Finding 2d ham sandwich cuts in linear time *

Benjamin Armbruster†

March 2, 2008

Abstract

A *ham sandwich cut* in d dimensions is a $(d-1)$-dimensional hyperplane that divides each of d objects in half. While the existence of such a hyperplane was shown in 1938, little is known about how to find one. We are the first to show how this can be done in 2 dimensions when both objects are (possibly overlapping) convex polygons. Our algorithm runs in $O(N)$ time where N is the sum of the number of sides of the two polygons. We also give a linear time algorithm for the case when the first object is a convex polygon and the second object is a finite set of points.

1 Introduction

Traditionally a *ham sandwich cut* in d dimensions is a $(d-1)$-dimensional hyperplane that divides each of d objects (i.e., sets of finite Lebesgue outer measure) in half (i.e., separates each object into two sets of equal outer measure). The corresponding *ham sandwich theorem* states that a ham sandwich cut always exists (though it may not be unique). In two dimensions this theorem is also known as the *pancake theorem*. Beyer and Zardecki (2004) relate the early history of the theorem which goes back to Steinhaus et al. (1938). A simple proof of the ham sandwich theorem is in the Wikipedia while Stone and Tukey (1942) prove a generalization. Proofs of the ham sandwich theorem do not give exact or efficient algorithms for finding a ham sandwich cut. Since the ham sandwich theorem is well known and has a long history, we believe that an exact and efficient algorithm for finding a ham sandwich cut is of interest.

*B.A. gratefully acknowledges support of an NSF Graduate Research Fellowship. The author thanks John Gunnar Carlsson, Erick Delage, and Yinyu Ye for helpful comments.

†Department of Management Science and Engineering, Stanford University, Stanford, California 94305-4026, USA

Stojmenović (1991) provides the first algorithm for finding a ham sandwich cut that we are aware of.[1] He considers two *disjoint* convex polygons in the plane with N sides in total and shows how to find a 2-dimensional ham sandwich cut in $O(N)$ time. This article extends his result by removing the requirement that the polygons must be disjoint.

Related work has shown how to find a d-dimensional ham sandwich cut when the d objects are finite sets of points. In this version of the problem, a hyperplane h is a ham sandwich cut if at most half the points in each set lie in either of the two open half-space defined by h. Lo et al. (1994) provide an algorithm to construct such ham sandwich cuts which for the case of two sets in the plane with N points in total runs in $O(N)$ time.

There is also a semi-discrete ham sandwich problem in d dimensions where only some of the d objects are finite sets of points. In that case we seek a hyperplane h dividing in half each set of points (i.e., the open half-planes defined by h each contain at most half the points) and dividing in half (in terms of volume) each of the other objects. Lemma 7 of Carlsson et al. (2007) considers the planar case where the first object is a convex polygon with m vertices and the second object is a set containing n points where n is even. It gives an algorithm that finds a ham sandwich cut in $O(N \log N)$ time where $N := m + n$. In this paper we remove the restriction that the number of points must be even and improve the running time to $O(N)$. Table 1 summarizes the previous results for the planar case and the scope of this work.

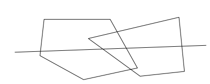
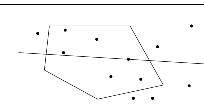
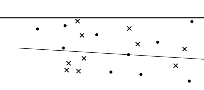
object 1	object 2	running time	note	
convex polygon	convex polygon	$O(N)$	objects disjoint; Stojmenović (1991)	
convex polygon	convex polygon	$O(N)$	this article	
convex polygon	set of points	$O(N \log N)$	even number of points; Carlsson et al. (2007)	
convex polygon	set of points	$O(N)$	this article	
set of points	set of points	$O(N)$	Lo et al. (1994)	

Table 1: Finding different types of 2-dimensional ham sandwich cuts. Here N is the sum of the number of sides or points of the two objects.

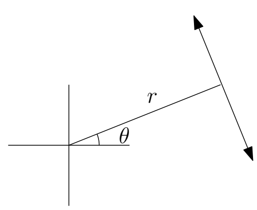
This paper focuses on the 2-dimensional case. We assume that the first object is a convex polygon and the second is either another convex polygon or a finite set of points. Convex polygons

[1] There is earlier work on the discrete version of the ham sandwich problem described below.

are specified by a list of vertices (and their locations) in counter-clockwise order. Our algorithms take $O(N)$ time to find a ham sandwich cut where N is the sum of the number of sides or points of the two objects. The only approximation in our algorithms is computing the roots of a quadratic equation. Because there are algorithms for computing square roots with quadratic convergence, the ϵ-dependency of the solution time is $\log\log\frac{1}{\epsilon}$ (which is negligible for typical floating point precisions). In the next section we describe the algorithm and sketch the proof for the simpler case of two convex polygons. In section 3 we deal with the more difficult case of a convex polygon and a set of point.

2 Two Convex Polygons

We will write points and vectors in boldface and unit vectors with hats. Define $\boldsymbol{x}\times\boldsymbol{y} := x_1y_2 - x_2y_1$ for two-dimensional vectors. Its magnitude equals the area of the parallelogram spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. We will denote the unit vector with angle θ by $\hat{\boldsymbol{\theta}} := (\cos\theta, \sin\theta)$. Many of our arguments will be in Hough space where we specify a line $\{\boldsymbol{x} : \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} = r\}$ by the angle θ of its normal and its distance from the origin r. Define the closed half-plane $H(t,\theta) := \{\boldsymbol{x} : \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} \le t\}$ and the open half-plane $H^o(t,\theta) := \{\boldsymbol{x} : \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} < t\}$.



We say a line ℓ bisects a polygon C if half the area of C is on either side of ℓ. For such a polygon, the area of $H(t,\theta)\cap C$ is 0 for very small t and strictly increasing for larger t until it equals the area of C. Hence for any angle θ, there is a unique value $r(\theta;C)$ such that the line $\{\boldsymbol{x} : \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} = r(\theta;C)\}$ bisects C. Since the area of $H(t,\theta)\cap C$ is a continuous function of t and θ, it follows that $r(\cdot;C)$ is a continuous function describing the bisectors of C. The following lemma gives a more explicit description of the bisectors when C is a convex polygon. Its proof is in section 2.1.

Lemma 1. *If C is a convex polygon with k sides, then in $O(k)$ time we can find an increasing sequence of angles $\theta_0 := 0 \le \theta_1 \le \cdots \le \theta_k < \theta_{k+1} := \pi$ and constants $(\boldsymbol{v}_i, \boldsymbol{e}_{1i}, \boldsymbol{e}_{2i}, B_i)$ for*

$i = 0, \ldots, k$ *such that for any angle* $\theta \in [\theta_i, \theta_{i+1}]$,

$$r(\theta; C) = \boldsymbol{v}_i \cdot \hat{\boldsymbol{\theta}} + \operatorname{sign}(\boldsymbol{e}_{1i} \cdot \hat{\boldsymbol{\theta}}) B_i \sqrt{(\boldsymbol{e}_{1i} \cdot \hat{\boldsymbol{\theta}})(\boldsymbol{e}_{2i} \cdot \hat{\boldsymbol{\theta}})} \tag{1}$$

We use the following lemma to simultaneously solve a pair of equations of the form (1). Its proof is in section 2.2.

Lemma 2. *Suppose* $r(\cdot; C_1)$ *and* $r(\cdot; C_2)$ *obey* (1) *with constants* $(\boldsymbol{v}_1, \boldsymbol{e}_{11}, \boldsymbol{e}_{21}, B_1)$ *and* $(\boldsymbol{v}_2, \boldsymbol{e}_{12}, \boldsymbol{e}_{22}, B_2)$ *respectively. Then in* $O(1)$ *time we can find all the ham sandwich cuts of* C_1 *and* C_2 *and for a given* $\boldsymbol{p}$*, all the ham sandwich cuts of* C_1 *and* $\{\boldsymbol{p}\}$.

We use the above two lemmas to find in the following theorem a ham sandwich cut for two convex polygons. Line ℓ is a ham sandwich cut of polygons C_1 and C_2 if it simultaneously bisects both polygons. Finding a ham sandwich cut means finding an angle θ such that $g(\theta) = 0$ where we define $g(\theta) := r(\theta; C_1) - r(\theta; C_2)$. By the intermediate value theorem, such an angle exists because $g(\cdot)$ is continuous and $g(0) = -g(\pi)$.

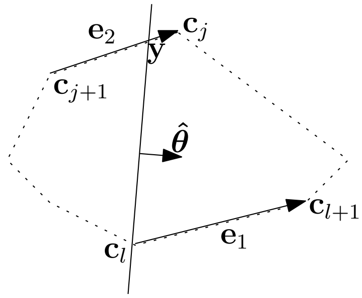
Theorem 3. *For two convex polygons* C_1 *and* C_2 *with* m *and* n *sides respectively, we can find a ham sandwich cut in* $O(N)$ *time, where* $N := m + n$.

Proof. We first apply lemma 1 to both polygons in $O(N)$ time. Then we merge the two sequences of angles into a list $\theta_0 := 0 \leq \theta_1 \leq \cdots \leq \theta_N < \theta_{N+1} := \pi$. This can be done in $O(N)$ time as the original sequences are already sorted. Now we calculate $g(\theta_i)$ for $i = 0, \ldots, N+1$ in $O(N)$ time. Since $g(\theta_0)$ and $g(\theta_{N+1})$ have opposite signs, we can find in $O(N)$ time i^* such that 0 is between $g(\theta_{i^*})$ and $g(\theta_{i^*+1})$. Let $(\boldsymbol{v}_1, \boldsymbol{e}_{11}, \boldsymbol{e}_{21}, B_1)$ and $(\boldsymbol{v}_2, \boldsymbol{e}_{12}, \boldsymbol{e}_{22}, B_2)$ be the parameters given in (1) of $r(\theta; C_1)$ and $r(\theta; C_2)$ respectively, for $\theta \in [\theta_{i^*}, \theta_{i^*+1}]$. We now apply lemma 2 to find possible ham sandwich cuts. Those whose normals have angles in $[\theta_{i^*}, \theta_{i^*+1}]$ are ham sandwich cuts (above, we have established that at least one exists). $\square$

2.1 Proving Lemma 1

Let $(\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{k-1}, \boldsymbol{c}_k := \boldsymbol{c}_0)$ be the vertices of C in counter-clockwise order. First, find in $O(k)$ time a point $\boldsymbol{y}$ on some edge $\overline{\boldsymbol{c}_j \boldsymbol{c}_{j+1}}$ such that $\overleftrightarrow{\boldsymbol{c}_0 \boldsymbol{y}}$ bisects C. Now let $i := 1$ and $l := 0$.

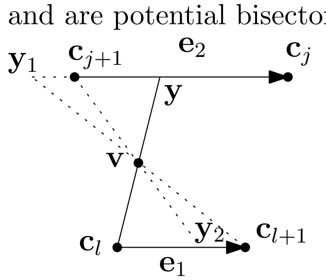
1. Let $\boldsymbol{e}_{1i} := \boldsymbol{e}_1 := \boldsymbol{c}_{l+1} - \boldsymbol{c}_l$, $\boldsymbol{e}_{2i} := \boldsymbol{e}_2 := \boldsymbol{c}_j - \boldsymbol{c}_{j+1}$, $s := \operatorname{sign}(\boldsymbol{e}_1 \times \boldsymbol{e}_2)$, and choose θ_i such that $\hat{\boldsymbol{\theta}}_i \cdot (\boldsymbol{y} - \boldsymbol{c}_l) = 0$ and $\hat{\boldsymbol{\theta}}_i \cdot \boldsymbol{e}_1 \geq 0$.



2. If $s = 0$, then

   (a) Since the edges $\overline{\boldsymbol{c}_l\boldsymbol{c}_{l+1}}$ and $\overline{\boldsymbol{c}_j\boldsymbol{c}_{j+1}}$ are parallel, any lines bisecting C and intersecting both these edges will pass through $\boldsymbol{v}_i := (\boldsymbol{c}_l + \boldsymbol{y})/2$. Hence $r = \boldsymbol{v}_i \cdot \hat{\boldsymbol{\theta}}$ for such bisectors, and (1) holds when we set $B_i := 0$.

   (b) Let $\boldsymbol{y}_1 := \boldsymbol{y} - \boldsymbol{e}_1$ and $\boldsymbol{y}_2 := \boldsymbol{c}_l + \boldsymbol{y} - \boldsymbol{c}_{j+1}$. The lines $\overleftrightarrow{\boldsymbol{c}_{l+1}\boldsymbol{y}_1}$ and $\overleftrightarrow{\boldsymbol{y}_2\boldsymbol{c}_{j+1}}$ pass through $\boldsymbol{v}$ and are potential bisectors of C.



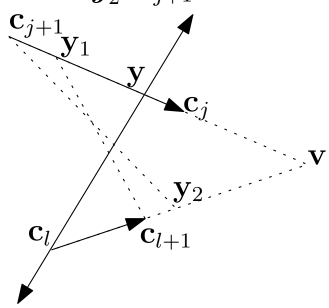
3. else,

   (a) Define $\boldsymbol{v}_i := \boldsymbol{v}$ to be intersection of $\overleftrightarrow{\boldsymbol{c}_l\boldsymbol{c}_{l+1}}$ and $\overleftrightarrow{\boldsymbol{c}_j\boldsymbol{c}_{j+1}}$. Let A be the area of $\triangle \boldsymbol{c}_l\boldsymbol{v}\boldsymbol{y}$. The line $\{\boldsymbol{x} : \boldsymbol{x} \cdot \hat{\boldsymbol{\theta}} = r\}$ is a bisector of C intersecting $\overleftrightarrow{\boldsymbol{c}_l\boldsymbol{c}_{l+1}}$ and $\overleftrightarrow{\boldsymbol{c}_j\boldsymbol{c}_{j+1}}$ if the area of the triangle formed by this line and $\overleftrightarrow{\boldsymbol{c}_l\boldsymbol{c}_{l+1}}$ and $\overleftrightarrow{\boldsymbol{c}_j\boldsymbol{c}_{j+1}}$ equals A. The line $\{\boldsymbol{x} : \boldsymbol{x} \cdot \hat{\boldsymbol{\theta}} = r\}$ intersects $\overleftrightarrow{\boldsymbol{c}_l\boldsymbol{c}_{l+1}}$ and $\overleftrightarrow{\boldsymbol{c}_j\boldsymbol{c}_{j+1}}$ at $\boldsymbol{v} + t_1\boldsymbol{e}_1$ and $\boldsymbol{v} + t_2\boldsymbol{e}_2$ respectively where $t_1 = \frac{r - \boldsymbol{v}\cdot\hat{\boldsymbol{\theta}}}{\boldsymbol{e}_1\cdot\hat{\boldsymbol{\theta}}}$ and $t_2 = \frac{r - \boldsymbol{v}\cdot\hat{\boldsymbol{\theta}}}{\boldsymbol{e}_2\cdot\hat{\boldsymbol{\theta}}}$. Since we require that $A = \frac{1}{2}\left|t_1\boldsymbol{e}_1 \times t_2\boldsymbol{e}_2\right| = \frac{1}{2}\left|\boldsymbol{e}_1 \times \boldsymbol{e}_2\right| t_1 t_2$, it follows that

$$\frac{2A}{|\boldsymbol{e}_1 \times \boldsymbol{e}_2|} = t_1 t_2 = \frac{(r - \boldsymbol{v}\cdot\hat{\boldsymbol{\theta}})^2}{(\boldsymbol{e}_1\cdot\hat{\boldsymbol{\theta}})(\boldsymbol{e}_2\cdot\hat{\boldsymbol{\theta}})}, \quad \text{and hence} \quad r = \boldsymbol{v}\cdot\hat{\boldsymbol{\theta}} + B_i\sqrt{(\boldsymbol{e}_1\cdot\hat{\boldsymbol{\theta}})(\boldsymbol{e}_2\cdot\hat{\boldsymbol{\theta}})}$$

   where $B_i := s\sqrt{\frac{2A}{|\boldsymbol{e}_1\times\boldsymbol{e}_2|}}$. The sign of B_i follows from simple geometric considerations.

(b) Let $\boldsymbol{y}_1 := \boldsymbol{v} + \boldsymbol{e}_2 sA/((\boldsymbol{c}_{l+1} - \boldsymbol{v}) \times \boldsymbol{e}_2)$ and $\boldsymbol{y}_2 := \boldsymbol{v} + \boldsymbol{e}_1 sA/(\boldsymbol{e}_1 \times (\boldsymbol{c}_{j+1} - \boldsymbol{v}))$. The lines $\overleftrightarrow{\boldsymbol{c}_{l+1}\boldsymbol{y}_1}$ and $\overleftrightarrow{\boldsymbol{y}_2\boldsymbol{c}_{j+1}}$ are potential bisectors of C because the areas of triangles $\triangle \boldsymbol{c}_{l+1}\boldsymbol{v}\boldsymbol{y}_1$ and $\triangle \boldsymbol{y}_2\boldsymbol{v}\boldsymbol{c}_{j+1}$ is A.



4. If $(\boldsymbol{y}_1 - \boldsymbol{c}_{j+1}) \cdot \boldsymbol{e}_2 \geq 0$, then $\boldsymbol{y}' := \boldsymbol{y}_1$, $l' := l+1$, and $j' := j$. Else, $\boldsymbol{y}' := \boldsymbol{y}_2$, $l' := j+1$ and $j' := l$.

5. Let $i' := i+1$, and if $i' \leq k$ then go back to step 1 using the new i', l', j', and $\boldsymbol{y}'$.

Since $\boldsymbol{c}_l$ is a different vertex every iteration, we will have characterized all the bisectors after k iterations and arrived at $\boldsymbol{c}_0$ again. Now take the angles mod π (the $\mathrm{sign}(\boldsymbol{e}_{1e} \cdot \hat{\boldsymbol{\theta}})$ factor in (1) compensates for this). Then find the smallest angle, and renumber the angles and constants starting with it. Finally, we explicitly evaluate angle $\theta_0 = 0$ using (1) with $(\boldsymbol{v}_k, \boldsymbol{e}_{1k}, \boldsymbol{e}_{2k}, B_k)$. This algorithm performs k iterations and hence runs in $O(k)$ time.

2.2 Proving lemma 2

Although we can solve the equations for θ^* (using Newton's method) we can find the ham sandwich cuts directly using geometric considerations.

The line through $\boldsymbol{v}_1 + t\boldsymbol{e}_{11}$, $\boldsymbol{v}_1 + t'\boldsymbol{e}_{21}$), and $\boldsymbol{p}$ is a ham sandwich cut of C_1 and $\{\boldsymbol{p}\}$ if these three points are collinear (i.e., $(\boldsymbol{v}_1 + t\boldsymbol{e}_{11} - \boldsymbol{p}) \times (\boldsymbol{v}_1 + t'\boldsymbol{e}_{21} - \boldsymbol{p}) = 0$), and cut-off the correct area (i.e., $tt' = B_1^2$). With some algebra we can show that the line $\overleftrightarrow{(\boldsymbol{v}_1 + t\boldsymbol{e}_{11})\boldsymbol{p}}$ is a ham sandwich cut of C_1 and $\{\boldsymbol{p}\}$ when t solves the quadratic equation $(\boldsymbol{e}_{11} \times (\boldsymbol{v}_1 - \boldsymbol{p}))t^2 + B_1^2(\boldsymbol{e}_{11} \times \boldsymbol{e}_{21})t + B_1^2(\boldsymbol{v}_1 - \boldsymbol{p}) \times \boldsymbol{e}_{21} = 0$.

Similarly, the line through $\boldsymbol{v}_1 + t_1\boldsymbol{e}_{11}$, $\boldsymbol{v}_1 + t_1'\boldsymbol{e}_{21}$, $\boldsymbol{v}_2 + t_2\boldsymbol{e}_{12}$, and $\boldsymbol{v}_2 + t_2'\boldsymbol{e}_{22}$ is a ham sandwich cut of C_1 and C_2 if these points are collinear and cut-off the correct area (i.e., $t_1 t_1' = B_1^2$ and $t_2 t_2' = B_2^2$). With some algebra we can show that the line $\overleftrightarrow{\boldsymbol{u}_1\boldsymbol{u}_2}$ is a ham sandwich cut of C_1 and

C_2 where $\boldsymbol{u}_1 := \boldsymbol{v}_1 + t_1\boldsymbol{e}_{11}$, $\boldsymbol{u}_2 := \boldsymbol{v}_2 + t_2\boldsymbol{e}_{22}$, t_1^2 solves the quadratic equation

$$\begin{aligned}
&B_2^2(\boldsymbol{e}_{11}\times\boldsymbol{e}_{12})(\boldsymbol{e}_{11}\times\boldsymbol{e}_{21})t_1^4 \\
&\qquad - B_1^2((\boldsymbol{e}_{11}\times\boldsymbol{e}_{21})^2 + B_2^2((\boldsymbol{e}_{11}\times\boldsymbol{e}_{12})(\boldsymbol{e}_{12}\times\boldsymbol{e}_{22}) + (\boldsymbol{e}_{21}\times\boldsymbol{e}_{12})(\boldsymbol{e}_{11}\times\boldsymbol{e}_{21})))t_1^2 \\
&\qquad\qquad + B_2^2(\boldsymbol{e}_{21}\times\boldsymbol{e}_{12})(\boldsymbol{e}_{21}\times\boldsymbol{e}_{22}) = 0,
\end{aligned}$$

and $t_2 := \frac{t_1 B_1^2(\boldsymbol{e}_{21}\times\boldsymbol{e}_{11})}{B_1^2(\boldsymbol{e}_{21}\times\boldsymbol{e}_{12}) - t_1(\boldsymbol{e}_{11}\times\boldsymbol{e}_{12})}$.

3 A Convex Polygon and a Set of Points

Now consider the more complicated case of a convex polygon C with m sides and a set of n points P. Again we define $N := m + n$. We say that a line ℓ bisects P if at most $|P|/2$ points lie in each of the open half-planes on either side.

So when n is odd, then any line bisecting P will intersect some point in P (because $n/2$ is not an integer), and will also bisect $P \cup \{\boldsymbol{x}\}$ (the set where we add some point $\boldsymbol{x}$ to P). Hence we will assume that the number of points n is odd (as otherwise we may just remove some point of P). Hence for any angle θ, there is a unique value $r(\theta; P)$ such that the line $\{\boldsymbol{x} : \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} = r(\theta; P)\}$ bisects P.

The set of lines going through some point $\boldsymbol{x} := (x_1, x_2)$, the Hough transform of $\boldsymbol{x}$, traces out a sinusoid in angle-distance (Hough) space: $r(\theta; \{\boldsymbol{x}\}) = \boldsymbol{x}\cdot\hat{\boldsymbol{\theta}} := x_1\cos\theta + x_2\sin\theta$. Let $L(\theta; q, P)$ be the q-smallest element of $\{\boldsymbol{p}\cdot\hat{\boldsymbol{\theta}} : \boldsymbol{p} \in P\}$. Due to the continuity of $r(\cdot; \{\boldsymbol{x}\})$, it follows that $L(\cdot; q, P)$ is continuous and composed of sinusoidal pieces with kinks at some angles in $\Theta_P := \{\theta : \boldsymbol{p}_1, \boldsymbol{p}_2 \in P,\ \boldsymbol{p}_1\cdot\hat{\boldsymbol{\theta}} = \boldsymbol{p}_2\cdot\hat{\boldsymbol{\theta}}\}$.

Line ℓ is a ham sandwich cut of C and P if it simultaneously bisects both. Finding a ham sandwich cut means finding an angle θ such that $g(\theta) = 0$ where we define $g(\theta) := r(\theta; C) - r(\theta; P)$. By the intermediate value theorem, such an angle exists because $g(\cdot)$ is continuous and $g(0) = -g(\pi)$. The following lemma gives a simple (but not particularly fast) way of finding a ham sandwich cut when we take $\alpha = 0$, $\beta = \pi$, and $q = \frac{n+1}{2}$.

Lemma 4. *Consider an interval $[\alpha, \beta] \subseteq [0, \pi]$ with $L(\alpha; q, P) - r(\alpha; C)$ and $L(\beta; q, P) - r(\beta; C)$ having opposite signs. In $O(N^3)$ time, we can find $\theta^* \in [\alpha, \beta]$ such that $L(\theta^*; q, P) = r(\theta^*; C)$.*

Proof. Our approach is to find a small interval containing θ^* in which $L(\cdot; q, P)$ and $r(\cdot; C)$ are analytic. First we find Θ_P in $O(n^2)$ time. Then we use lemma 1 to find a set of angles Θ_C in $O(m)$ time. Combining these sets we create $\Theta := (\Theta_P \cup \Theta_C \cup \{\alpha, \beta\}) \cap [\alpha, \beta]$ and sort it in $O(N^2 \log N)$ time. Then we perform a binary search in $O(N \log N)$ time to find an adjacent pair of angles $\theta_1, \theta_2 \in \Theta$ such that $L(\theta_1; q, P) - r(\theta_1; C)$ and $L(\theta_2; q, P) - r(\theta_2; C)$ have opposite signs (we can evaluate $L(\cdot; q, P)$ in $O(n)$ time using a selection algorithm). Since these angles are adjacent, there is $p \in P$ so that for $\theta \in [\theta_1, \theta_2]$, $L(\cdot; q, P)$ is a simple sinusoid with $L(\theta; q, P) = \boldsymbol{p} \cdot \hat{\boldsymbol{\theta}}$ and $r(\theta; C)$ has the analytic form given in (1) with parameters $(\boldsymbol{v}, \boldsymbol{e}_1, \boldsymbol{e}_2, B)$. We now apply lemma 2 to find possible ham sandwich cuts. Those whose normals have angles in $[\theta_1, \theta_2]$ are ham sandwich cuts (above, we have established that at least one exists). □

Our approach to quickly finding a ham sandwich cut is to reduce in $O(N)$ time the problem to one of $O(1)$ size and then apply the above lemma. Our reduction is similar to that of Lo et al. (1994) and assumes that the points P are in general position.

Let $\theta_0 := 0 \leq \theta_1 \leq \cdots \leq \theta_m < \theta_{m+1} := \pi$ be the sequence of angles obtained by applying lemma 1 to C. We say angles $\alpha, \beta \in [0, \pi]$, integers i, k, q, and points $Q \subseteq P$ obey *the invariant* if

1. $\theta_i \leq \alpha \leq \beta \leq \theta_{i+k}$,
2. $g(\alpha) g(\beta) \leq 0$, and
3. $r(\theta; P) = L(\theta; q, Q)$ for all $\theta \in [\alpha, \beta]$.

The following lemma is analogous to lemma 3.2 of Lo et al. (1994).

Lemma 5. *Suppose $\alpha, \beta \in [0, \pi]$, $i, k, q \in \mathbb{Z}$, and $Q \subseteq P$ obey the invariant. Then if $|Q| + k \geq 17$ we can compute in $O(|Q| + k)$ time $\alpha', \beta' \in [0, \pi]$, $i', k', q' \in \mathbb{Z}$, and $Q' \subseteq P$ obeying the invariant such that $|Q'| + k' \leq \lceil 0.75(|Q| + k) \rceil$.*

Theorem 6. *We can find a ham sandwich cut of C and P in $O(N)$ time.*

Algorithm First, we apply lemma 1 to C and obtain a sequence of angles $\theta_0 := 0 \leq \theta_1 \leq \cdots \leq \theta_m < \theta_{m+1} := \pi$. We initially choose $\alpha := 0$, $\beta := \pi$, $i := 0$, $k := m + 1$, $q := \frac{n+1}{2}$, and $Q := P$. This satisfies the invariant because $g(0) = -g(\pi)$.

1. We compute $g(\alpha)$ and $g(\beta)$. If either equals 0, then we are done.

2. If $|Q| + k < 17$, then we finish by applying lemma 4.

3. Otherwise we apply lemma 5 to find new $(\alpha, \beta, i, k, q, Q)$ and go back to step 1.

Running Time The application of lemma 1 takes $O(m)$ time. For $j \geq 17$, it holds that $\lceil 0.75j \rceil \leq (13/17)j$. Hence the jth application of lemma 5 in our algorithm costs $(13/17)^{j-1}(m+n+1)O(1)$, and the cost of all the applications of the lemma is $(m+n+1)O(1)\sum_{j=0}^{\infty}(13/17)^j = O(m+n)$. Adding the $O(17^3)$ cost of lemma 4 we have a total cost of $O(N)$ for our algorithm.

Extension The approach we use to prove theorem 6 extends cleanly to the case of a weighted set of points where each point $p_i \in P$ has a positive weight w_i, normalized so that $\sum_{i=1}^{n} w_i = 1$. In that case, we say that a line ℓ bisects P if the points lying strictly on one side of the line have a total weight of at most 0.5 (and the same being true for the points lying strictly on the other side of the line). The only change we need to make is to replace q by a threshold weight w and evaluate $L(\cdot; w, Q)$ in $O(|Q|)$ time using a *weighted* selection algorithm.

3.1 Proving lemma 5

Suppose that $k \geq |Q|$. We aim to halve k. Choose $Q' := Q$ and $q' := q$. We then evaluate $r(\theta_{i+\lceil k/2 \rceil}; C)$ in $O(1)$ time and $r(\theta_{i+\lceil k/2 \rceil}; P) = L(\theta_{i+\lceil k/2 \rceil}; q, Q)$ in $O(|Q|)$ time. If $g(\alpha)g(\theta_{i+\lceil k/2 \rceil}) \leq 0$, then choose $\alpha' := \alpha$, $\beta' := \theta_{i+\lceil k/2 \rceil}$, $i' := i$, and $k' := \lceil k/2 \rceil$. Otherwise $g(\theta_{i+\lceil k/2 \rceil})g(\beta) \leq 0$ and we choose $\alpha' := \theta_{i+\lceil k/2 \rceil}$, $\beta' := \beta$, $i' := i + \lceil k/2 \rceil$, and $k' := \lfloor k/2 \rfloor$. Note that

$$k' + \left|Q'\right| \leq \lceil k/2 \rceil + |Q| = \lceil k/2 + 0.25\,|Q| + 0.75\,|Q| \rceil \leq \lceil 0.75(k + |Q|) \rceil .$$

Suppose on the other hand that $k < |Q|$. By assumption, $|Q| \geq 9$. Then we aim to halve the number of points in Q. Our approach uses the following lemma to choose α' and β' closer together; uses these angles to construct a set $\mathcal{M}$; discards the points not lying in it, $Q' := Q \cap \mathcal{M}$; shows that $|Q'| \leq \lceil |Q|/2 \rceil$; and then picks a q' such that we continue to satisfy the invariant.

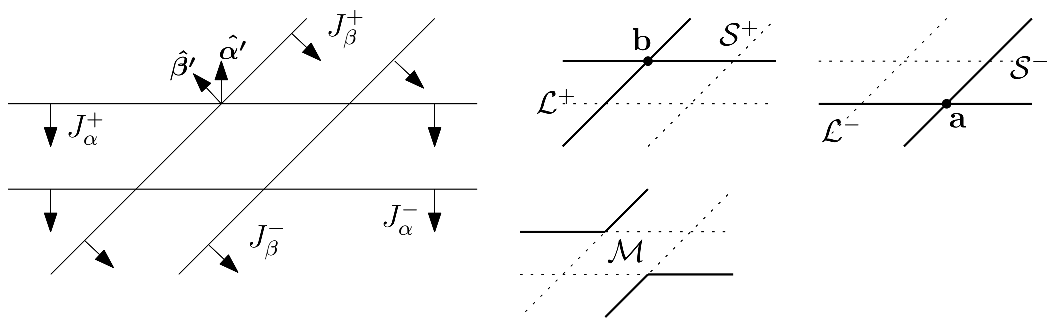
Choosing $\delta := 1/32$ we use lemma 7 to divide the possible angles into 64 intervals. Here we use the assumption that the points P are in general position. Lemma 3.3 from Lo et al. (1994) gives a proof of this lemma (after interchanging the terms *point* and *line*).

Lemma 7. *Consider a fixed constant $\delta < 1$ and a set of points $Q \subseteq P$ with $\delta\binom{|Q|}{2} \geq 1$. Then in $O(|Q|)$ time, we can find angles $\varphi_0 := 0 \leq \varphi_1 \leq \cdots \leq \varphi_{\lfloor 2/\delta \rfloor} := \pi$ so that for all j, (i) $\varphi_j \notin \Theta_P$, and (ii) $|\Theta_P \cap [\varphi_j, \varphi_{j+1}]| \leq \delta\binom{|Q|}{2}$.*

We evaluate $r(\cdot; C)$ and $L(\cdot; q, Q)$ at angles $\{\varphi_j\} \cup \{\alpha, \beta\}$ in $O(k)$ and $O(|Q|)$ time respectively. We then find adjacent angles α' and β' such that $g(\alpha')g(\beta') \leq 0$. Choose i' and k' such that $\theta_i \leq \theta_{i'} \leq \alpha' \leq \beta' \leq \theta_{i'+k'} \leq \theta_{i+k}$.

Let $\epsilon := 1/8$. Now define the closed half-planes, $J^+_\alpha := H(\alpha', L(\alpha'; q + \lfloor \epsilon\, |Q| \rfloor, Q))$, $J^-_\alpha := H(\alpha', L(\alpha'; q - \lceil \epsilon\, |Q| \rceil, Q))$, $J^+_\beta := H(\beta', L(\beta'; q + \lfloor \epsilon\, |Q| \rfloor, Q))$, and $J^-_\beta := H(\beta', L(\beta'; q - \lceil \epsilon\, |Q| \rceil, Q))$. Let $\mathcal{L}^+ := J^+_\alpha \setminus J^+_\beta$, $\mathcal{S}^+ := J^+_\beta \setminus J^+_\alpha$, $\mathcal{L}^- := J^-_\alpha \setminus J^-_\beta$, and $\mathcal{S}^- := J^-_\beta \setminus J^-_\alpha$. Also define $\mathcal{M} := \mathcal{L}^+ \cup (J^+_\beta \setminus J^-_\beta) \cup \mathcal{S}^-$ and $Q' := Q \cap \mathcal{M}$. Hence

$$\left|Q'\right| \leq \left|Q \cap \mathcal{L}^+\right| + \left|Q \cap (J^+_\beta \setminus J^-_\beta)\right| + \left|Q \cap \mathcal{S}^-\right|.$$



By lemma 7, $\alpha', \beta' \notin \Theta_P$. Hence there is only one point each on the boundary of J^+_α, J^-_α, J^+_β, and J^-_β. So $|Q \cap J^+_\alpha| = \left|Q \cap J^+_\beta\right| = q + \lfloor \epsilon\, |Q| \rfloor$ and $|Q \cap J^-_\alpha| = \left|Q \cap J^-_\beta\right| = q - \lceil \epsilon\, |Q| \rceil$. Since $\mathcal{L}^+ \cup J^+_\beta = \mathcal{S}^+ \cup J^+_\alpha$, it follows that $|Q \cap \mathcal{L}^+| = |Q \cap \mathcal{S}^+|$. Similarly, $|Q \cap \mathcal{L}^-| = |Q \cap \mathcal{S}^-|$. Note that the normal to any line $\overleftrightarrow{p_1 p_2}$ where $p_1 \in \mathcal{L}^+$ and $p_2 \in \mathcal{S}^+$ has an angle in (l', u'). Hence by lemma 7, $|Q \cap \mathcal{L}^+| \cdot |Q \cap \mathcal{S}^+| \leq \delta\binom{|Q|}{2} < \delta\frac{|Q|^2}{2}$ (as $|Q| \geq 9$). As $|Q \cap \mathcal{L}^+| = |Q \cap \mathcal{S}^+|$, it follows that $|Q \cap \mathcal{L}^+| < |Q|\sqrt{\delta/2}$. Similarly, $|Q \cap \mathcal{S}^-| < |Q|\sqrt{\delta/2}$. Further, $J^-_\beta \subseteq J^+_\beta$, and hence $\left|Q \cap (J^+_\beta \setminus J^-_\beta)\right| = \lfloor \epsilon\, |Q| \rfloor + \lceil \epsilon\, |Q| \rceil$. So

$$\left|Q'\right| < 2\,|Q|\sqrt{\delta/2} + \lfloor \epsilon\, |Q| \rfloor + \lceil \epsilon\, |Q| \rceil = \frac{|Q|}{4} + \left\lfloor \frac{|Q|}{8} \right\rfloor + \left\lceil \frac{|Q|}{8} \right\rceil$$

by our choice of δ and ϵ. Hence

$$\left|Q'\right| \leq \lceil |Q|/2 \rceil .$$

Choose $q' = q - \left|Q \cap J_\alpha^- \cap J_\beta^-\right|$.

Now we need to show that it satisfies the invariant, $L(\theta; q, Q) = L(\theta; q', Q')$ for $\theta \in [l', u']$. The definition of L states that $t = L(\theta; q, Q)$ iff $|H^o(t, \theta) \cap Q| < q \leq |H(t, \theta) \cap Q|$. So we aim to show that

$$\left|H^o(L(\theta; q, Q), \theta) \cap Q'\right| < q' \leq \left|H(L(\theta; q, Q), \theta) \cap Q'\right| .$$

Note that $\mathcal{M} = (J_\alpha^+ \cup J_\beta^+) \setminus (J_\alpha^- \cap J_\beta^-)$ where $J_\alpha^- \cap J_\beta^- \subseteq J_\alpha^+ \cup J_\beta^+$. So for any set X,

$$\left|X \cap Q'\right| = |X \cap Q \cap \mathcal{M}| = \left|X \cap Q \cap (J_\alpha^+ \cup J_\beta^+)\right| - \left|X \cap Q \cap J_\alpha^- \cap J_\beta^-\right| .$$

We will later show that

$$J_\alpha^- \cap J_\beta^- \subseteq H^o(L(\theta; q, Q), \theta) \subseteq H(L(\theta; q, Q), \theta) \subseteq J_\alpha^+ \cup J_\beta^+ . \tag{2}$$

Hence

$$\left|H^o(L(\theta; q, Q), \theta) \cap Q'\right| = |H^o(L(\theta; q, Q), \theta) \cap Q| - \left|J_\alpha^- \cap J_\beta^- \cap Q\right| ,$$

and similarly,

$$\left|H(L(\theta; q, Q), \theta) \cap Q'\right| = |H(L(\theta; q, Q), \theta) \cap Q| - \left|J_\alpha^- \cap J_\beta^- \cap Q\right| ,$$

proving our claim:

$$\left|H^o(L(\theta; q, Q), \theta) \cap Q'\right| < q - \left|J_\alpha^- \cap J_\beta^- \cap Q\right| = q' \leq \left|H(L(\theta; q, Q), \theta) \cap Q'\right| .$$

Now we prove (2). Because $\epsilon = \sqrt{\delta/2}$ it follows that

$$\begin{aligned} \left|(J_\alpha^- \cup \mathcal{S}^-) \cap Q\right| &\leq \left|J_\alpha^- \cap Q\right| + \left|\mathcal{S}^- \cap Q\right| < q - \lceil \epsilon |Q| \rceil + \sqrt{\delta/2}\, |Q| = q - \lceil \epsilon |Q| \rceil + \epsilon |Q| < q \\ &= q + \lfloor \epsilon |Q| \rfloor - \left\lfloor \sqrt{\delta/2}\, |Q| \right\rfloor \leq \left|J_\alpha^+ \cap Q\right| - \left|\mathcal{L}^+ \cap Q\right| \leq \left|(J_\alpha^+ \setminus \mathcal{L}^+) \cap Q\right| . \end{aligned} \tag{3}$$

That is, $|(J_\alpha^- \cup \mathcal{S}^-) \cap Q| < q \le |(J_\alpha^+ \setminus \mathcal{L}^+) \cap Q|$. Let $\boldsymbol{a}$ be the corner of $J_\alpha^- \cap J_\beta^-$ and $\boldsymbol{b}$ the corner of $J_\alpha^+ \cap J_\beta^+$. Since $H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta) \subseteq J_\alpha^- \cup \mathcal{S}^-$ and $J_\alpha^+ \setminus \mathcal{L}^+ \subseteq H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta)$ it follows that

$$\left|Q \cap H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta)\right| < q \le \left|H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta) \cap Q\right|.$$

Then from the definition of $L(\theta; q, Q)$,

$$\left|Q \cap H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta)\right| < |H(L(\theta; q, Q), \theta) \cap Q| \quad \text{and} \quad |H^o(L(\theta; q, Q), \theta) \cap Q| < \left|H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta) \cap Q\right|,$$

$$H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta) \subset H(L(\theta; q, Q), \theta) \quad \text{and} \quad H^o(L(\theta; q, Q), \theta) \subset H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta).$$

Since $H^o(s, \theta) \subset H(s, \theta) \subset H^o(t, \theta) \subset H(t, \theta)$ iff $s < t$, it follows that

$$H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta) \subset H^o(L(\theta; q, Q), \theta) \subset H(L(\theta; q, Q), \theta) \subseteq H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta).$$

Noting that $J_\alpha^- \cap J_\beta^- \subseteq H(\boldsymbol{a} \cdot \hat{\boldsymbol{\theta}}, \theta)$ and $H(\boldsymbol{b} \cdot \hat{\boldsymbol{\theta}}, \theta) \subseteq J_\alpha^+ \cup J_\beta^+$ then proves (2):

$$J_\alpha^- \cap J_\beta^- \subseteq H^o(L(\theta; q, Q), \theta) \subseteq H(L(\theta; q, Q), \theta) \subseteq J_\alpha^+ \cup J_\beta^+.$$

References

W. A. Beyer and A. Zardecki. The early history of the ham sandwich theorem. *Amer. Math. Monthly*, 111(1):58–61, 2004. URL http://www.findarticles.com/p/articles/mi_qa3742/is_200401/ai_n9371126.

J. G. Carlsson, B. Armbruster, and Y. Ye. Finding equitable convex partitions of points in a polygon efficiently, 2007. URL http://www.stanford.edu/~barmbrus/regionpartition.pdf. Under review.

C.-Y. Lo, J. Matoušek, and W. Steiger. Algorithms for ham-sandwich cuts. *Discrete Comput. Geom.*, 11(4):433–452, 1994.

H. Steinhaus et al. A note on the ham sandwich theorem. *Mathesis Polska*, 11:26–28, 1938. Translated in Beyer and Zardecki (2004).

I. Stojmenović. Bisections and ham-sandwich cuts of convex polygons and polyhedra. *Information Processing Letters*, 38:15–21, 1991. URL http://www.site.uottawa.ca/~ivan/F17-bisections.PDF.

A. H. Stone and J. W. Tukey. Generalized "sandwich" theorems. *Duke Math. J.*, 9:356–359, 1942. URL http://www.ams.org/mathscinet-getitem?mr=MR0007036.

Wikipedia. Ham sandwich theorem. 2006. URL http://en.wikipedia.org/w/index.php?title=Ham_sandwich_theorem&oldid=78163874.