Lecture 3: Annuities

Last class, we were particularly interested in a stream of equal cashflows occurring at equal intervals: \$11,268 of Social Security benefits every year, from age 65 to 79.

This pattern often arises: in loans, bonds, mortgages, and some business projects. It's called an *annuity*.

There is a useful formula (and Excel function) for computing the present value of an annuity.

Warning! As we've said, "present value" is a relative concept: relative to the point in time that serves as the "present." Later cashflows are discounted back to this time.

The formula we'll use gives the present value of the annuity as of 1 period before the first cashflow of the annuity, whose cashflows are all 1 period apart.

This is the standard convention for taking the present value of annuity because of their role in financial securities.

Suppose when you retire you want to use your savings to buy an annuity that will provide you with equal monthly income for the next 30 years. There would be no point in paying for the annuity and receiving some of your money back right away.

Likewise, suppose a business wanted to raise money by selling a financial security and paying the buyer equal quarterly payments for the next 10 years. Again, it wouldn't be convenient to raise money and then pay some of it out right away.

Where C is the cashflow each period, r is the discount rate per period, and there are n payments (all one period apart), the present value is

$$X = \frac{C}{r}\left(1 - \left(\frac{1}{1+r}\right)^n\right).$$

This formula is derived in Section 3.2 of Luenberger.

This formula can be evaluated in Excel using `pv(r,n,-C)`. Excel's convention is that the third argument is a payment, which is why we use $-C$. This relates to a cashflow diagram for a loan: if you borrow X now (a positive cashflow of X) then your n future payments will be C (a cashflow of $-C$ representing money you must pay out).

Go to the Proposal page of the Social Security spreadsheet, and type `pv(growth, nyears, -yearlyincome)`. The result is \$135,499, the same as we computed in cell `C67` as the PV at age 64 of the 15-year annuity paying \$11,268 per year.

If we want the PV of the annuity at age 65, at the time of the first payment, it must be $Y = X(1 + r)$. This is because

- Having Y at age 65 is the same as getting \$11,268 per year from ages 65 through 79, and
- having Y at age 65 is the same as having $X = Y/(1+r) = Y(1+r)^{64-65}$ at age 64 (the basic formula for present value).

So the PV of Social Security's obligation to a 65-year-old who has just retired and is going to live 15 years is $\$139,428 = \$135,499 \times 1.029$, the same as we computed in cell `C68`.

We can also compute this in Excel via `pv(growth, nyears, -yearlyincome, 0, 1)` where the last argument being 1 means that the payments begin right away, not after 1 period.

People often comment that changing demographics have greatly increased the burden on the Social Security system since its founding in 1935, so that it is no longer sustainable in its present form.[1] Let's investigate the impact of life expectancy. Nowadays the average 65-year-old man (not woman) has a life expectancy of 15 years; back then, it was more like 13. If we replace `nyears` with 13, we get \$124,106 (the same as in cell `C70`, where the remaining life expectancy is 13 more years).

This is not a huge difference. The more important factors affecting the burden are:

- More workers are living to retirement age. Men who turned 21 around 1900 had only slightly better than even odds of reaching age 65; nowadays it's well over 70%, maybe 80%.
- A decreasing birth rate has also caused the "population pyramid" to become more like a "population house," i.e. a rectangle with a pyramid on top. See `http://www.nationmaster.com/country/us/Age_distribution`.

Both of these (as well as increasing remaining life expectancy at retirement) contribute a lot to the increasing number of retirees per worker (see Projections sheet: right now each worker also supports 0.24 of a retiree; in 2050 it's projected to be 0.41 of a retiree).

Observe the non-linearity in the PV of annuity vs. its duration: increasing the duration by 15%, from 13 to 15 years, increases the PV only by 9%.

The two extra years further in the future don't count for as much now. Indeed, the distant future hardly counts for anything—to such an extent that an infinitely long annuity (called a *perpetuity*) has a finite PV! It is given by the formula

$$X = \frac{C}{r}$$

[1] Source for the following numbers: `http://www.ssa.gov/history/lifeexpect.html`.

for a perpetuity of C per period, starting in 1 period, when the discount rate is r per period.

This is also derived in Section 3.2 of Luenberger. It makes a lot of sense if you think of it this way: if you have X now and you can earn interest at a rate of r per period, then each period you can consume the interest $C = Xr$ and be left with the principal X intact to provide more interest in the remaining periods.

EXAMPLE: The immortal millionaire. Suppose you have \$1 million in the bank and want to spend an equal amount every year forever (in inflation-adjusted terms). You will withdraw a constant amount yearly, starting now (not 1 year from now). How much can you withdraw, assuming a real interest rate of 2.9% per year forever?

ANSWER: The PV of this perpetuity whose 1st payment is now is $X = (1 + r)C/r$. We know r=2.9% and want to find C such that X is \$1 million. The answer is $C = Xr/(1+r) = \$28,183$.