$

for all $i \neq B$. These bounds are based on the fact that the two-sided MCB intervals (6) imply that

$$\max_{\ell \neq i} \mu_\ell - \mu_i \geq \max_{\ell \neq i} \hat{\mu}_\ell - \hat{\mu}_i - \delta,$$

for $i = 1, 2, \ldots, k$ with probability $\geq 1 - \alpha$. In the Appendix we carefully justify the use of this result to provide LCBs on $\mu_{[k]} - \mu_{[i]}$.

Given these LCBs, we treat $\hat{\mu}_B - \hat{\mu}_i - \delta$ as constants and assert a $(1 - \alpha)100\%$ LCB for PGS of the form

$$\text{PGS} \geq \Pr\{D_i < \max[\delta, \hat{\mu}_B - \hat{\mu}_i - \delta], \quad \forall i : i \neq B\}, \quad (10)$$

and for PCS of the form

$$\text{PCS} \geq \Pr\{D_i < \hat{\mu}_B - \hat{\mu}_i - \delta, \quad \forall i : i \neq B\}. \tag{11}$$

To compute these bounds we must be able to evaluate the right-hand sides of (10) and (11), which is the topic of the next section.

4. Computing lower confidence bounds on PGS

In this section we show how to evaluate PGS for each version of the two-stage procedure *if the true means are known*. To evaluate PCS we simply replace $\max[\delta, \mu_{[k]} - \mu_{[i]}]$ by $\mu_{[k]} - \mu_{[i]}$ in each derivation. LCBs on PGS and PCS are obtained by replacing the unknown differences $\mu_{[k]} - \mu_{[i]}$ by the LCBs for them obtained in the previous section.

All the derivations that follow begin with the statement

$$\text{PGS} \geq \Pr\left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1\right\}. \tag{12}$$

Since the right-hand side of (12) is guaranteed to be $\geq 1 - \alpha$ by the design of our procedures, the LCB on PGS will never be less than $1 - \alpha$. In fact, the LCB on PGS (PCS) is only meaningful if it is greater than $1 - \alpha$ (respectively, $1/k$).

4.1. *Procedure S: sample means, unequal variance*

Let $\xi = \delta/(\sqrt{2}v_{n_0-1}^{(1-\alpha)})$, and let $\sigma_{[i]}^2$ be the variance of an observation from system $[i]$. Then

$$\begin{aligned}
\text{PGS} &\geq \Pr\left\{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1\right\}, \\
&= \Pr\left\{\frac{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]})}{\sqrt{(\sigma_{[i]}^2/N_{[i]}) + (\sigma_{[k]}^2/N_{[k]})}} < \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\sqrt{(\sigma_{[i]}^2/N_{[i]}) + (\sigma_{[k]}^2/N_{[k]})}}, \quad i = 1, 2, \ldots, k-1\right\}, \\
&\geq \Pr\left\{Z_i \leq \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi}{\sqrt{(\sigma_{[i]}^2/S_{[i]}^2) + (\sigma_{[k]}^2/S_{[k]}^2)}}, \quad i = 1, 2, \ldots, k-1\right\},
\end{aligned} \tag{13}$$

Table 1. N^S/N^E for $\alpha = 0.10$

$df \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14.2	35.7	64.7	97.7	139.0	189.4	238.4	290.8	353.4	389.8	502.7	568.0	644.8
2	2.74	4.13	5.37	6.52	7.63	8.87	9.78	10.73	12.06	12.54	14.12	15.03	15.96
3	3.63	2.40	2.86	3.36	3.72	4.11	4.57	4.87	5.21	5.68	6.03	6.28	6.68
4	1.55	1.91	2.26	2.52	2.81	3.02	3.34	3.59	3.79	4.00	4.23	4.44	4.61
5	1.42	1.73	1.96	2.20	2.37	2.65	2.83	3.02	3.21	3.37	3.53	3.65	3.79
6	1.35	1.59	1.82	1.98	2.21	2.39	2.54	2.71	2.85	2.97	3.09	3.22	3.37
7	1.30	1.51	1.70	1.92	2.06	2.23	2.37	2.50	2.62	2.75	2.87	2.98	3.12
8	1.28	1.49	1.67	1.84	1.98	2.14	2.24	2.37	2.48	2.61	2.73	2.84	2.96
9	1.26	1.43	1.62	1.76	1.90	2.03	2.15	2.27	2.39	2.51	2.62	2.74	2.89
10	1.23	1.40	1.58	1.71	1.86	1.97	2.09	2.22	2.31	2.44	2.55	2.68	2.77
11	1.21	1.41	1.55	1.68	1.80	1.92	2.04	2.16	2.26	0.09	2.52	2.60	2.71
12	1.21	1.38	1.53	1.66	1.77	1.89	2.00	2.12	2.23	2.40	2.44	2.54	2.64
13	1.19	1.38	1.51	1.63	1.74	1.86	1.98	2.10	2.22	2.35	2.41	2.51	2.59
14	1.19	1.37	1.51	1.60	1.72	1.84	1.95	2.07	2.18	2.30	2.37	2.47	2.55
15	1.17	1.35	1.49	1.59	1.71	1.82	1.94	2.05	2.16	2.28	2.34	2.44	2.51
16	1.19	1.34	1.46	1.58	1.69	1.81	1.94	2.04	2.12	2.24	2.30	2.41	2.48
17	1.20	1.32	1.44	1.56	1.68	1.79	1.91	2.01	2.10	2.21	2.28	2.38	2.46
18	1.18	1.33	1.44	1.56	1.67	1.80	1.89	1.99	2.08	2.19	2.26	2.35	2.44
19	1.17	1.31	1.43	1.54	1.66	1.79	1.88	1.98	2.06	2.18	2.25	2.33	2.42
20	1.18	1.32	1.42	1.53	1.66	1.78	1.86	1.96	2.04	2.14	2.24	2.32	2.40
30	1.15	1.25	1.38	1.49	1.60	1.69	1.79	1.86	1.96	2.05	2.13	2.21	2.29
40	1.13	1.26	1.37	1.46	1.56	1.66	1.74	1.83	1.93	2.00	2.08	2.17	2.25
50	1.13	1.25	1.35	1.46	1.54	1.64	1.72	1.81	1.90	1.97	2.07	2.15	2.22
60	1.13	1.24	1.34	1.43	1.53	1.62	1.71	1.79	1.88	1.97	2.05	2.13	2.20
70	1.13	1.23	1.33	1.42	1.52	1.61	1.69	1.79	1.87	1.94	2.03	2.10	2.18
80	1.13	1.23	1.32	1.43	1.51	1.60	1.69	1.78	1.86	1.94	2.03	2.10	2.18
90	1.12	1.22	1.32	1.42	1.52	1.60	1.68	1.77	1.86	1.93	2.01	2.10	2.17
100	1.11	1.22	1.32	1.42	1.50	1.59	1.68	1.76	1.85	1.93	2.01	2.09	2.16

Table 2. N^S/N^E for $\alpha = 0.05$

$df\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15
1	27.43	82.70	162.0	274.39	406.46	567.42	713.73	852.40	1105.82	1252.63	1566.43	1823.19	2072.28
2	3.42	5.56	7.52	9.37	11.15	13.06	14.79	16.00	18.24	19.11	21.19	22.84	24.15
3	2.07	2.77	3.29	3.97	4.49	4.95	5.55	5.88	6.27	6.95	7.30	5.82	8.20
4	1.63	2.07	2.46	2.79	3.14	3.37	3.73	3.95	4.29	4.46	4.72	5.42	5.13
5	1.47	1.82	2.10	2.35	2.54	2.83	3.05	3.24	3.43	3.61	3.78	3.90	4.03
6	1.38	1.66	1.90	2.08	2.33	2.51	2.69	2.83	2.98	3.08	3.22	3.35	3.50
7	1.32	1.55	1.75	2.00	2.17	2.31	2.47	2.59	2.71	2.83	2.95	3.07	3.20
8	1.28	1.53	1.72	1.90	2.05	2.20	2.29	2.41	2.54	2.67	2.78	2.90	3.01
9	1.27	1.46	1.66	1.81	1.97	2.08	2.18	2.31	2.45	2.54	2.66	2.79	2.92
10	1.23	1.41	1.62	1.75	1.92	1.99	2.12	2.26	2.34	2.47	2.58	2.71	2.80
11	1.22	1.43	1.58	1.72	1.83	1.95	2.07	2.18	2.29	2.43	2.53	2.62	2.72
12	1.21	1.41	1.56	1.71	1.78	1.91	2.02	2.13	2.25	2.38	2.45	2.57	2.64
13	1.20	1.39	1.52	1.65	1.76	1.88	1.99	2.10	2.23	2.32	2.41	2.50	2.58
14	1.19	1.38	1.51	1.62	1.74	1.86	1.97	2.09	2.19	2.29	2.36	2.46	2.54
15	1.17	1.36	1.50	1.60	1.71	1.85	1.95	2.06	2.16	2.25	2.34	2.43	2.50
16	1.20	1.34	1.47	1.59	1.70	1.83	1.96	2.04	2.13	2.22	2.29	2.40	2.47
17	1.21	1.34	1.45	1.56	1.70	1.79	1.91	2.02	2.10	2.19	2.27	2.37	2.44
18	1.20	1.33	1.44	1.55	1.67	1.79	1.88	1.99	2.08	2.18	2.25	2.33	2.42
19	1.19	1.31	1.43	1.54	1.66	1.78	1.88	1.99	2.05	2.15	2.22	2.32	2.40
20	1.19	1.32	1.41	1.54	1.64	1.78	1.86	1.96	2.04	2.13	2.21	2.30	2.37
30	1.16	1.26	1.38	1.50	1.59	1.68	1.78	1.85	1.94	2.03	2.10	2.18	2.25
40	1.13	1.26	1.37	1.45	1.55	1.64	1.73	1.81	1.90	1.98	2.04	2.14	2.21
50	1.12	1.25	1.35	1.45	1.53	1.62	1.71	1.79	1.87	1.95	2.03	2.11	2.18
60	1.12	1.24	1.33	1.43	1.52	1.60	1.68	1.78	1.85	1.94	2.01	2.09	2.16
70	1.13	1.23	1.33	1.41	1.51	1.60	1.67	1.76	1.84	1.91	1.99	2.06	2.13
80	1.13	1.23	1.31	1.42	1.50	1.58	1.67	1.75	1.83	1.91	1.98	2.05	2.13
90	1.12	1.22	1.32	1.40	1.50	1.58	1.66	1.75	1.83	1.90	1.97	2.06	2.12
100	1.12	1.22	1.31	1.40	1.48	1.57	1.66	1.74	1.82	1.90	1.97	2.05	2.12

where

$$Z_i = \frac{\hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]})}{\sqrt{(\sigma^2_{[i]}/N_{[i]}) + (\sigma^2_{[k]}/N_{[k]})}}$$

and (13) follows because $N_i \geq S_i^2/\xi^2$ for all i. We then follow the steps in the proof of Proposition 1 in Rinott (1978) to show that

$$\text{PGS} \geq \int_{b=0}^{\infty} \prod_{i=1}^{k-1} \left[\int_{a=0}^{\infty} \Phi\left(\frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi}{\sqrt{(n_0 - 1)(1/a + 1/b)}} \right) \times \mathrm{d}G_{n_0-1}(a) \right] \mathrm{d}G_{n_0-1}(b), \tag{14}$$

where G_{n_0-1} is the cdf of a χ^2 random variable with $n_0 - 1$ degrees of freedom. To obtain a $(1-\alpha)100\%$ lower confidence bound on PGS we substitute $(1-\alpha)100\%$ lower confidence bounds on $\mu_{[k]} - \mu_{[i]}$ in (14) and evaluate the integral numerically.

Certainly (14) will be difficult to evaluate. However, in the Appendix we show that

$$(14) \geq \prod_{i=1}^{k-1} \left[\int_{t=-\infty}^{\infty} F_{n_0-1}\left(t + \max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi\right) \times \mathrm{d}F_{n_0-1}(t) \right], \tag{15}$$

where F_{n_0-1} is the cdf of the standard t distribution with $n_0 - 1$ degrees of freedom. This expression is easier to evaluate.

4.2. *Procedure G: generalized means*

Let $\xi = \delta / r_{n_0-1}^{(1-\alpha)}$. Then

$$\begin{aligned} \text{PGS} &\geq \Pr\left\{ \hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}], \right. \\ &\qquad \left. i = 1, 2, \ldots, k-1 \right\}, \\ &= \Pr\left\{ \frac{\hat{\mu}_{[i]} - \mu_{[i]}}{\xi} \leq \frac{\hat{\mu}_{[k]} - \mu_{[k]}}{\xi} + \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\xi}, \right. \\ &\qquad \left. i = 1, 2, \ldots, k-1 \right\}, \end{aligned}$$

Table 3. N^S/N^E for $\alpha = 0.01$

$df\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15
1	223.4	989.0	2196.9	3883.1	6620.8	9121.0	11604.3	14509.0	18559.5	20723.7	28693.0	36037.3	37635.2
2	10.82	18.71	27.24	33.84	43.34	55.81	57.98	63.81	74.54	75.45	84.22	94.12	100.62
3	4.83	6.66	8.44	10.06	10.76	12.10	13.90	14.45	15.07	17.22	18.12	18.69	19.86
4	3.28	4.38	5.23	5.93	6.63	6.73	7.47	7.83	8.50	8.49	9.07	9.15	9.45
5	2.90	3.62	4.01	4.49	4.72	5.22	5.50	5.71	5.93	6.19	6.30	6.50	6.45
6	2.63	3.18	3.49	3.72	4.12	4.31	4.52	4.61	4.76	4.76	4.88	4.94	5.05
7	2.47	2.90	3.17	3.50	3.66	3.78	3.90	3.99	4.04	4.14	4.16	4.31	4.34
8	2.42	2.80	3.06	3.24	3.40	3.49	3.48	3.56	3.67	3.69	3.82	3.86	3.93
9	2.36	2.65	2.92	3.01	3.17	3.19	3.20	3.34	3.40	3.46	3.50	3.60	3.67
10	2.32	2.54	2.82	2.89	3.03	3.02	3.02	3.17	3.22	3.23	3.33	3.40	3.42
11	2.29	2.55	2.71	2.78	2.83	2.88	2.88	3.03	3.07	3.12	3.20	3.21	3.29
12	2.27	2.50	2.64	2.75	2.73	2.79	2.77	2.91	2.95	3.05	3.06	3.11	3.11
13	2.24	2.46	2.56	2.67	2.68	2.71	2.68	2.84	2.94	2.94	2.95	2.99	3.00
14	2.20	2.43	2.54	2.58	2.61	2.73	2.57	2.79	2.83	2.88	2.87	2.88	2.91
15	2.17	2.40	2.51	2.54	2.56	2.65	2.57	2.74	2.76	2.77	2.78	2.81	2.82
16	2.27	2.35	2.45	2.48	2.52	2.62	2.62	2.68	2.71	2.72	2.70	2.75	2.74
17	2.26	2.36	2.38	2.46	2.51	2.54	2.54	2.64	2.65	2.68	2.65	2.71	2.68
18	2.23	2.35	2.39	2.45	2.48	2.48	2.49	2.62	2.61	2.61	2.64	2.64	2.68
19	2.22	2.33	2.35	2.41	2.45	2.46	2.46	2.57	2.58	2.60	2.56	2.59	2.62
20	2.22	2.32	2.29	2.39	2.45	2.41	2.44	2.52	2.53	2.55	2.52	2.56	2.58
30	2.18	2.17	2.24	2.29	2.29	2.26	2.25	2.31	2.33	2.35	2.37	2.36	2.34
40	2.12	2.15	2.22	2.19	2.23	2.15	2.17	2.24	2.24	2.25	2.24	2.26	2.26
50	2.10	2.17	2.17	2.17	2.17	2.12	2.11	2.18	2.18	2.20	2.20	2.20	2.21
60	2.10	2.14	2.14	2.14	2.16	2.07	2.06	2.15	2.16	2.17	2.16	2.16	2.17
70	2.11	2.10	2.14	2.14	2.13	2.06	2.06	2.13	2.13	2.12	2.16	2.13	2.13
80	2.07	2.12	2.09	2.10	2.12	2.02	2.06	2.10	2.12	2.12	2.12	2.13	2.13
90	2.08	2.09	2.09	2.08	2.11	2.04	2.04	2.11	2.11	2.10	2.10	2.13	2.11
100	2.08	2.09	2.09	2.09	2.08	2.01	2.03	2.07	2.10	2.10	2.11	2.10	2.10

$$= \Pr\left\{ T_i \le T_k + \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\xi}, \quad i = 1, 2, \ldots, k-1 \right\},$$

$$= \int_{t=-\infty}^{\infty} \prod_{i=1}^{k-1} F_{n_0-1}\left(t + \max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi\right) \mathrm{d}F_{n_0-1}(t), \tag{16}$$

where $T_1, T_2, \ldots, T_k$, are independent standard t random variables, each with $n_0 - 1$ degrees of freedom, and F_{n_0-1} is the cdf of the t distribution with $n_0 - 1$ degrees of freedom. The fact that $(\hat{\mu}_{[i]} - \mu_{[i]})/\xi, i = 1, 2, \ldots, k$ are independent t random variables follows from Stein (1945).

4.3. *Procedure E: sample means, equal variance*

Let $\xi = \delta/q_{k(n_0-1)}^{(1-\alpha)}$. Then

$$\mathrm{PGS} \ge \Pr\left\{ \hat{\mu}_{[i]} - \hat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}], \quad i = 1, 2, \ldots, k-1 \right\},$$

$$= \Pr\left\{ \frac{\hat{\mu}_{[i]} - \mu_{[i]}}{\sqrt{S_p^2/N}} \le \frac{\hat{\mu}_{[k]} - \mu_{[k]}}{\sqrt{S_p^2/N}} + \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\sqrt{S_p^2/N}}, \quad i = 1, 2, \ldots, k-1 \right\},$$

$$\ge \Pr\left\{ T_i \le T_k + \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\xi}, \quad i = 1, 2, \ldots, k-1 \right\},$$

$$= \int_{t=-\infty}^{\infty} \prod_{i=1}^{k-1} F_{k(n_0-1)}\left(t + \max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi\right) \times \mathrm{d}F_{k(n_0-1)}(t),$$

where $T_1, T_2, \ldots, T_k$ are independent t random variables, each with $k(n_0 - 1)$ degrees of freedom, and $F_{k(n_0-1)}$ is the cdf of the t distribution with $k(n_0 - 1)$ degrees of freedom. The fact that $(\hat{\mu}_{[i]} - \mu_{[i]})/\sqrt{S_p^2/N}, i = 1, 2, \ldots, k$ are independent standard t random variables follows from Stein (1945).

Table 4. Critical values $v_{df}^{0.99}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	162.3	231.9	289.9	350.8	434.5	491.7	542.2	601.0	659.3	688.9	805.1	903.0	921.00
2	14.66	17.54	20.43	22.42	25.13	28.40	29.18	30.71	33.31	33.56	35.39	37.71	38.88
3	7.47	8.56	9.62	10.63	11.08	11.90	12.77	13.11	13.66	14.37	14.80	15.05	15.68
4	5.40	6.32	6.98	7.60	8.07	8.39	8.79	9.14	9.59	9.78	10.19	10.33	10.66
5	4.72	5.45	5.89	6.39	6.75	7.07	7.35	7.70	7.96	8.21	8.42	8.68	8.83
6	4.29	4.94	5.31	5.73	6.07	6.32	6.65	6.86	7.02	7.28	7.51	7.67	7.88
7	4.03	4.59	4.99	5.34	5.64	5.89	6.12	6.41	6.61	6.81	6.97	7.20	7.34
8	3.89	4.42	4.75	5.09	5.39	5.64	5.87	6.07	6.29	6.47	6.69	6.82	6.98
9	3.79	4.25	4.59	4.88	5.18	5.43	5.67	5.88	6.06	6.26	6.41	6.61	6.75
10	3.69	4.13	4.48	4.76	5.03	5.26	5.51	5.70	5.89	6.05	6.26	6.42	6.58
11	3.64	4.03	4.38	4.67	4.92	5.14	5.37	5.59	5.78	5.90	6.11	6.28	6.45
12	3.56	3.98	4.30	4.58	4.82	5.07	5.30	5.49	5.66	5.84	6.00	6.19	6.33
13	3.52	3.90	4.22	4.54	4.76	4.99	5.25	5.39	5.58	5.75	5.92	6.08	6.24
14	3.46	3.86	4.19	4.45	4.72	4.99	5.15	5.33	5.52	5.70	5.87	6.01	6.15
15	3.42	3.82	4.15	4.43	4.66	4.90	5.14	5.27	5.46	5.63	5.80	5.96	6.10
16	3.42	3.77	4.12	4.39	4.62	4.86	5.08	5.24	5.42	5.59	5.73	5.89	6.03
17	3.40	3.77	4.06	4.34	4.60	4.80	5.00	5.20	5.38	5.56	5.68	5.86	5.98
18	3.36	3.74	4.05	4.32	4.57	4.77	4.97	5.16	5.34	5.50	5.67	5.82	5.96
19	3.33	3.71	4.04	4.29	4.53	4.74	4.94	5.14	5.30	5.48	5.62	5.77	5.93
20	3.32	3.71	3.98	4.26	4.49	4.73	4.91	5.11	5.27	5.43	5.56	5.75	5.91
30	3.22	3.58	3.87	4.13	4.35	4.58	4.77	4.95	5.11	5.28	5.45	5.57	5.72
40	3.18	3.51	3.82	4.05	4.30	4.50	4.69	4.85	5.03	5.18	5.34	5.48	5.62
50	3.15	3.50	3.78	4.03	4.24	4.48	4.64	4.82	4.99	5.14	5.31	5.45	5.59
60	3.14	3.49	3.76	4.01	4.23	4.42	4.60	4.80	4.95	5.13	5.28	5.42	5.54
70	3.13	3.45	3.75	4.00	4.21	4.43	4.59	4.77	4.93	5.09	5.25	5.37	5.51
80	3.09	3.46	3.72	3.97	4.21	4.38	4.58	4.73	4.92	5.07	5.23	5.38	5.51
90	3.10	3.44	3.71	3.96	4.22	4.39	4.56	4.76	4.92	5.06	5.22	5.37	5.51
100	3.09	3.43	3.71	3.95	4.16	4.38	4.56	4.72	4.91	5.06	5.21	5.35	5.49

4.4. *Procedure C: sample means, sphericity*

Let $\xi = \delta / q_{k,(k-1)(n_0-1)}^{(1-\alpha)}$. Then

$$\begin{aligned}
\text{PGS} &\geq \Pr\left\{\widehat{\mu}_{[i]} - \widehat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]}) < \max[\delta, \mu_{[k]} - \mu_{[i]}],\right. \\
&\qquad \left. i = 1, 2, \ldots, k-1\right\}, \\
&= \Pr\left\{\frac{\widehat{\mu}_{[i]} - \widehat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]})}{\sqrt{2S_d^2/N}} \leq \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\sqrt{2S_d^2/N}},\right. \\
&\qquad \left. i = 1, 2, \ldots, k-1\right\}, \\
&\geq \Pr\left\{T_i \leq \frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]}{\sqrt{2}\xi}, \quad i = 1, 2, \ldots, k-1\right\}, \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \left[\prod_{i=1}^{k-1} \Phi\left(x + y \max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi\right)\right] \\
&\quad \times \mathrm{d}\Phi(x) \mathrm{d}H_{(k-1)(n_0-1)}(y),
\end{aligned}$$

where $T_1, T_2, \ldots, T_{k-1}$ are multivariate t random variables with common correlation $1/2$ and $(k-1)(n_0-1)$ degrees of freedom, and $H_{(k-1)(n_0-1)}$ is the cdf of $\sqrt{\chi_v^2/v}$, with $v = (k-1)(n_0-1)$. The fact that $\left(\widehat{\mu}_{[i]} - \widehat{\mu}_{[k]} - (\mu_{[i]} - \mu_{[k]})\right) \Big/ \sqrt{2S_d^2/N}, i = 1, 2, \ldots, k$ are multivariate t, and the validity of the last integral expression, can be deduced from Theorem 2 of Nelson and Matejcik (1995).

5. Critical values and comparison of procedures

One penalty for adopting a confidence-interval procedure that allows unequal variances across systems is that the resulting intervals tend to be wider than those that would be obtained for the same systems when assuming equal variances. This is primarily due to the loss of degrees of freedom in the variance estimators of the individual variances as opposed to the pooled variance estimator. Since, for our procedures, we assure that the width of the confidence intervals will be fixed at δ, the consequence of allowing unequal variances is that we tend to take more replications than we would if we assumed equal variances.

Table 5. Critical values $v_{df}^{0.95}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	30.9	43.9	56.6	70.6	84.1	98.0	109.0	118.5	134.5	142.9	159.8	172.4	183.80
2	6.59	8.17	9.48	10.63	11.69	12.75	13.68	14.34	15.43	15.92	16.73	17.54	18.18
3	4.42	5.20	5.86	6.48	7.00	7.47	7.95	8.31	8.71	9.08	9.40	9.59	10.10
4	3.65	4.28	4.80	5.26	5.65	5.99	6.32	6.59	6.91	7.16	7.44	7.68	7.87
5	3.32	3.90	4.33	4.70	5.02	5.33	5.63	5.89	6.14	6.37	6.57	6.79	6.98
6	3.13	3.64	4.04	4.39	4.70	4.97	5.23	5.46	5.66	5.88	6.08	6.27	6.47
7	3.00	3.48	3.85	4.20	4.48	4.74	4.98	5.22	5.42	5.61	5.80	5.98	6.16
8	2.92	3.40	3.74	4.05	4.33	4.59	4.81	5.01	5.23	5.43	5.60	5.78	5.94
9	2.88	3.30	3.65	3.94	4.23	4.47	4.68	4.90	5.12	5.27	5.46	5.65	5.80
10	2.81	3.23	3.58	3.86	4.14	4.37	4.60	4.83	4.99	5.18	5.36	5.51	5.68
11	2.78	3.19	3.53	3.81	4.07	4.31	4.54	4.73	4.92	5.09	5.27	5.43	5.60
12	2.74	3.15	3.49	3.78	4.01	4.26	4.47	4.67	4.85	5.04	5.19	5.37	5.52
13	2.71	3.12	3.44	3.74	3.98	4.21	4.42	4.62	4.80	4.97	5.15	5.30	5.45
14	2.70	3.10	3.42	3.70	3.95	4.18	4.39	4.57	4.76	4.95	5.09	5.25	5.41
15	2.67	3.07	3.39	3.67	3.91	4.16	4.36	4.54	4.73	4.90	5.07	5.23	5.37
16	2.66	3.05	3.37	3.66	3.89	4.13	4.34	4.52	4.69	4.86	5.02	5.19	5.34
17	2.66	3.04	3.35	3.62	3.88	4.08	4.29	4.49	4.66	4.84	4.99	5.16	5.30
18	2.64	3.02	3.34	3.61	3.84	4.06	4.26	4.46	4.64	4.82	4.97	5.12	5.28
19	2.62	3.00	3.32	3.58	3.83	4.05	4.26	4.46	4.61	4.79	4.94	5.11	5.26
20	2.61	3.00	3.30	3.58	3.80	4.05	4.24	4.43	4.59	4.77	4.93	5.08	5.23
30	2.55	2.92	3.23	3.49	3.72	3.93	4.14	4.30	4.49	4.65	4.81	4.95	5.10
40	2.53	2.90	3.19	3.44	3.67	3.89	4.08	4.25	4.43	4.60	4.74	4.90	5.04
50	2.51	2.87	3.17	3.43	3.64	3.87	4.06	4.23	4.40	4.57	4.72	4.87	5.01
60	2.50	2.86	3.15	3.41	3.64	3.84	4.03	4.21	4.38	4.55	4.70	4.84	4.98
70	2.49	2.85	3.15	3.39	3.62	3.83	4.01	4.20	4.36	4.52	4.68	4.81	4.96
80	2.48	2.85	3.13	3.40	3.61	3.81	4.01	4.19	4.35	4.52	4.66	4.80	4.95
90	2.48	2.84	3.13	3.38	3.61	3.81	4.00	4.19	4.35	4.50	4.65	4.81	4.94
100	2.48	2.83	3.12	3.38	3.59	3.80	4.00	4.17	4.34	4.51	4.66	4.80	4.94

In this section we compare the additional sampling effort required for the unequal-variance procedures relative to the equal-variance procedure. In order to make this comparison, we consider the case in which the variances across systems are *actually equal*, since this is the case in which we would lose the most in terms of the extra sampling effort. We do this both for the ordinary sample-mean-based procedure (Procedure S) and the generalized-sample-mean-based procedure (Procedure G). We comment briefly on the potential benefit of the sample-mean procedure that allows for common random numbers (Procedure C).

This section also provides tables of the critical values $r^{(1-\alpha)}$ and $v^{(1-\alpha)}$ since few such values appear in the published literature.

5.1. *Procedure S*

Let the total number of replications required for system i when using Procedure S be denoted by N_i^S. This is given by expression (4). Similarly let N^E denote the total number of replications required for each system when we assume equal variance, Procedure E. This is given by the expression (8). In what follows we assume that the indifference level δ is small enough or the variance from the first n_0 samples is large enough so that the total number of replications N_i required is much larger than the initial sample size n_0; i.e., $N_i \gg n_0$. Thus, the ratio of the additional samples required in the two cases is approximately (neglecting integrality)

$$\frac{N_i^S}{N^E} \simeq \left[\frac{\sqrt{2}v_{n_0-1}^{(1-\alpha)}S_i}{q_{k,k(n_0-1)}^{(1-\alpha)}S_p}\right]^2.$$

Now when the variances for the systems are all equal, $E[S_i^2] = E[S_p^2]$. Therefore,

$$\frac{E[N_i^S]}{E[N^E]} \simeq \left[\frac{\sqrt{2}v_{n_0-1}^{(1-\alpha)}}{q_{k,k(n_0-1)}^{(1-\alpha)}}\right]^2.$$

This quantity is tabulated at confidence levels of 90, 95 and 99% in Tables 1, 2 and 3, respectively. See Tables 4, 5 and 6 for the values of $v_{n_0-1}^{(1-\alpha)}$ used for this comparison. These values were obtained by simulation using 100 000 replications for each value, and almost all are accurate to the second decimal place.

From these tables we see that Procedure S is less efficient than Procedure E when variances are in fact equal. The number of additional samples increases rapidly with

Table 6. Critical values $v_{df}^{0.9}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	15.3	22.0	28.7	34.8	41.4	48.4	54.5	60.4	66.9	71.9	80.5	86.0	91.93
2	4.66	5.85	6.84	7.69	8.50	9.31	9.93	10.54	11.32	11.78	12.44	12.99	13.52
3	3.45	4.12	4.68	5.25	5.69	6.10	6.53	6.87	7.22	7.53	7.83	8.08	8.43
4	2.97	3.54	4.04	4.44	4.80	5.13	5.44	5.74	5.99	6.23	6.49	6.73	6.92
5	2.76	3.29	3.70	4.06	4.37	4.68	4.95	5.20	5.45	5.67	5.87	6.09	6.27
6	2.63	3.11	3.51	3.83	4.14	4.41	4.66	4.89	5.10	5.32	5.51	5.70	5.88
7	2.55	3.00	3.37	3.70	3.98	4.24	4.47	4.71	4.91	5.10	5.29	5.47	5.64
8	2.49	2.94	3.29	3.59	3.88	4.13	4.36	4.57	4.76	4.96	5.14	5.32	5.47
9	2.46	2.87	3.22	3.52	3.79	4.04	4.26	4.47	4.67	4.85	5.02	5.20	5.37
10	2.42	2.83	3.18	3.46	3.73	3.97	4.19	4.40	4.58	4.76	4.94	5.11	5.26
11	2.39	2.80	3.13	3.42	3.68	3.91	4.13	4.33	4.52	0.90	4.89	5.03	5.20
12	2.37	2.77	3.10	3.39	3.64	3.88	4.08	4.28	4.48	4.70	4.81	4.97	5.13
13	2.34	2.75	3.07	3.36	3.61	3.84	4.05	4.25	4.43	4.65	4.78	4.94	5.09
14	2.33	2.74	3.06	3.32	3.58	3.81	4.02	4.21	4.40	4.60	4.73	4.90	5.05
15	2.31	2.71	3.03	3.31	3.56	3.79	4.00	4.18	4.37	4.58	4.71	4.87	5.01
16	2.30	2.70	3.02	3.30	3.54	3.77	3.98	4.17	4.33	4.54	4.67	4.84	4.98
17	2.30	2.67	3.00	3.27	3.52	3.74	3.95	4.14	4.32	4.51	4.65	4.81	4.96
18	2.29	2.68	2.99	3.27	3.50	3.73	3.92	4.12	4.30	4.49	4.63	4.78	4.94
19	2.27	2.65	2.98	3.24	3.49	3.72	3.92	4.11	4.27	4.48	4.61	4.77	4.92
20	2.28	2.66	2.97	3.23	3.48	3.71	3.90	4.09	4.26	4.44	4.60	4.75	4.89
30	2.22	2.59	2.91	3.16	3.41	3.62	3.83	3.99	4.17	4.35	4.49	4.64	4.78
40	2.20	2.59	2.88	3.13	3.37	3.58	3.77	3.95	4.13	4.29	4.44	4.60	4.74
50	2.20	2.56	2.86	3.12	3.34	3.56	3.75	3.93	4.10	4.26	4.42	4.57	4.71
60	2.19	2.55	2.85	3.10	3.34	3.54	3.73	3.91	4.09	4.26	4.40	4.55	4.69
70	2.18	2.54	2.84	3.09	3.32	3.53	3.72	3.91	4.07	4.23	4.38	4.52	4.67
80	2.18	2.54	2.83	3.09	3.31	3.52	3.71	3.90	4.06	4.23	4.38	4.52	4.66
90	2.17	2.53	2.83	3.08	3.32	3.51	3.71	3.88	4.06	4.22	4.37	4.52	4.65
100	2.16	2.53	2.82	3.09	3.30	3.50	3.70	3.88	4.05	4.22	4.37	4.50	4.65

the number of systems we are comparing. For example, with $n_0 = 10$ initial replications (that is df = 9) at the 95% confidence level we need around 27% more replications per system for Procedure S as compared to Procedure E when $k = 3$ systems, whereas we need almost double the number of experiments comparing $k = 7$ systems. At higher degrees of freedom – that is, with larger n_0 – this difference is less. For example, with $n_0 = 31$ (df = 30) and at the 95% confidence level, we need 16% more replications per system for Procedure S as compared to Procedure E when $k = 3$ systems versus 60% more replications for $k = 7$ systems.

Also notice that as we increase the number of initial replications, n_0, we need fewer additional replications beyond n_0. This is as expected, since a larger initial sample provides a more precise variance estimator.

5.2. *Procedure G*

Let N_i^G denote the total number of replications required for system i when using the generalized-sample-mean procedure, Procedure G. This is given by the expression (5). Then under the same assumptions as in the previous section we have the following relationship between N_i^G and N^E,

$$\frac{E[N_i^G]}{E[N^E]} \simeq \left[\frac{r_{n_0-1}^{(1-\alpha)}}{q_{k,k(n_0-1)}^{(1-\alpha)}}\right]^2.$$

This quantity is tabulated at confidence levels of 90, 95 and 99% in Tables 7, 8 and 9, respectively. See Tables 10, 11 and 12 for the values of $r_{n_0-1}^{(1-\alpha)}$ used for this comparison. These values are also obtained via simulation and almost all are accurate to the second decimal place.

From the tables we see that the general trend in behavior is similar to that of Procedure S. However, we observe that the number of replications is considerably smaller for this procedure as compared to the ordinary sample-mean-based procedure. The difference is more apparent at higher confidence levels. For example, using Procedure S with $n_0 = 10$ and $1 - \alpha = 0.99$, we need almost two and a half times more replications, relative to Procedure E, when $k = 3$ systems and more than three times more experiments when $k = 7$ systems. The corresponding values for the Procedure G are 18 and 59%, respectively. Thus, this method is much more economical than the one based on ordinary sample means, although still less efficient than Procedure E when variances are equal.

Table 7. N^G/N^E for $\alpha = 0.10$

$df\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15
1	11.92	27.22	46.91	68.11	95.03	126.46	156.93	188.53	227.03	250.44	319.13	360.11	410.22
2	2.45	3.43	4.26	4.98	5.72	6.43	6.95	7.56	8.31	8.57	9.56	10.06	10.61
3	3.30	2.03	2.28	2.56	2.76	2.94	3.20	3.33	3.50	3.73	3.90	3.97	4.20
4	1.42	1.62	1.80	1.92	2.04	2.12	2.28	2.37	2.44	2.51	2.61	2.66	2.73
5	1.30	1.46	1.55	1.65	1.71	1.82	1.89	1.96	2.01	2.05	2.10	2.11	2.14
6	1.23	1.35	1.45	1.49	1.59	1.64	1.67	1.72	1.75	1.77	1.79	1.82	1.86
7	1.19	1.28	1.34	1.43	1.46	1.51	1.55	1.57	1.59	1.61	1.63	1.65	1.68
8	1.17	1.26	1.31	1.37	1.40	1.45	1.45	1.47	1.49	1.51	1.53	1.55	1.58
9	1.14	1.21	1.29	1.31	1.34	1.37	1.38	1.40	1.42	1.45	1.46	1.47	1.50
10	1.12	1.18	1.25	1.27	1.31	1.32	1.33	1.35	1.37	1.39	1.41	1.44	1.44
11	1.11	1.19	1.23	1.24	1.27	1.29	1.29	1.32	1.33	1.36	1.37	1.38	1.39
12	1.10	1.17	1.21	1.23	1.24	1.26	1.27	1.29	1.30	1.32	1.33	1.34	1.35
13	1.09	1.16	1.18	1.21	1.22	1.23	1.25	1.27	1.29	1.29	1.31	1.31	1.32
14	1.09	1.15	1.18	1.18	1.20	1.21	1.23	1.25	1.27	1.27	1.28	1.28	1.29
15	1.07	1.14	1.17	1.17	1.19	1.21	1.22	1.23	1.25	1.25	1.26	1.27	1.27
16	1.09	1.13	1.14	1.16	1.17	1.19	1.22	1.22	1.23	1.23	1.23	1.25	1.24
17	1.09	1.12	1.13	1.15	1.17	1.18	1.19	1.21	1.21	1.22	1.22	1.23	1.23
18	1.08	1.12	1.13	1.15	1.16	1.18	1.18	1.19	1.20	1.20	1.20	1.21	1.21
19	1.08	1.10	1.12	1.13	1.15	1.18	1.17	1.19	1.18	1.19	1.19	1.20	1.21
20	1.08	1.11	1.11	1.12	1.15	1.17	1.16	1.17	1.17	1.18	1.19	1.19	1.19
30	1.06	1.05	1.08	1.09	1.10	1.10	1.11	1.10	1.11	1.12	1.12	1.12	1.12
40	1.04	1.05	1.06	1.06	1.07	1.08	1.07	1.08	1.08	1.08	1.08	1.09	1.09
50	1.04	1.05	1.05	1.06	1.05	1.07	1.06	1.06	1.06	1.06	1.07	1.07	1.07
60	1.03	1.04	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.06	1.06	1.06	1.06
70	1.03	1.04	1.04	1.04	1.04	1.05	1.04	1.05	1.04	1.04	1.05	1.05	1.05
80	1.03	1.03	1.03	1.04	1.04	1.04	1.03	1.04	1.04	1.04	1.04	1.05	1.04
90	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.04	1.03	1.04	1.04	1.04
100	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.04	1.03	1.03

5.3. *Procedure C*

The critical values for Procedure E, assuming independent systems and equal variance, and the sample-mean-based procedure assuming that CRN has been used to induce dependence, Procedure C, are $q^{(1-\alpha)}_{k,k(n_0-1)}$ and $q^{(1-\alpha)}_{k,(k-1)(n_0-1)}$, respectively. A quick check of these values (see, for instance, Hochberg and Tamhane (1987)) shows that they are very close provided n_0 is not too small. Thus, the relative efficiency of the two procedures will depend on $E[S_d^2]/E[S_p^2]$.

If we assume that the true variances are equal, and further that the correlation induced between any pair of systems under CRN is $\rho \geq 0$, then we can show that $E[S_d^2]/E[S_p^2] = 1 - \rho$. Therefore, Procedure C will tend to be more efficient than Procedure E if CRN achieves any positive correlation, and much more efficient if ρ is close to 1. Of course, the assumption of sphericity, just like that of equal variance, is an approximation that is rarely if ever precisely true in practice.

6. Examples

In this section we present two examples: The first is a toy example in which the true means and variances are under our control, while the second arose in research on agile manufacturing systems. In both cases we apply Procedure G.

To illustrate the performance of these procedures, consider $k = 5$ independent systems represented by normal distributions with means and variances as shown in Table 13. Suppose we take $n_0 = 10$ initial replications from each, apply Procedure G with $1 - \alpha = 0.9$ and $\delta = 0.5, 1$ or 2, and compute the LCBs on PCS and PCS from the data. The last two columns of Table 13 show the average of these LCBs across 100 replications of the entire procedure.

The primary feature to notice in these results is that the LCB on PGS always remains larger than 0.9, while the LCB on PCS may be larger or smaller than $1 - \alpha$ depending on how large or small is our indifference zone δ. A large value of δ implies little second-stage sampling and a small chance of selecting the unique best system (remember that these are realizations of lower confidence bounds on PCS, which is why the estimated PCS can be below $1/k = 0.2$); while a small value of δ delivers precise estimates and a larger chance of selecting the unique best.

As a second, more realistic example, consider a question that arose in research on agile manufacturing systems. Suppose a portion of such a system consists of two stations in tandem, station 1 and station 2, but just *one*

Table 8. N^G/N^E for $\alpha = 0.05$

df\k	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
1	22.4	60.8	111.7	182.2	266.0	361.8	447.1	534.5	678.5	769.6	952.2	1097.9	1244.1
2	3.00	4.48	5.86	7.04	8.19	9.40	10.50	11.24	12.62	13.13	14.47	15.39	16.26
3	1.854	2.305	2.637	3.065	3.361	3.616	3.967	4.120	4.315	4.738	4.917	3.866	5.388
4	1.470	1.763	1.985	2.151	2.335	2.424	2.615	2.721	2.886	2.912	3.061	3.432	3.203
5	1.334	1.553	1.684	1.810	1.874	2.019	2.107	2.175	2.243	2.307	2.364	2.373	2.406
6	1.258	1.412	1.532	1.586	1.714	1.767	1.823	1.861	1.904	1.927	1.952	1.992	2.035
7	1.199	1.320	1.402	1.524	1.574	1.608	1.655	1.689	1.712	1.729	1.759	1.789	1.805
8	1.168	1.302	1.375	1.437	1.485	1.528	1.540	1.551	1.592	1.615	1.633	1.663	1.675
9	1.159	1.238	1.326	1.362	1.414	1.435	1.445	1.473	1.503	1.516	1.545	1.567	1.606
10	1.122	1.196	1.289	1.324	1.374	1.370	1.398	1.427	1.430	1.458	1.483	1.509	1.513
11	1.121	1.214	1.257	1.295	1.312	1.330	1.355	1.377	1.388	1.423	1.437	1.444	1.460
12	1.110	1.191	1.235	1.284	1.271	1.304	1.316	1.332	1.355	1.384	1.389	1.399	1.403
13	1.095	1.185	1.206	1.241	1.257	1.274	1.294	1.309	1.342	1.348	1.354	1.362	1.368
14	1.098	1.174	1.199	1.211	1.236	1.259	1.271	1.300	1.308	1.328	1.317	1.329	1.338
15	1.074	1.152	1.184	1.201	1.216	1.245	1.257	1.273	1.285	1.297	1.298	1.305	1.313
16	1.099	1.137	1.164	1.189	1.202	1.228	1.261	1.260	1.264	1.272	1.266	1.288	1.283
17	1.107	1.136	1.152	1.167	1.197	1.208	1.233	1.247	1.245	1.256	1.252	1.262	1.264
18	1.090	1.125	1.144	1.164	1.179	1.209	1.215	1.225	1.232	1.239	1.238	1.249	1.248
19	1.090	1.111	1.130	1.146	1.169	1.202	1.200	1.220	1.210	1.228	1.218	1.227	1.233
20	1.092	1.119	1.125	1.143	1.157	1.195	1.195	1.202	1.200	1.206	1.203	1.219	1.218
30	1.057	1.064	1.086	1.116	1.113	1.115	1.128	1.121	1.125	1.140	1.132	1.134	1.133
40	1.039	1.062	1.078	1.073	1.083	1.083	1.087	1.087	1.097	1.093	1.093	1.105	1.099
50	1.030	1.060	1.059	1.071	1.062	1.071	1.076	1.077	1.077	1.076	1.079	1.082	1.079
60	1.029	1.051	1.055	1.055	1.056	1.055	1.052	1.061	1.066	1.070	1.067	1.067	1.060
70	1.035	1.042	1.045	1.043	1.052	1.053	1.040	1.059	1.051	1.050	1.054	1.056	1.051
80	1.033	1.042	1.034	1.046	1.043	1.039	1.037	1.044	1.047	1.048	1.048	1.050	1.047
90	1.027	1.028	1.037	1.036	1.039	1.038	1.036	1.045	1.040	1.038	1.039	1.045	1.043
100	1.028	1.029	1.026	1.028	1.029	1.030	1.037	1.037	1.040	1.042	1.041	1.041	1.035

Table 9. N^G/N^E for $\alpha = 0.01$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	111.7	494.5	1098.4	1941.5	3310.4	4560.5	5802.2	7254.5	9279.8	10361.8	14346.5	18018.6	18817.6
2	5.41	9.35	13.62	16.92	21.67	27.91	28.99	31.90	37.27	37.72	42.11	47.06	50.31
3	2.41	3.33	4.22	5.03	5.38	6.05	6.95	7.22	7.54	8.61	9.06	9.34	9.93
4	1.64	2.19	2.62	2.96	3.31	3.37	3.73	3.91	4.25	4.24	4.54	4.58	4.72
5	1.45	1.81	2.00	2.25	2.36	2.61	2.75	2.86	2.96	3.09	3.15	3.25	3.23
6	1.32	1.59	1.75	1.86	2.06	2.15	2.26	2.31	2.38	2.38	2.44	2.47	2.52
7	1.24	1.45	1.59	1.75	1.83	1.89	1.95	2.00	2.02	2.07	2.08	2.15	2.17
8	1.21	1.40	1.53	1.62	1.70	1.74	1.74	1.78	1.84	1.84	1.91	1.93	1.96
9	1.18	1.32	1.46	1.50	1.59	1.60	1.60	1.67	1.70	1.73	1.75	1.80	1.83
10	1.16	1.27	1.41	1.45	1.52	1.51	1.51	1.58	1.61	1.61	1.67	1.70	1.71
11	1.14	1.27	1.35	1.39	1.42	1.44	1.44	1.51	1.54	1.56	1.60	1.60	1.65
12	1.13	1.25	1.32	1.37	1.36	1.40	1.39	1.46	1.48	1.52	1.53	1.55	1.55
13	1.12	1.23	1.28	1.33	1.34	1.35	1.34	1.42	1.47	1.47	1.48	1.50	1.50
14	1.10	1.22	1.27	1.29	1.30	1.36	1.29	1.40	1.41	1.44	1.44	1.44	1.45
15	1.09	1.20	1.26	1.27	1.28	1.32	1.28	1.37	1.38	1.39	1.39	1.41	1.41
16	1.13	1.18	1.23	1.24	1.26	1.31	1.31	1.34	1.35	1.36	1.35	1.37	1.37
17	1.13	1.18	1.19	1.23	1.26	1.27	1.27	1.32	1.33	1.34	1.33	1.35	1.34
18	1.11	1.17	1.20	1.23	1.24	1.24	1.25	1.31	1.30	1.30	1.32	1.32	1.34
19	1.11	1.17	1.18	1.21	1.23	1.23	1.23	1.28	1.29	1.30	1.28	1.29	1.31
20	1.11	1.16	1.15	1.20	1.22	1.20	1.22	1.26	1.26	1.27	1.26	1.28	1.29
30	1.09	1.09	1.12	1.14	1.14	1.13	1.12	1.16	1.16	1.18	1.18	1.18	1.17
40	1.06	1.07	1.11	1.09	1.11	1.08	1.09	1.12	1.12	1.12	1.12	1.13	1.13
50	1.05	1.08	1.08	1.08	1.09	1.06	1.06	1.09	1.09	1.10	1.10	1.10	1.10
60	1.05	1.07	1.07	1.07	1.08	1.03	1.03	1.07	1.08	1.08	1.08	1.08	1.08
70	1.06	1.05	1.07	1.07	1.07	1.03	1.03	1.06	1.06	1.06	1.08	1.06	1.06
80	1.03	1.06	1.05	1.05	1.06	1.01	1.03	1.05	1.06	1.06	1.06	1.06	1.06
90	1.04	1.05	1.05	1.04	1.05	1.02	1.02	1.05	1.05	1.05	1.05	1.06	1.06
100	1.04	1.05	1.04	1.04	1.04	1.01	1.01	1.04	1.05	1.05	1.05	1.05	1.05

Table 10. Critical values $r_{df}^{0.99}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	200.8	271.3	330.1	392.6	478.7	536.2	585.0	638.8	709.0	738.0	858.8	954.4	967.10
2	18.89	21.50	24.47	26.45	29.42	33.07	33.49	35.02	37.79	38.08	39.70	42.23	43.82
3	9.84	10.88	12.00	13.01	13.43	14.27	15.15	15.61	16.06	16.75	17.27	17.54	18.19
4	7.22	8.14	8.88	9.49	9.98	10.22	10.57	10.89	11.40	11.47	11.93	12.04	12.29
5	6.34	7.07	7.49	7.95	8.30	8.59	8.88	9.15	9.38	9.64	9.76	10.11	10.13
6	5.80	6.42	6.77	7.15	7.45	7.68	7.98	8.12	8.28	8.45	8.64	8.76	8.89
8	5.27	5.76	6.07	6.34	6.62	6.78	7.01	7.10	7.29	7.36	7.56	7.65	7.73
9	5.11	5.56	5.87	6.08	6.34	6.52	6.72	6.84	6.98	7.09	7.19	7.34	7.38
10	4.99	5.41	5.71	5.93	6.15	6.32	6.51	6.64	6.76	6.81	6.98	7.04	7.13
11	4.91	5.27	5.58	5.78	6.00	6.15	6.35	6.46	6.58	6.61	6.77	6.84	6.99
12	4.80	5.19	5.49	5.71	5.87	6.04	6.22	6.31	6.42	6.53	6.61	6.73	6.80
13	4.76	5.12	5.38	5.66	5.81	5.93	6.11	6.20	6.34	6.41	6.50	6.61	6.67
14	4.69	5.06	5.34	5.55	5.71	5.93	5.97	6.10	6.22	6.35	6.41	6.48	6.57
15	4.65	5.01	5.28	5.50	5.65	5.83	5.96	6.04	6.14	6.23	6.31	6.41	6.48
16	4.63	4.94	5.25	5.42	5.59	5.78	5.91	5.98	6.09	6.17	6.21	6.33	6.38
17	4.61	4.94	5.17	5.39	5.57	5.68	5.82	5.92	6.02	6.12	6.16	6.28	6.31
18	4.56	4.91	5.17	5.37	5.51	5.65	5.76	5.91	5.97	6.04	6.15	6.21	6.23
20	4.51	4.85	5.04	5.28	5.45	5.58	5.70	5.79	5.88	5.97	6.00	6.11	6.19
30	4.37	4.67	4.91	5.09	5.22	5.40	5.47	5.55	5.64	5.74	5.82	5.87	5.90
40	4.32	4.59	4.84	4.98	5.15	5.27	5.38	5.46	5.53	5.61	5.66	5.74	5.79
50	4.28	4.58	4.79	4.96	5.08	5.23	5.31	5.38	5.46	5.54	5.62	5.67	5.72
60	4.25	4.55	4.76	4.93	5.07	5.17	5.23	5.35	5.44	5.51	5.55	5.61	5.67
70	4.24	4.51	4.75	4.92	5.04	5.16	5.23	5.33	5.40	5.45	5.57	5.57	5.62
80	4.19	4.53	4.70	4.88	5.02	5.11	5.24	5.29	5.39	5.45	5.51	5.57	5.62
90	4.20	4.50	4.70	4.85	5.01	5.13	5.21	5.30	5.37	5.42	5.48	5.57	5.60
100	4.20	4.50	4.70	4.86	4.97	5.09	5.20	5.25	5.36	5.42	5.49	5.54	5.59

Table 11. Critical values $r_{df}^{0.95}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	39.4	53.2	66.5	81.4	96.2	110.7	122.0	132.7	149.0	158.4	176.2	189.2	201.40
2	8.72	10.37	11.84	13.03	14.17	15.30	16.30	17.00	18.15	18.66	19.55	20.36	21.10
3	5.91	6.71	7.42	8.05	8.56	9.03	9.51	9.84	10.22	10.60	10.91	11.06	11.58
4	4.90	5.58	6.10	6.53	6.89	7.19	7.48	7.74	8.02	8.18	8.47	8.64	8.79
5	4.48	5.09	5.49	5.83	6.11	6.37	6.62	6.83	7.02	7.20	7.35	7.49	7.63
6	4.23	4.75	5.13	5.42	5.69	5.89	6.10	6.26	6.40	6.57	6.70	6.84	6.98
7	4.05	4.53	4.87	5.18	5.40	5.59	5.76	5.96	6.09	6.20	6.33	6.45	6.54
8	3.95	4.43	4.72	4.99	5.21	5.41	5.57	5.69	5.86	5.97	6.07	6.19	6.27
9	3.89	4.30	4.61	4.84	5.06	5.25	5.39	5.53	5.67	5.76	5.88	5.98	6.08
10	3.79	4.21	4.52	4.75	4.96	5.12	5.29	5.43	5.51	5.63	5.74	5.82	5.91
11	3.76	4.16	4.45	4.67	4.88	5.03	5.19	5.32	5.41	5.51	5.62	5.70	5.80
12	3.70	4.10	4.40	4.63	4.79	4.97	5.10	5.22	5.33	5.44	5.53	5.61	5.69
13	3.67	4.07	4.33	4.59	4.76	4.90	5.05	5.15	5.27	5.36	5.46	5.53	5.62
14	3.67	4.05	4.31	4.52	4.71	4.87	4.99	5.10	5.20	5.32	5.38	5.47	5.55
15	3.62	4.00	4.27	4.50	4.66	4.83	4.95	5.04	5.16	5.26	5.34	5.41	5.50
16	3.60	3.97	4.25	4.47	4.63	4.78	4.93	5.02	5.12	5.21	5.28	5.38	5.44
17	3.60	3.96	4.22	4.42	4.61	4.74	4.88	4.99	5.08	5.18	5.25	5.32	5.40
18	3.56	3.93	4.20	4.41	4.57	4.72	4.84	4.95	5.05	5.14	5.22	5.30	5.36
19	3.55	3.90	4.17	4.37	4.54	4.70	4.81	4.94	5.00	5.12	5.18	5.25	5.33
20	3.54	3.90	4.16	4.36	4.51	4.69	4.80	4.90	4.98	5.07	5.15	5.23	5.30
30	3.45	3.80	4.04	4.26	4.40	4.53	4.66	4.73	4.83	4.93	4.99	5.05	5.11
40	3.43	3.77	4.01	4.17	4.34	4.47	4.58	4.66	4.77	4.83	4.90	4.98	5.03
50	3.40	3.74	3.97	4.17	4.30	4.44	4.55	4.64	4.72	4.79	4.87	4.93	4.99
60	3.38	3.72	3.97	4.14	4.29	4.41	4.50	4.60	4.70	4.78	4.85	4.90	4.94
70	3.37	3.71	3.95	4.12	4.28	4.40	4.48	4.60	4.67	4.74	4.81	4.87	4.92
80	3.36	3.71	3.93	4.12	4.26	4.37	4.47	4.57	4.66	4.73	4.80	4.86	4.91
90	3.35	3.68	3.93	4.10	4.25	4.37	4.47	4.57	4.64	4.71	4.78	4.85	4.90
100	3.36	3.68	3.91	4.09	4.23	4.35	4.47	4.55	4.64	4.72	4.79	4.84	4.88

operator; see Fig. 1. Jobs come in to the system at station 1 at the rate of λ per hour. After arrival a job needs to be set up at the station by the operator. After setting up the job, the station processes it to completion *without requiring any assistance by the operator*. Following completion at station 1, the job needs to be set up by the operator at station 2, and then is processed at station 2 without requiring any assistance from the operator. After completion at station 2 the job leaves the system. The processing rate and the set up rate at the two machines are μ_1, β_1 and μ_2, β_2, respectively. We assume that there is no walking time between the two stations. Holding costs of c_1 and c_2 per job per unit time are incurred while the jobs are in stations 1 and 2, respectively. The operator has to decide which station to set up first when there are jobs waiting at both stations. The problem is to find a policy for the operator that minimizes the expected holding cost of the system.

We considered the following policies, some of which have been examined previously in Desruelle and Steudel (1996) and Nakade *et al.* (1997):

FIFO: The worker sets up the jobs on a First-Come-First-Serve basis. This is the simplest policy.

SEIZE1: The worker gives priority to station 1. That is, at the end of a set up completion if there is a job in buffer 1 she shifts to buffer 1 irrespective of the situation in buffer 2.

SEIZE2: The worker gives priority to buffer 2 irrespective of number of jobs in buffer 1. The opposite of the SEIZE1 policy.

PREEMPT1: Whenever there are jobs waiting to be set up at buffer 1, the operator sets up a job there. Even when she is currently setting up a job at station 2, she abandons that job and finishes all the jobs in buffer 1 before returning to complete the set up at 2.

PREEMPT2: The same as above except the preemption is in favor of buffer 2.

TH1(n): The operator follows the FIFO policy until the number of jobs in buffer 1 reaches n, at which point she switches to the PREEMPT1 policy (and switches back to FIFO when the number of jobs in buffer 1 falls below n). For this example we used $n = 3$.

Table 12. Critical Values $r_{df}^{0.9}$

df\k	3	4	5	6	7	8	9	10	11	12	13	14	15
1	19.8	27.1	34.5	41.1	48.5	55.9	62.5	68.8	75.8	81.5	90.8	96.8	103.70
2	6.23	7.54	8.60	9.50	10.40	11.21	11.84	12.51	13.29	13.77	14.48	15.03	15.59
3	4.65	5.35	5.92	6.48	6.93	7.30	7.73	8.03	8.36	8.64	8.91	9.08	9.45
4	4.01	4.61	5.10	5.47	5.78	6.07	6.35	6.60	6.79	6.98	7.21	7.37	7.53
5	3.73	4.27	4.66	4.99	5.25	5.49	5.73	5.92	6.10	6.26	6.40	6.54	6.67
6	3.55	4.05	4.42	4.69	4.96	5.17	5.35	5.51	5.65	5.81	5.93	6.06	6.19
7	3.45	3.91	4.23	4.52	4.74	4.94	5.11	5.28	5.40	5.53	5.65	5.75	5.86
8	3.37	3.83	4.13	4.38	4.61	4.80	4.96	5.10	5.22	5.34	5.45	5.56	5.65
9	3.32	3.74	4.07	4.29	4.50	4.69	4.82	4.96	5.09	5.21	5.31	5.40	5.48
10	3.26	3.68	3.99	4.21	4.42	4.60	4.73	4.86	4.99	5.09	5.19	5.28	5.36
11	3.23	3.64	3.94	4.16	4.36	4.54	4.65	4.79	4.90	5.01	5.10	5.19	5.28
12	3.19	3.59	3.89	4.12	4.30	4.48	4.60	4.72	4.84	4.93	5.02	5.11	5.19
13	3.16	3.57	3.84	4.09	4.26	4.41	4.56	4.68	4.79	4.87	4.98	5.06	5.14
14	3.16	3.55	3.83	4.04	4.22	4.38	4.51	4.62	4.74	4.84	4.92	4.99	5.08
15	3.13	3.52	3.79	4.02	4.20	4.36	4.48	4.59	4.71	4.79	4.88	4.96	5.04
16	3.11	3.51	3.77	4.00	4.17	4.33	4.46	4.56	4.67	4.76	4.83	4.93	4.98
17	3.12	3.47	3.75	3.97	4.15	4.29	4.42	4.54	4.63	4.73	4.81	4.89	4.96
18	3.09	3.47	3.74	3.96	4.12	4.27	4.39	4.51	4.61	4.71	4.77	4.85	4.93
19	3.08	3.44	3.72	3.93	4.10	4.27	4.38	4.50	4.57	4.68	4.74	4.82	4.91
20	3.08	3.45	3.71	3.91	4.09	4.24	4.36	4.47	4.56	4.66	4.74	4.81	4.87
30	3.01	3.36	3.63	3.82	3.99	4.12	4.27	4.34	4.44	4.53	4.60	4.66	4.72
40	2.98	3.35	3.59	3.77	3.94	4.08	4.19	4.29	4.38	4.46	4.53	4.60	4.66
50	2.97	3.32	3.57	3.76	3.91	4.06	4.17	4.26	4.34	4.42	4.50	4.56	4.62
60	2.96	3.30	3.56	3.75	3.90	4.02	4.14	4.23	4.32	4.42	4.48	4.54	4.60
70	2.95	3.30	3.55	3.73	3.89	4.02	4.12	4.23	4.30	4.38	4.45	4.51	4.57
80	2.95	3.30	3.53	3.73	3.88	4.00	4.10	4.21	4.30	4.37	4.44	4.51	4.56
90	2.94	3.28	3.53	3.71	3.88	3.99	4.10	4.20	4.29	4.36	4.44	4.50	4.55
100	2.93	3.28	3.52	3.72	3.86	3.99	4.11	4.20	4.28	4.36	4.43	4.49	4.54

TH2(n): The operator follows the FIFO policy until the number of jobs in buffer 2 reaches n, at which point she switches to the PREEMPT2 policy (and switches back to FIFO when the number of jobs in buffer 2 falls below n). For this example we used $n = 3$.

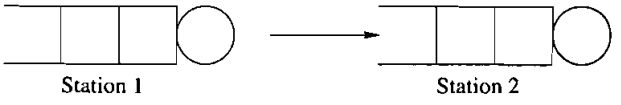

Fig. 1. Layout of stations for the tandem queueing system used in the example.

Interarrival, service and set up distributions are taken to be exponential. The arrival rate is fixed at $\lambda = 2$ jobs/hour; the service distribution parameters are shown in Table 14 along with the holding costs. Notice that with the cost in both buffers being equal, minimization of the total holding cost is equivalent to the minimization of the total WIP in the system; our goal is to find the policy with the lowest expected holding cost per unit time.

A cost reduction of more than \$1 was considered significant, so we set $\delta = 1$. Using confidence level $1 - \alpha = 0.9$ and $n_0 = 10$ initial replications for the $k = 7$ alternatives, the critical value for Procedure G is $r_9^{(0.9)} = 4.50$. Since smaller cost is better, we applied

Table 13. Average LCB on PGS and PCS over 100 replications of Procedure G with $n_0 = 10$, $1 - \alpha = 0.9$ and $r_9^{(0.9)} = 4.07$ when the data are normally distributed

$\mu_1, \ldots, \mu_5$	$\sigma_1, \ldots, \sigma_5$	δ	$\widehat{LCB}_{PGS}$	$\widehat{LCB}_{PCS}$
0,0,0,0,1	1,2,3,4,5	0.5	0.98	0.91
		1.0	0.97	0.21
		2.0	0.97	0.03
1,2,3,4,5	1,1,1,1,1	0.5	0.99	0.96
		1.0	0.99	0.47
		2.0	0.98	0.15

Table 14. Service and cost parameters for the agile manufacturing example

Station	*Processing rate (jobs/hour)*	*Set up rate (jobs/hour)*	*Holding cost (\$/job/hour)*
Station 1	4	4	1
Station 2	6	6	1

Table 15. Results from the agile manufacturing system simulation

Policy	*FIFO*	*PREEMPT1*	*PREEMPT2*	*SEIZE1*	*SEIZE2*	*TH1(3)*	*TH2(3)*
index i	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$\widehat{\mu}_i$	5.43	8.06	4.43	4.63	7.59	6.05	5.13
N_i	28	129	24	23	123	45	32

Procedure G to the negative of the simulation output data, then converted the final results back to positive costs.

Table 15 summarizes the simulation results. The policy PREEMPT2 has the smallest estimated expected cost and would be selected as the best. Our 90% LCB on PGS is 0.98, showing that we have added assurance that PREEMPT2 is a good selection, meaning that its expected cost is within \$1 of the true minimum expected cost, even if it is not the true best. However, since our LCB on PCS is only 0.01, there is no strong evidence that PREEMPT2 is the unique best.

But we can say more. With confidence level at least 90%, we can claim that all policies whose sample means are 4.43 + 1 = 5.43 or greater are clearly not the best; this includes PREEMPT1, SEIZE2 and TH1(3). Some of these – specifically PREEMPT1 and SEIZE2 – can be claimed to be more than \$1 from the true best. For instance, since $\widehat{\mu}_2 - \min_{\ell \neq 2} \widehat{\mu}_\ell - \delta = 8.06 - 4.43 - 1 = 2.63$, we can claim that PREEMPT1 is at least \$2.63 more costly than the true best policy. If we decided that it was useful to determine the unique best, then this analysis would justify eliminating PREEMPT1, SEIZE2 and TH1(3) from consideration in any follow-up analysis using a smaller indifference zone, δ.

7. Conclusions

The procedures presented in this paper provide quite a bit of useful information to the experimenter, much more than other indifference-zone procedures that have been proposed for this problem. However, this inference comes at a price that indicates areas for further research.

All of the procedures in this paper are conservative if the primary interest is in selecting a good system (recall that each procedure also provides inference on all-pairwise comparisons). In other words, PGS $> 1 - \alpha$ for *all* configurations of the means $\mu_1, \mu_2, \ldots, \mu_k$. Procedures that are much tighter (require less sampling) while still providing the same inference are desirable.

The method for constructing the LCBs on PCS and PGS are parametric, meaning that they depend strongly on the assumption of normally distributed data. Nonparametric versions, based on bootstrapping for instance, would be extremely valuable since the LCB could account for both the possible increase in confidence due to encountering a favorable configuration of the means, and also the possible degradation in confidence due to violation of the assumptions of the procedure.

Acknowledgements

This research was partially supported by National Science Foundation Grant number DMI-9622065, and by Symix Corporation/Pritsker Division and Systems Modeling Corporation.

References

Anderson, P.O., Bishop, T.A. and Dudewicz, E.J. (1977) Indifference-zone ranking and selection: confidence intervals for true achieved P(CD). *Communications in Statistics*, **A6**, 1121–1132.

Bechhofer, R.E., Santner, T.J. and Goldsman, D.M. (1995) *Design and Analysis for Statistical Selection, Screening and Multiple Comparisons*, John Wiley, New York.

Bofinger, E. (1994) Should we estimate the probability of a correct selection? *Australian Journal of Statistics*, **38**, 59–66.

Desruelle, P. and Steudel, H.J. (1996) A queuing network model of a single operator manufacturing workcell with machine/operator interference. *Management Science*, **42**, 576–590.

Goldsman, D.M. and Nelson, B.L. (1998) Comparing systems via simulation, in *Handbook of Simulation*, Banks, J. (ed), John Wiley, New York, Ch. 8.

Hochberg, Y. (1975) Simultaneous inference under Behrens-Fisher conditions: a two stage approach. *Communications in Statistics*, **A4**, 1109–1119.

Hochberg, Y. and Tamhane, A.C. (1987) *Multiple Comparison Procedures*, John Wiley, New York.

Hsu, J.C. (1984) Ranking and selection and multiple comparisons with the best, in *Design of Experiments: Ranking and Selection*, Santner, T.J. and Tamhane, A.J. (eds), Marcel Dekker, New York.

Hsu, J.C. (1996) *Multiple Comparisons: Theory and methods*, Chapman & Hall, New York.

Jeon, J.-W., Kim, W.-C. and Jeong, G.-J. (1988) Confidence bounds on superiority. *Journal of the Korean Society for Quality Control*, **16**, 10–17.

Jeong, G.-J., Kim, W.-C. and Jeon, J.-W. (1989) A lower confidence bound on the probability of a correct selection of the t best populations. *Journal of the Korean Statistical Society*, **18**, 26–37.

Kim, W.-C. (1986) A lower confidence bound on the probability of a correct selection. *Journal of the American Statistical Association*, **81**, 1012–1017.

Kim, W.-C. (1988) On detecting the best treatment. *Journal of the Korean Statistical Society*, **17**, 82–92.

Nakade, K., Ohno, K. and Shantikumar, J.G. (1997) Bounds and approximations for cycle time of a U-shaped production line. *Operations Research Letters*, **21**, 191–200.

Nelson, B.L. and Goldsman, D.M. (1998) Comparisons with a standard in simulation experiments. Technical Report, Department of

Industrial Engineering and Management Sciences, Northwestern University.

Nelson, B.L. and Matejcik, F.J. (1995) Using common random numbers for indifference-zone selection and multiple comparisons in simulation. *Management Science*, **41**, 1935–1945.

Rinott, Y. (1978) On two-stage selection procedures and related probability-inequalities. *Communications in Statistics*, **A7**, 799–811.

Stein, C. (1945) A two-sample test for a linear hypothesis whose power is independent of the variance. *Annals of Mathematical Statistics*, **16**, 243–258.

Tamhane, A.C. (1977) Multiple comparisons in model I one-way anova with unequal variances. *Communications in Statistics*, **A6**, 15–32.

Appendix

Calculating PCS and PGS for Procedure S

To establish the computational formula (15) presented in Section 4.1 we need the following lemma:

Lemma 2. (Tamhane, 1977) *Let $Q_1, Q_2, \ldots, Q_k$ be independent random variables, and let $g_j(q_1, q_2, \ldots, q_k)$, $j = 1, 2, \ldots, p$, be non-negative, real-valued functions, each one non-decreasing in each of its arguments. Then*

$$\mathrm{E}\left[\prod_{j=1}^{p} g_j(Q_1, Q_2, \ldots, Q_k)\right] \geq \prod_{j=1}^{p} \mathrm{E}\left[g_j(Q_1, Q_2, \ldots, Q_k)\right].$$

Notice that (14) can be written as

$$\mathrm{E}\left\{\prod_{i=1}^{k-1}\left[\int_{a=0}^{\infty} \Phi\left(\frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi}{\sqrt{(n_0 - 1)(1/a + 1/C)}}\right) \times \mathrm{d}G_{n_0-1}(a)\right]\right\}, \tag{A1}$$

where C is a χ^2 random variable with $n_0 - 1$ degrees of freedom. Since this expected value is non-decreasing in C, the lemma may be applied to obtain

$$(\mathrm{A1}) \geq \prod_{i=1}^{k-1}\left[\int_{b=0}^{\infty}\int_{a=0}^{\infty} \Phi\left(\frac{\max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi}{\sqrt{(n_0 - 1)(1/a + 1/b)}}\right) \times \mathrm{d}G_{n_0-1}(a)\mathrm{d}G_{n_0-1}(b)\right].$$

But the term inside [ ] is just Rinott's integral for the case $k = 2$ systems, which by the proof of Proposition 3 in Rinott (1978) is equal to

$$\int_{t=-\infty}^{\infty} F_{n_0-1}\left(t + \max[\delta, \mu_{[k]} - \mu_{[i]}]/\xi\right)\mathrm{d}F_{n_0-1}(t).$$

Lower confidence bounds on $\mu_{[k]} - \mu_{[i]}$

We propose simultaneous $(1 - \alpha)100\%$ LCBs for $\mu_{[k]} - \mu_i$, for all $i \neq [k]$, of the form

$$\{\mu_{[k]} - \mu_i \geq \widehat{\mu}_B - \widehat{\mu}_i - \delta, \quad \forall i : i \neq B\}. \tag{A2}$$

Here we will prove that the event (A2) holds whenever the event $\mathscr{A}$ holds, and since by the design of our procedures $\Pr\{\mathscr{A}\} \geq 1 - \alpha$, the bounds have the desired confidence level.

Theorem 1. *The event $\mathscr{A}$ implies the event* (A2).

Proof. We know that the event $\mathscr{A}$ implies that

$$\widehat{\mu}_i - \max_{\ell \neq i} \widehat{\mu}_\ell - \delta \leq \mu_i - \max_{\ell \neq i} \mu_\ell \leq \widehat{\mu}_i - \max_{\ell \neq i} \widehat{\mu}_\ell + \delta,$$

for $i = 1, 2, \ldots, k$, which can be rewritten as

$$\max_{\ell \neq i} \widehat{\mu}_\ell - \widehat{\mu}_i + \delta \geq \max_{\ell \neq i} \mu_\ell - \mu_i \geq \max_{\ell \neq i} \widehat{\mu}_\ell - \widehat{\mu}_i - \delta. \tag{A3}$$

There are two cases to consider:

1. If $B = [k]$, then this together with (A3) implies that
$$\begin{aligned}\max_{\ell \neq i} \mu_\ell - \mu_i &= \mu_{[k]} - \mu_i, \\ &\geq \max_{\ell \neq i} \widehat{\mu}_\ell - \widehat{\mu}_i - \delta, \\ &= \widehat{\mu}_B - \widehat{\mu}_i - \delta,\end{aligned}$$
for all $i \neq [k]$ (since $B = [k]$). Therefore (A2) holds.
2. If $B \neq [k]$, the same reasoning as in Case 1 implies that
$$\mu_{[k]} - \mu_i \geq \widehat{\mu}_B - \widehat{\mu}_i - \delta,$$
for $i \neq B, [k]$. For $i = B$ we need to establish that
$$\mu_{[k]} - \mu_B \geq \widehat{\mu}_B - \widehat{\mu}_{[k]} - \delta. \tag{A4}$$
Notice that the left-hand side of (A4) is the parameter we need to bound, and the right-hand side is the lower bound that we actually use. Notice also that $\mu_{[k]} - \mu_B \geq 0$ by the definition of $[k]$, so that any non-positive lower bound will be valid. But since the event $\mathscr{A}$ holds, we are guaranteed that $\widehat{\mu}_B - \widehat{\mu}_{[k]} - (\mu_B - \mu_{[k]}) < \delta$, which implies that $\widehat{\mu}_B - \widehat{\mu}_{[k]} < \delta$ since $-(\mu_B - \mu_{[k]})$ is non-negative. Therefore, the right-hand side of (A4) is less-than-or-equal-to zero. ■

Biographies

Barry L. Nelson is a Professor in the Department of Industrial Engineering and Management Sciences at Northwestern University, and is

Director of the Master of Engineering Management Program there. His research centers on the design and analysis of computer simulation experiments on models of stochastic systems. He has published numerous papers and two books, including *Discrete-Event System Simulation* (Prentice Hall, 1995). In 1997 he received the Institute of Industrial Engineers Operations Research Division Award. Nelson has served the profession as the Simulation Area Editor of *Operations Research* and President of the INFORMS (then TIMS) College on Simulation. He has held many positions for the annual Winter Simulation Conference, including Program Chair in 1997.

Souvik Banerjee is a Graduate Student in the Department of Industrial Engineering and Management Sciences at Northwestern University. He obtained his Bachelors degree in Industrial Engineering from the Indian Institute of Technology, India and his Masters degree in Industrial Engineering and Operations Research from the University of Massachusetts at Amherst. His research interests include strategic resource allocation and agile workforce systems. He has also conducted research on global optimization and has written commercial software for automatic code generation for system dynamics modeling. He is currently a student member of INFORMS.