Equilibrium in Prediction Markets with Buyers and Sellers

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Abstract

Prediction markets with buyers and sellers of contracts on multiple outcomes are shown to have unique equilibrium prices, which can be computed in polynomial time.

1 Introduction

Prediction markets are increasingly seen as efficient belief-aggregation devices. In these markets, traders buy and sell contracts on possible outcomes of some "experiment," e.g., sports competition or election. A contract will pay $1 if the actual outcome satisfies the conditions specified in the contract. Thus, if the subjective beliefs of the traders give rise to unique market prices, then these prices may be interpreted as the market-generated probabilities of outcomes. Prediction markets have grown in popularity as there is increasing empirical evidence that predictions from these markets are at least as accurate, and often outperform traditional polls [2]. However, the existing theory does not suffice to support this practice. A market with multiple risk-neutral buyers and one organizer that acts as a seller ("parimutuel betting markets") was analyzed by Eisenberg and Gale [3]. They showed that the equilibrium price of a contract is equal to the (budget-weighted) fraction of traders who believe that the respective outcome is most likely. However, for a general exchange market, such an analysis exists only for the simpler case of binary outcomes [4, 5]. Here, the general prediction market with buyers and sellers is reduced to parimutuel betting markets. Thus, prediction markets have unique equilibrium prices, which reflect market belief in the same sense as proven by Eisenberg and Gale for parimutuel betting. The results are also extended to traders with concave utility functions.

2 Equilibrium prices in prediction markets

Let $M = \{1, \ldots, m\}$ be the set of all the *outcomes* of an "experiment," e.g., sports competition or election. Let $N = \{1, \ldots, n\}$ the set of *traders* who buy and sell betting contracts. For $j \in M$, a single contract C_j entitles the buyer to receive from the seller one dollar when the outcome is j. It is important to note that in our model the contracts can be traded in fractions. This assumption is necessary for proving existence of an equilibrium like in many other economic models. A buyer must pay the current market price of a contract. A seller of a set of contracts must deposit the worst-case amount he may have to pay on the contracts he sells. Denote the market prices by $\pi_1, \ldots, \pi_m$. If trader i buys b_{ij} C_j-contracts, he must pay $\sum_j \pi_j b_{ij}$. If i sells s_{ij} C_j-contracts, he receives $\sum_j \pi_j s_{ij}$, but must deposit $\max_j\{s_{ij}\}$.

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Each i has an initial budget $m_i > 0$ and a subjective probability distribution $\boldsymbol{p}^i = (p_{i1}, \ldots, p_{im})$, i.e, $\sum_j p_{ij} = 1$ and $p_{ij} \geq 0$, $j \in M$. We first assume the traders are risk-neutral, i.e, trader i's utility is equal to his subjective expectation of the amount of money he will have after the experiment has been performed and the obligations are have been settled. A generalization to risk-averse traders is discussed in Section 5. Thus, trader i wishes to solve the following problem:

$$
(P_i) \qquad \begin{array}{rl} \text{Maximize} & \displaystyle\sum_j (p_{ij} - \pi_j)\,(b_{ij} - s_{ij}) \\ \text{subject to} & \displaystyle\sum_j \pi_j\,(b_{ij} - s_{ij}) + \max_j\{s_{ij}\} \leq m_i \qquad (i \in N) \\ & b_{ij},\ s_{ij} \geq 0 \qquad\qquad (j \in M)\ . \end{array}
$$

We call $\pi_1, \ldots, \pi_m$ *equilibrium prices* if there exist b_{ij}s and s_{ij}s that solve the individual optimization problems (P_i), such that the market clears, i.e., for every $j \in M$:

$$
\sum_i b_{ij} = \sum_i s_{ij}\ . \tag{1}
$$

For prices to represent probabilities, they must sum to 1. Indeed,

Proposition 1. *If $\pi_1, \ldots, \pi_n$ are equilibrium prices, then $\sum_{j \in M} \pi_j = 1$.*

Proof. If i sells one C_j-contract, then from i's point of view, his final wealth will be $m_i + \pi_j - 1$ with probability p_{ij} and $m_i + \pi_j$ with probability $1 - p_{ij}$. On the other hand, if i buys one C_k-contract for each $k \neq j$, then from i's point of view, his final wealth will be $m_i - \sum_{k \neq j} \pi_k$ with probability p_{ij} and $m_i - \sum_{k \neq j} \pi_k + 1$ with probability $1 - p_{ij}$.

If $\sum_{k \in M} \pi_k < 1$, then both

$$
m_i + \pi_j - 1 < m_i - \sum_{k \neq j} \pi_k
$$

and

$$
m_i + \pi_j < m_i - \sum_{k \neq j} \pi_k + 1\ .
$$

It follows that i would not sell C_j. This conclusion holds for every $i \in N$ and $j \in M$ and, therefore, in this case there would be no selling and no buying. However, this can be an equilibrium only if $p_{ij} = \pi_j$ for every $i \in N$ and $j \in M$, which implies $\sum_j \pi_j = 1$.

Suppose $\sum_{j \in M} \pi_j > 1$. If i sells x C_j-contracts for each $j \in M$, then i receives $x \cdot \sum_{j \in M} \pi_j$, which is more than the required deposit x, so every value of x is feasible. The final wealth of i is equal to $m_i + x \cdot \sum_{j \in M} \pi_j - x$, which can be arbitrarily large, and hence not in equilibrium. $\square$

Note that the proof of Proposition 1 holds for nonlinear utility functions. The idea of the proof is also instrumental in proving equivalence of equilibrium prices in prediction markets to equilibrium prices in parimutuel betting markets. The latter was studied in [3].

3 Equivalence to Parimutuel Betting

In parimutuel betting, each trader acts only as a buyer. Thus, the optimization problem for trader i is:

$$\begin{aligned} \text{Maximize } & \sum_j (p_{ij} - \pi_j)\, b_{ij} \\ \text{subject to } & \sum_j \pi_j\, b_{ij} \le m_i \\ & b_{ij} \ge 0 \quad (j \in M)\ . \end{aligned} \tag{2}$$

Prices $\pi_1, \ldots, \pi_m$ are *parimutuel betting equilibrium prices*(PBEP) if there exist b_{ij}s that solve the individual optimization problems (2), respectively, so that under every outcome j, the buyers are exactly paid off by the total money collected, i.e., for every j,

$$\sum_i b_{ij} = \sum_{i,k} \pi_k b_{ik} \tag{3}$$

Proposition 2. *If $\pi_1, \ldots, \pi_m$ are PBEP, then $\sum_j \pi_j = 1$.*

Proof. By definition, at equilibrium,

$$\sum_i m_i = \sum_i \sum_j \pi_j b_{ij} = \sum_j \pi_j \sum_i b_{ij} = \sum_j \pi_j \cdot \sum_i m_i\ .$$

This implies the claim. ☐

Proposition 3. *Prices $\pi_1, \ldots, \pi_m$ are equilibrium prices in the prediction market if and only if they are PBEP in the corresponding parimutuel betting market.*

Proof. Let $\{b_{ij}\}$ be optimal purchases, satisfying (1), made by traders at equilibrium prices $\pi_1, \ldots, \pi_m$ in the parimutuel betting market. Define We show that the b'_{ij}s and s'_{ij}s satisfy the conditions required for

$$b'_{ij} = \begin{cases} b_{ij} & \text{if } j < m \\ 0 & \text{if } j = m \end{cases} \qquad s'_{ij} = \begin{cases} b_{im} & \text{if } j < m \\ 0 & \text{if } j = m\ . \end{cases}$$

$\pi_1, \ldots, \pi_m$ to be equilibrium prices in the prediction market. First, trader i's wealth under every outcome j is the same in both formulations. Trader i's wealth under $j < m$ is:

$$m_i - \sum_k \pi_j b'_{ik} + b'_{ij} - s'_{ij} = m_i - \sum_{k<m} \pi_j b_{ik} + b_{ij} - b_{im} = m_i - \sum_k \pi_j b_{ik} + b_{ij}\ ,$$

and under $j = m$,

$$m_i - \sum_k \pi_j b'_{ik} + b'_{ij} - s'_{ij} = m_i - \sum_{k<m} \pi_j b_{ik} = m_i - \sum_k \pi_j b_{ik} + b_{ij}\ .$$

Second, the amounts of money spent in both optimization problems are equal as well;

$$\sum_{k=1}^{m} \pi_k (b'_{ik} - s'_{ik}) + \max_k s'_{ik} = \sum_{k=1}^{m-1} \pi_k (b_{ik} - \bar{b}_{im}) + b_{im} = \sum_{k=1}^{m-1} \pi_k \bar{b}_{ik} + \left(1 - \sum_{k=1}^{m-1} \pi_k\right) \bar{b}_{im} = \sum_{k=1}^{m} \pi_k b_{ik}\ .$$

Therefore, each $i \in N$ is indifferent between the purchases $b_{ij}, \ldots, b_{im}$ and the transactions $b'_{i1}, \ldots, b'_{im}$ and $s'_{i1}, \ldots, s'_{im}$.

Furthermore, the b'_{ij}s and s'_{ij}s satisfy (1) because, for $j = m$,

$$\sum_i b'_{im} = \sum_i s'_{im} = 0 \ ,$$

and for every $j < m$,

$$\sum_i (b'_{ij} - s'_{ij}) = \sum_i (b_{ij} - b_{im}) = \sum_i b_{ij} - \sum_i b_{im} = 0 \ .$$

For the converse, suppose $\boldsymbol{\pi} \equiv (\pi_1, \ldots, \pi_m)$ is a vector equilibrium prices in the prediction market. Denote

$$A_i \equiv \{j \in M \ : \ \pi_j > p_{ij}\} \ .$$

First, we show that there exist b'_{ij}s and s'_{ij}s that satisfy the equilibrium conditions with respect to $\boldsymbol{\pi}$, such that for all $j \in A_i$, $s'_{ij} = s_i$, and for all $j \notin A_i$, $s'_{ij} = 0$. Suppose, on the contrary, that for some $k \in A_i$, $s'_{ik} < \max_{j \in A_i} s'_{ij}$. By increasing s'_{ik} and b'_{ik} simultaneously, the objective value and the left-hand side of (2) remain unchanged. Therefore, we can simultaneously increase s'_{ik} and b'_{ik} until $s'_{ik} = s'_i$, while maintaining (1). Similarly, if $s'_{ij} > 0$ for some $j \notin A_i$, then necessarily $b'_{ij} \geq s'_{ij}$ (refer to (2)), so we can simultaneously decrease s'_{ik} and b'_{ik} until $s'_{ik} = 0$. Therefore, w.l.o.g., $s'_{ij} = 0$ if $j \notin A_i$, and $s'_{ij} = s_i$ if $j \in A_i$.

For every $i \in N$, define replacements

$$\bar{b}_{ij} = \begin{cases} b'_{ij} + s'_i & j \notin A_i \\ b'_{ij} & j \in A_i \ . \end{cases}$$

We show that, from the point of view of trader i, the probability distribution over his final wealth is unchanged in the replacement. In the original prediction market, with probability $\sum_{j \in A_i} p_{ij}$, the net profit from sales was $s'_i \sum_{j \in A_i} \pi_j - s'_i$, and with probability $\sum_{j \notin A_i} p_{ij}$, it was $s'_i \sum_{j \in A_i} \pi_j$. Now, if i buys s'_i C_j-contracts for every $j \notin A_i$, then with probability $\sum_{j \in A_i} p_{ij}$ the net profit from these purchases is $-s'_i \sum_{j \notin A_i} \pi_j$, and with probability $\sum_{j \notin A_i} p_{ij}$ it is $s'_i - s'_i \sum_{j \notin A_i} \pi_j$. Since $\sum_j \pi_j = 1$, the net profits under these two scenarios are equal. The total costs for each of these sets of contracts are also equal:

$$\sum_j \pi_j b'_{ij} + s'_i - s'_i \sum_{j \in A_i} \pi_j \ = \ \sum_j \pi_j b'_{ij} + s'_i \sum_{j \notin A_i} \pi_j = \sum_j \pi_j \bar{b}_{ij} \ .$$

Furthermore, the $\bar{b}_{ij}$s satisfy (3) since for every $j \in M$,

$$\sum_i \bar{b}_{ij} = \sum_i b'_{ij} + \sum_{i : j \notin A_i} s'_i = \sum_{i : j \in A_i} s'_{ij} + \sum_{i : j \notin A_i} s'_i = \sum_i s'_i \ .$$

On the other hand, for every j,

$$\sum_{i,j} \pi_i \bar{b}_{ij} = \sum_j \pi_j \sum_i s'_i = \sum_i s'_i \ . \qquad \square$$

4 Equilibrium prices reflect market belief

Eisenberg and Gale [3] presented a concave maximization problem for computing an equilibrium in parimutuel betting, where the prices are obtained as Karush-Kuhn-Tucker (KKT) multipliers, and proved existence and uniqueness of equilibrium prices. By our equivalence mapping, we obtain following result for prediction market equilibrium:

Theorem 1. *There exists a unique vector of equilibrium prices for prediction markets as defined by* (2), (1). *The equilibrium prices can be computed in polynomial time. Furthermore, the equilibrium price of C_j is equal to the fraction of money spent on C_j.*

Proof. Existence and uniqueness follow from [3] and Proposition 3. Polynomial-time computability also follows from the concave-maximization formulation [3]. Denote by m_{ij} the amount of money spent by i on C_j at equilibrium. Thus, $b_{ij} = m_{ij}/\pi_j$. By (3),

$$\pi_j = \frac{\sum_i m_{ij}}{\sum_{ij} m_{ij}} \ . \qquad \square$$

To get a precise relation between equilibrium prices and traders' beliefs, in line with the results of [4, 5], consider a market with a nonatomic continuum of traders. Each trader is represented by a probability distribution $\boldsymbol{p} = (p_1, \ldots, p_m)$ over a fixed M. The set of all such $\boldsymbol{p}$ is a simplex $\boldsymbol{\Delta}$. Consider a nonatomic measure $\mu : \boldsymbol{\Delta} \to \Re_+$, so that $\mu(\boldsymbol{p})$ is the density of money (i.e., budget) per unit volume at $\boldsymbol{p}$. W.l.o.g, assume

$$\int_{\boldsymbol{\Delta}} \mu(\boldsymbol{p})\, d\boldsymbol{p} = 1 \ . \tag{4}$$

$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ are equilibrium prices if there exists a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ of measures $\beta_j : \boldsymbol{\Delta} \to \Re_+$, $j \in M$, where $\beta_j(\boldsymbol{p})$ represents the density of C_j-contracts bought per unit volume at $\boldsymbol{p}$; two conditions must hold:

- *Individual optimality.* Consider a given vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ and a vector of beliefs $\boldsymbol{p} = (p_1, \ldots, p_m)$. The optimization problem of traders in the neighborhood of $\boldsymbol{p}$ is to choose the values of $\beta_1(\boldsymbol{p}), \ldots, \beta_m(\boldsymbol{p})$ so as to

$$\begin{aligned} \text{Maximize} \qquad & \sum_{j \in M} (p_j - \pi_j)\, \beta_j(\boldsymbol{p}) && (5) \\ \text{subject to} \qquad & \sum_{j \in M} \pi_j\, \beta_j(\boldsymbol{p}) \le \mu(\boldsymbol{p}) && (6) \\ & \beta_j(\boldsymbol{p}) \ge 0 \quad (j \in M) \ . \end{aligned}$$

  Here, the cost per unit volume is constrained by the budget per unit volume, and the objective is to maximize the net profit per unit volume.

- *Balance constraint.*

$$\int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = 1 \quad (j \in M) \ . \tag{7}$$

The individual-optimality condition implies that, in equilibrium,

$$\frac{p_j}{\pi_j} < \max_{k \in M} \frac{p_k}{\pi_k} \quad \Rightarrow \quad \beta_j(\boldsymbol{p}) = 0 \ . \tag{8}$$

For every $\mathbf{0} < \boldsymbol{\pi} \in \boldsymbol{\Delta}$ and $j \in M$, denote

$$\boldsymbol{\Delta}_j(\boldsymbol{\pi}) \equiv \left\{ \boldsymbol{p} \in \boldsymbol{\Delta} \,:\, (\forall k \in M) \left(k \ne j \Rightarrow \frac{p_j}{\pi_j} > \frac{p_k}{\pi_k} \right) \right\} \ . \tag{9}$$

Theorem 2. *If μ is nonatomic, then, in equilibrium, for every $j \in M$,*

$$\int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \mu(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \ . \tag{10}$$

Thus, at equilibrium, for every $j \in M$, the fraction of the total budget that comes from traders who prefer C_j is equal to π_j.

Proof. Denote

$$\boldsymbol{\Delta}_0(\boldsymbol{\pi}) = \bigcup_{j\in M} \boldsymbol{\Delta}_j(\boldsymbol{\pi}) \ .$$

Since the Lesbegue-measure of the set $\boldsymbol{\Delta} \setminus \boldsymbol{\Delta}_0(\boldsymbol{\pi})$ is zero,

$$\int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \int_{\boldsymbol{\Delta}_0(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} \ .$$

Therefore, by (7)–(9), for every $j \in M$,

$$1 = \int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \sum_{k\in M} \int_{\boldsymbol{\Delta}_k(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} \ . \tag{11}$$

On the other hand, by (6), in equilibrium,

$$\sum_{k\in M} \pi_k \beta_k(\boldsymbol{p}) = \mu(\boldsymbol{p}) \ . \tag{12}$$

It follows from (12) and (11) that for every j,

$$\int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \mu(\boldsymbol{p})\, d\boldsymbol{p} = \sum_{k\in M} \pi_k \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_k(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \ . \qquad \square$$

In the case of two outcomes, $\boldsymbol{\Delta}_1([\pi, 1-\pi])$ and $\boldsymbol{\Delta}_2([\pi, 1-\pi])$ are exactly the set of p's such that $\pi < p$ and $\pi \geq p$, respectively. Thus, in this case, Theorem 9 proves that the equilibrium price π is the (budget-weighted) π-tile of traders' beliefs, consistent with the observations in [4, 5] for this special case.

5 Extension to nonlinear utilities

So far we considered risk-neutral traders, i.e., the utility equals expected profit. Our observations can be extended to nonlinear utilities, for example, when traders are risk averse, i.e., the utilities are concave functions of money (see [1] for details). Here is a sketch of the major ideas involved. Note that the proof of Proposition 3 holds without the assumption of linear utility functions. Therefore, we can reduce our problem to considering equilibrium in the parimutuel betting markets. The latter are equivalent to Fisher market with nonlinear utilities. Thus, results on existence and uniqueness of equilibrium for those markets can be applied.

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