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Equilibrium in Prediction Markets with Buyers and Sellers

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1 Introduction

We consider a situation where a certain "experiment" is planned, and traders are betting on its outcome. Typical examples could be a horse race and an election. Denote by $M = \{1, \ldots, m\}$ the set of *outcomes* of the experiment and by $N = \{1, \ldots, n\}$ the set of *traders*. Traders are interested in buying or selling betting contracts. For $j \in M$, a single contract C_j entitle the purchaser receive from the seller of the contract one dollar in the event where the outcome is j.[1]

Each trader $i \in N$ has an initial amount of money $m_i > 0$ and a subjective probability distribution $\boldsymbol{p}^i = (p_{i1}, \ldots, p_{im})$ over M, so that $p_{ij} \geq 0$ for every j and $\sum_{j=1}^{m} p_{ij} = 1$. The number p_{ij} is the subjective probability of trader i that the outcome will be j. We assume that the utility of trader i, before the experiment has taken place, is equal to his subjective expectation of the amount of money he will have after the experiment has been performed and the obligations are have been settled. Thus, trader i is willing to pay at most p_{ij} for contract C_j and sell this contract for at least p_{ij}. Denote $\boldsymbol{P} = (p_{ij})$ and $\boldsymbol{m}(= (m_1, \ldots, m_n)$.

Each trader can be either a seller or a buyer of each contract. The market determines prices for contracts, as an equilibrium according the definition stated below, as a function of the subjective probability distribution $\boldsymbol{p}^i$ and the amounts m_i. If the market determines a price π_j for contract C_j, then each trader i, such that $p_{ij} > \pi_j$, is willing in principle to buy the contract C_j be-

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[1] We assume that contracts can be traded in fractions, so that a fraction $\alpha > 0$ of a contract C_j at the price π_j costs $\alpha\,\pi_j$ and the seller pays α if the outcome is j.

cause it yields an expected positive net profit. Similarly, each trader i, such that $p_{ij} < \pi_j$, is willing in principle to sell the contract C_j. However, traders have limited budgets, and therefore wish to optimize their trades, so the numbers of particular contracts they wish to buy and sell depend on the budget. Also, a seller must put on deposit the worst-case amount he may have to pay on the contracts he sells. The tuple $\langle N, M, \boldsymbol{P}, \boldsymbol{m} \rangle$ is called a *prediction market.*

In a paper published in 1959, Eisenberg and Gale [2] introduced a model for betting on the outcome of an experiment, where there are only buyers. In their model, the money that collected from buyers is returned to the winners, possibly after deduction of a certain percentage.

Manski [3] considered a model where the distribution of beliefs is continuous and the budgets are stochastically independent of beliefs. Under these assumptions, Manski characterized the unique equilibrium price in the case of two possible outcomes.

In this paper we discuss the following concept of a prediction-market equilibrium:

Definition 1 (**Equilibrium**). *Given a prediction market* $\langle N, M, \boldsymbol{P}, \boldsymbol{m} \rangle$, *a* prediction-market equilibrium *is a triple*

$$\langle \boldsymbol{\pi} = (\pi_1, \ldots, \pi_m), \boldsymbol{B} = (b_{ij}), \boldsymbol{S} = (s_{ij}) \rangle$$

that satisfies the conditions stated below, where π_j *is the price of contract* C_j, b_{ij} *is the number of contracts* C_j *bought by trader* i, *and* s_{ij} *is the number of contracts* C_j *sold by trader* i*; the following conditions must be satisfied:*

(i) *For every trader* $i \in N$, *the vector*

$$\boldsymbol{u}_i \equiv (b_{i1}, \ldots, b_{im}, s_{i1}, \ldots, s_{im})$$

is an optimal solution of the following optimization problem:

$$\text{Maximize } \boldsymbol{u}_i \quad \sum_{j \in M} (p_{ij} - \pi_j)\, b_{ij} + \sum_{j \in M} (\pi_{ij} - p_{ij})\, s_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_j \pi_j\, b_{ij} - \sum_j \pi_j\, s_{ij} + \max_{j \in M}\{s_{ij}\} \le m_i \quad (i \in N) \tag{2}$$

$$b_{ij},\ s_{ij} \ge 0 \quad (j \in M)\ , \tag{3}$$

(ii) *for every outcome* $j \in M$,

$$\sum_{i \in N} b_{ij} = \sum_{i \in N} s_{ij}\ . \tag{4}$$

Note that in (2), $\sum_j b_{ij}\, \pi_j$ is the total amount of money trader i has to pay for all the contracts he buys, $\sum_j s_{ij}\, \pi_j$ is the total amount that trader i receives for all the contracts he sells, and $\max_j\{s_{ij}\}$ is the amount that trader i has to set aside in order to guarantee that he will always be able pay his obligations.

An m-tuple $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is called a *vector of equilibrium prices* if there exist $\boldsymbol{B}$ and $\boldsymbol{S}$ such that $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S})\rangle$ is an equilibrium.

2 Individual optimization

In equilibrium, each trader achieves the maximum possible expected net profit under the equilibrium prices. In this section we characterize these optimal trades.

2.1 Individual buyer optimization

Proposition 1. *Given nonnegative prices $(\pi_1, \ldots, \pi_m)$, for every trader $i \in N$:*

(*i*) *if there is a $j \in M$ such that $p_{ij} > \pi_j = 0$, then the expected net profit that trader i can achieve by only buying contracts is unbounded, and*

(*ii*) *if $p_{ij} = 0$ for every j such that $\pi_j = 0$, then the maximum expected net profit that trader i can achieve by only buying contracts is equal to*

$$m_i \cdot \max\left\{0,\ \max\left\{\frac{p_{ij} - \pi_j}{\pi_j}\ :\ \pi_j > 0\right\}\right\} \ .$$

Proof. If $p_{ij} > \pi_j$, then by buying b_{ij} contracts C_j, trader i's expected net profit is equal to

$$\sum_{j \in M} (p_{ij} - \pi_j)\, b_{ij} \ .$$

If trader i only buys contracts, then he considers the following optimization problem:

$$P_i^b \qquad \begin{aligned} \text{Maximize}_{(b_{ij})} \quad & \sum_{j \in M} (p_{ij} - \pi_j)\, b_{ij} \\ \text{subject to} \quad & \sum_{j \in M} \pi_j\, b_{ij} \le m_i \\ & b_{ij} \ge 0 \quad (j \in M) \end{aligned}$$

If $p_{ij} > \pi_j = 0$, then the trader can buy as many C_j contracts as he wishes, and hence the expected net profit is unbounded. Otherwise, it is optimal for him to buy m_i/π_j units of any contract C_j that maximizes the ratio $(p_{ij} - \pi_j)/\pi_j$ (or , equivalently, the ratio p_{ij}/π_j) provided that $p_{ij} > \pi_j$. Contracts with $p_{ij} = \pi_j = 0$ yield zero net profit. $\square$

2.2 Individual seller optimization

Next, consider the optimization problem of a trader who wishes only to sell contracts.

Proposition 2. *Given nonnegative prices* $(\pi_1, \ldots, \pi_m)$*, for every trader* $i \in N$*:*

(*i*) *if there exists a* $j \in M$ *such that* $\pi_j > p_{ij}$*, then for every outcome* $j \in M$ *such that* $\pi_j > p_{ij}$*, in every optimal solution of problem* P_i^s*,*

$$s_{ij} = \max_k \{s_{ik}\} > 0 \ ;$$

(*ii*) *if* $\sum_{j:\pi_j > p_{ij}} \pi_j < 1$*, then the optimal common value of the* s_{ij}*s is*

$$s_i = \frac{m_i}{1 - \sum_{j:\pi_j > p_{ij}} \pi_j} \ ,$$

and the expected net profit is

$$m_i \sum_{j \in M} \frac{\max\{0, \pi_j - p_{ij}\}}{1 - \sum_{j:\pi_j > p_{ij}} \pi_j}$$

and

(*iii*) *if* $\sum_{j:\pi_j > p_{ij}} \pi_j \geq 1$*, then the profit to the seller is unbounded as long as there is demand.*

Proof. If $p_{ij} < \pi_j$, then by selling s_{ij} contracts C_j, trader i's expected net profit is equal to

$$\sum_{j \in M} (\pi_j - p_{ij})\, s_{ij} \ .$$

However, trader i must deposit the amount of $\max_j\{s_{ij}\}$ to guarantee that for any $j \in M$, he would be able to pay s_{ij} dollars if the outcome were j. Thus, if trader i only sells contracts, then he considers the following optimization problem:

$$P_i^s \qquad \begin{array}{rl} \text{Maximize}_{(s_{ij})} & \displaystyle\sum_{j \in M} (\pi_j - p_{ij})\, s_{ij} \\ \text{subject to} & s_{ij} - \displaystyle\sum_{k \in M} \pi_k\, s_{ik} \leq m_i \quad (j \in M) \\ & s_{ij} \geq 0 \quad (j \in M) \ . \end{array}$$

If there exists a j such that $\pi_j > p_{ij}$ then obviously $\max_k\{s_{ik}\} > 0$. Thus, if $s_{ij} < \max_k\{s_{ik}\}$, then by increasing s_{ij} below $\max_k\{s_{ik}\} > 0$, the objective value increases while the constraints continue to hold. There exists an optimal solution such that $s_{ij}=0$ for every j such that $\pi_j \le p_{ij}$. In such a solution, the maximum possible value s of $\max_k\{s_{ik}\}$ satisfies

$$s - s\cdot \sum_{j:\pi_j>p_{ij}} \pi_j \le m_i \ .$$

□

2.3 Individual optimization in general

We now discuss the full optimization problem of trader i when he considers both buying and selling. The decision variables are $b_{i1},\ldots,b_{im},s_{i1},\ldots,s_{im}$. Thus, the problem is the following:

$$P_i^{bs} \qquad \begin{array}{rl} \text{Maximize}_{(b_{ij}),(s_{ij})} & \displaystyle\sum_{j\in M}(p_{ij}-\pi_j)\,b_{ij} + \sum_{j\in M}(\pi_j - p_{ij})\,s_{ij} \\ \text{subject to} & \displaystyle\sum_{j\in M}\pi_j\,b_{ij} + s_{ij} - \sum_{k\in M}\pi_k\,s_{ik} \le m_i \quad (j\in M) \\ & b_{ij},\ s_{ij}\ge 0 \quad (j\in M)\ . \end{array}$$

Proposition 3. *Given nonnegative prices $(\pi_1,\ldots,\pi_m)$, for every trader $i\in N$:*

(i) *If $p_{ij}=\pi_j$ for every $j\in M$, then trader i is totally indifferent and has zero expected net profit.*

(ii) *If either there exists a $j\in M$ such that $p_{ij}>\pi_j=0$, or $\sum_{j:\pi_j>p_{ij}}\pi_j\ge 1$, then the expected net profit of trader i is unbounded.*

(iii) *If $p_{ij}=0$ for every $j\in M$ such that $\pi_j=0$, and if $\sum_{j:\pi_j>p_{ij}}\pi_j<1$, then the optimal expected net profit of trader i is equal to*

$$m_i\cdot\max\left\{\max\left\{\frac{p_{ij}-\pi_j}{\pi_j}\ :\ j\in M\right\},\ \frac{\sum_{j:\pi_j>p_{ij}}(\pi_j-p_{ij})}{1-\sum_{j:\pi_j>p_{ij}}\pi_j}\right\}\ . \qquad (5)$$

Proof. Suppose that at an optimal solution trader i spends $\beta\ge 0$ on buying contracts and $\gamma\ge 0$ on selling, so that $\beta+\gamma\le m_i$. It follows from Proposition 1 that the expected net profit from buying is equal to:

$$\beta\cdot\max\left\{0,\ \max\left\{\frac{p_{ij}-\pi_j}{\pi_j}\ :\ j\in M\right\}\right\}\ .$$

On the other hand, it follows from Proposition 2 that if $\sum_{j:\pi_j>p_{ij}}\pi_j<1$, then the expected net profit is

$$\gamma\cdot\sum_{j\in M}\frac{\max\{0,\pi_j-p_{ij}\}}{1-\sum_{j:\pi_j>p_{ij}}\pi_j}\ ,$$

and otherwise, the problem is unbounded[2]. It follows that if $p_{ij} = \pi_j$ for every $j \in M$, then trader i is totally indifferent and has zero expected net profit; otherwise, the optimal expected net profit is the one given in (5). □

Corollary 1. *For every trader $i \in N$, if $(\pi_1, \ldots, \pi_m)$ are nonnegative prices such that $\pi_j \neq p_{ij}$ for at least one $j \in M$, then at optimal solution of P_i^{bs}, trader i either uses his entire budget to buy, or uses his entire budge to sell, except in the special case where*

$$\max\left\{\frac{p_{ij} - \pi_j}{\pi_j} \ : \ j \in M\right\} = \frac{\sum_{j:\pi_j > p_{ij}}(\pi_j - p_{ij})}{1 - \sum_{j:\pi_j > p_{ij}} \pi_j} \ ,$$

i.e., the expected net profit per dollar from buying is equal to the expected net profit per dollar from selling.

Proposition 4. *If $(\pi_1, \ldots, \pi_m)$ are nonnegative prices such that $\sum_{j \in M} \pi_j = 1$, then for every trader $i \in N$, there is an optimal solution of Problem P_i^{bs} where i does not sell at all.*

Proof. Let $i \in N$ be fixed. Suppose that instead of selling one C_j-contract for each j such that $\pi_j > p_{ij}$, trader i buys one C_j-contract for each j such that $\pi_j \leq p_{ij}$. The expected net profit from buying these contracts is equal to

$$\sum_{j:\pi_j \leq p_{ij}} (p_{ij} - \pi_j) = 1 - \sum_{j:\pi_j > p_{ij}} p_{ij} - 1 + \sum_{j:\pi_j > p_{ij}} \pi_j = \sum_{j:\pi_j > p_{ij}} (p_{ij} - \pi_j) \ ,$$

which is the same as the net profit from selling those other contracts. Also, the cost of buying these contracts is equal to

$$\sum_{j:\pi_j \leq p_{ij}} \pi_j = 1 - \sum_{j:\pi_j > p_{ij}} \pi_j \ ,$$

which is the same as the cost of selling those other contracts. □

Proposition 5. *In Problem P_i^{bs}, there is always an optimal solution such that for every $j \in M$,*

$$\sum_{k \in M} \pi_k (b_{ik} - s_{ik}) + s_{ij} = m_i \ .$$

Proof. Suppose that at some optimal solution of P_i^{bs} there is a $j \in M$ such that

$$\sum_{k \in M} \pi_k (b_{ik} - s_{ik}) + s_{ij} < m_i \ .$$

Denote

$$\delta_j = m_i - \sum_{k \in M} \pi_k (b_{ik} - s_{ik}) - s_{ij} > 0 \ .$$

[2]However, as we show later, at equilibrium prices the problem is bounded.

Increasing each of b_{ij} and s_{ij} by δ_j does not change the value of the objective function, nor does it violate any of the constraints, and in the new solution the equality to m_i holds. □

3 Equilibrium prices sum to 1

Proposition 6. *If* $(\pi_1, \ldots, \pi_n)$ *are equilibrium prices, then* $\sum_{j \in M} \pi_j = 1$.

Proof. Let $i \in N$ be any trader and let $j \in M$ be any outcome. If i sells one C_j-contract, then from i's point of view, with probability p_{ij}, his final wealth will be $m_i + \pi_j - 1$, and with probability $1 - p_{ij}$, his final wealth will be $m_i + \pi_j$. On the other hand, if i buys one C_k-contract for each $k \neq j$, then from i's point of view, with probability p_{ij}, his final wealth will be $m_i - \sum_{k \neq j} \pi_k$, and with probability $1 - p_{ij}$ his final wealth will be $m_i - \sum_{k \neq j} \pi_k + 1$.

If $\sum_{k \in M} \pi_k < 1$, then both

$$m_i + \pi_j - 1 < m_i - \sum_{k \neq j} \pi_k$$

and

$$m_i + \pi_j < m_i - \sum_{k \neq j} \pi_k + 1 \ .$$

It follows, that i would not sell C_j. This conclusion holds for every $i \in N$ and $j \in M$ and, therefore, in this case there would be no selling and no buying. However, this can be an equilibrium only if $p_{ij} = \pi_j$ for every $i \in N$ and $j \in M$, which implies $\sum_j \pi_j = 1$.

If $\sum_{j \in M} \pi_j > 1$, if i sells x C_j-contracts for each C_j, then i receives $x \cdot \sum_{j \in M} \pi_j$, is more than the required deposit of x, so every value of x is feasible. The final wealth of i is equal to $m_i + x \cdot \sum_{j \in M} \pi_j - x$, which can be arbitrarily large, and hence not in equilibrium. □

Note that the proof of Proposition 6 holds *without the assumption of a linear utility function.*

4 An equivalent formulation with no sellers

In this section we discuss a different formulation of the betting problem, which was introduced by Eisenberg and Gale [2]. We show the equivalence of the prediction-market model we defined above to the Eisenberg-Gale model.

Definition 2. *An equilibrium* $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ *is called* regular *if for every* $i \in N$*,* (*i*) *there exists an* s_i *such that for every* $j \in M$*, if* $\pi_j \geq p_{ij}$*, then* $s_{ij} = s_i$*, and*

(*ii*) *for every* $j \in M$*, if* $\pi_j < p_{ij}$*, then* $s_{ij} = 0$*.*

Proposition 7. *If* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ *is a vector of equilibrium prices, then there exist transactions* $\boldsymbol{B}$ *and* $\boldsymbol{S}$ *such that* $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ *is a regular equilibrium.*

Proof. Let $\boldsymbol{B}$ and $\boldsymbol{S}$ be transactions such that $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ is an equilibrium. For every $i \in N$ and $j \in M$, if $\pi_j \geq p_{ij}$, then as described in the proof of Proposition 5, define new values b'_{ij} and s'_{ij} by increasing each of s_{ij} and b_{ij} by an equal amount δ_{ij}, until the new amounts $b'_{ij} = b_{ij} + \delta_{ij}$ and $s'_{ij} + \delta_{ij}$ satisfy

$$\sum_{k \in M} \pi_k (b'_{ijk} - s'_{ik}) + s'_{ij} = m_i \quad (i \in N,\ j \in M)\ .$$

These modified transactions are also individually optimal, obviously, s'_{ij} is independent of j, as long as $\pi_j \geq p_{ij}$. Next, for every $i \in N$ and $j \in M$, if $\pi_j < p_{ij}$ and $s_{ij} > 0$, then necessarily $b_{ij} \geq s_{ij}$. In this case we define $b'_{ij} = b_{ij} - s_{ij}$ and $s'_{ij} = s_{ij} - s_{ij} = 0$. These modified transactions too are individually optimal. Obviously,

$$\sum_{j \in M} b'_{ij} = \sum_{j \in M} s'_{ij}$$

so $\left\langle \boldsymbol{\pi}, \boldsymbol{B}', \boldsymbol{S}' \right\rangle$ is an equilibrium that satisfies conditions (*i*) and (*ii*) of this proposition. □

In view of Propositions 7, 4 and 6, the characterization of equilibrium prices can be simplified as described in the following proposition:

Proposition 8. *If* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ *is a vector of equilibrium prices, then there exists a matrix* $\bar{\boldsymbol{B}} = (\bar{b}_{ij})$ $(i \in N,\ j \in M)$ *such that*

(*i*) *for every* $i \in N$*, given* $\boldsymbol{\pi}$*, the maximum expected net profit that trader* i *can achieve by buying and selling is equal to* $\sum_{j \in M} (p_{ij} - \pi_j) \bar{b}_{ij}$*,*

(*ii*) *for every* $i \in N$*,*

$$\sum_{j \in M} \pi_j \bar{b}_{ij} = m_i\ ,$$

and

(*iii*) *for every* $j \in M$*,*

$$\sum_{i \in N} \bar{b}_{ij} = \sum_{i \in N} m_i\ .$$

Proof. Let $\boldsymbol{B}$ and $\boldsymbol{S}$ be transactions such that $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ is a regular equilibrium (see Proposition 7). We define a replacement of all the sales $s_{i1}, \ldots, s_{im}$ by purchases. For every $i \in N$, we replace the sales of $s_{ij} = s_i > 0$ C_j-contracts (where $\pi_j \geq p_{ij}$) by purchases of s_i C_j-contracts for each j such that $\pi_j < p_{ij}$. We denote by $\bar{b}_{ij}$ the resulting total purchase amounts (i.e, the previous amounts b_{ij} plus the ones that replaced the sales) and denote $\bar{\boldsymbol{B}} = (\bar{b}_{ij})$. Thus,

$$\bar{b}_{ij} = \begin{cases} b_{ij} + s_i & \text{if } \pi_j < p_{ij} \\ b_{ij} & \text{if } \pi_j \geq p_{ij} \ . \end{cases}$$

From the point of view of trader i, the probability distribution over his final wealth is unchanged as can be seen as follows. Denote by A the set of all j such that $\pi_j \geq p_{ij}$. If trader i sells s_i units of each contract C_j such that $j \in A$, then with probability $\sum_{j \in A} p_{ij}$, the net profit from these sales is equal to $s_i \sum_{j \in A} \pi_j - s_i$, and with probability $\sum_{j \notin A} p_{ij}$, the net profit from these sales is equal to $s_i \sum_{j \in A} \pi_j$. If i buys s_i units of each contract C_j such that $j \notin A$, then with probability $\sum_{j \in A} p_{ij}$ the net profit from these sales is equal to $-s_i \sum_{j \notin A} \pi_j$, and with probability $\sum_{j \notin A} p_{ij}$ the net profit from these sales is equal to $s_i - s_i \sum_{j \notin A} \pi_j$. Since $\sum_{j \in M} \pi_j = 1$, the net profits under these two scenarios are equal. In particular, the expected net profit from the sales that are replaced is equal to the expected net profit from the purchases that replace them:

$$\sum_{j : \pi_j \geq p_{ij}} (\pi_j - p_{ij})\, s_i = \sum_{j : \pi_j < p_{ij}} (p_{ij} - \pi_j)\, s_i$$

and the total costs for each of these sets of contracts are also equal:

$$s_i - s_i \sum_{j : \pi_j \geq p_{ij}} \pi_j \ = \ s_i \sum_{j : \pi_j < p_{ij}} \pi_j \ .$$

Therefore, given $\boldsymbol{p}$, every trader $i \in N$ is indifferent between the purchases $(\bar{b}_{i1}, \ldots, \bar{b}_{im})$ and the transactions $(b_{i1}, \ldots, b_{im}; s_{i1}, \ldots, s_{im})$. The balance constraints

$$\sum_{i \in N} b_{ij} = \sum_{i \in N} s_{ij} \quad (j \in M)$$

imply relations over the $\bar{b}_{ij}$s as follows. For every $j \in M$,

$$\begin{aligned} \sum_{i \in N} \bar{b}_{ij} &= \sum_{i \in N} b_{ij} \ + \sum_{i : \pi_j < p_{ij}} s_i \\ &= \sum_{i : s_{ij} > 0} s_{ij} \ + \sum_{i : \pi_j < p_{ij}} s_i \\ &= \sum_{i : s_{ij} > 0} s_i \ + \sum_{i : s_{ij} = 0} s_i \\ &= \sum_{i \in N} s_i \ . \end{aligned}$$

On the other hand, since $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ is a regular equilibrium, for every $i \in N$ and $j \in M$, if $\pi_j \geq p_{ij}$, then $s_{ij} = s_i$. It follows that for every $i \in N$,

$$\sum_{k \in M} \pi_k (b_{ik} - s_{ik}) + s_i = m_i \ ,$$

and hence

$$\begin{aligned} \sum_{i \in N} m_i &= \sum_{i \in N} \sum_{k \in M} \pi_k (b_{ik} - s_{ik}) + \sum_{i \in N} s_i \\ &= \sum_{k \in M} \pi_k \sum_{i \in N} (b_{ik} - s_{ik}) + \sum_{i \in N} s_i \\ &= \sum_{i \in N} s_i \ . \end{aligned}$$

Thus, for every $j \in M$,

$$\sum_{i \in N} \bar{b}_{ij} = \sum_{i \in N} s_i = \sum_{i \in N} m_i \ .$$

□

The above-mentioned relations give rise to the following definition:

Definition 3 (**Buyers equilibrium**). *A* buyers equilibrium *is a pair* $\langle \boldsymbol{\pi}, \bar{\boldsymbol{B}} \rangle$ *that satisfies the conditions stated below, where* π_j *is the price of contract* C_j*,* $\bar{b}_{ij}$ *is the number of contracts* C_j *bought by trader* i*; the following conditions must be satisfied:*

(*i*) *For every trader* $i \in N$*, the vector* $(\bar{b}_{i1}, \ldots, \bar{b}_{im})$ *is an optimal solution of the following optimization problem:*

$$\text{Maximize}_{\ \bar{b}_{i1},\ldots,\bar{b}_{im}} \quad \sum_j (p_{ij} - \pi_j)\, \bar{b}_{ij} \tag{6}$$

$$\text{subject to} \quad \sum_{j \in M} \pi_j\, \bar{b}_{ij} \leq m_i \quad (i \in N) \tag{7}$$

$$\bar{b}_{ij} \geq 0 \quad (j \in M) \ , \tag{8}$$

(*ii*) *For every outcome* $j \in M$*,*

$$\sum_{i \in N} \bar{b}_{ij} = \sum_{i \in N} m_i \ . \tag{9}$$

The following proposition was first proven by Eisenberg and Gale [2] (see below).

Proposition 9. *In a buyers equilibrium,* $\langle \boldsymbol{\pi}, \bar{\boldsymbol{B}} \rangle$*,* $\sum_{j \in M} \pi_j = 1$*.*

Proof. It follows from the definition that

$$\sum_{i \in N} m_i = \sum_{i \in N} \sum_{j \in M} \pi_j \bar{b}_{ij} = \sum_{j \in M} \pi_j \sum_{i \in N} \bar{b}_{ij} = \sum_{j \in M} \pi_j \cdot \sum_{i \in N} m_i \ .$$

This implies the claim. □

Proposition 10. *If $\langle \boldsymbol{\pi}, \bar{\boldsymbol{B}} \rangle$ is a buyers equilibrium, then $\boldsymbol{\pi}$ is also a vector of equilibrium prices in the prediction market with buyers and sellers.*

Proof. Given a buyers equilibrium, define

$$b'_{ij} = \begin{cases} \bar{b}_{ij} & \text{if } j < m \\ 0 & \text{if } j = m \end{cases}$$

and

$$s'_{ij} = \begin{cases} \bar{b}_{im} & \text{if } j < m \\ 0 & \text{if } j = m \ . \end{cases}$$

We now show that the triple $\langle \boldsymbol{\pi}, \boldsymbol{B}' = (b'_{ij}), \boldsymbol{S}' = (s'_{ij}) \rangle$ is an equilibrium. First, the expected net profits in the buyers-only formulation and in the regular formulation, respectively, are equal:

$$\sum_{j \in M} (p_{ij} - \pi_j)(b'_{ij} - s'_{ij}) = \sum_{j \in M} (p_{ij} - \pi_j) \bar{b}_{ij} \ .$$

Second, the amounts of money spent in both optimization problems are equal as well; if $j < m$, then

$$\begin{aligned} \sum_{k=1}^{m} \pi_k (b'_{ik} - s'_{ik}) + s'_{ij} &= \sum_{k=1}^{m-1} \pi_k (\bar{b}_{ik} - \bar{b}_{im}) + \bar{b}_{im} \\ &= \sum_{k=1}^{m-1} \pi_k \bar{b}_{ik} + \left(1 - \sum_{k=1}^{m-1} \pi_k\right) \bar{b}_{im} \\ &= \sum_{k=1}^{m} \pi_k \bar{b}_{ik} = m_i \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{m} \pi_k (b'_{ik} - s'_{ik}) + s'_{im} &= \sum_{k=1}^{m-1} \pi_k (\bar{b}_{ik} - \bar{b}_{im}) \\ &= \sum_{k=1}^{m-1} \pi_k \bar{b}_{ik} - \bar{b}_{im} \sum_{k=1}^{m-1} \pi_k \\ &\leq \sum_{k=1}^{m} \pi_k \bar{b}_{ik} = m_i \ . \end{aligned}$$

Furthermore, $\boldsymbol{B}'$ and $\boldsymbol{S}'$ satisfy the balance requirements; for $j = m$,

$$\sum_{i\in N} b'_{im} = \sum_{i\in N} s'_{im} = 0 \ ,$$

and for every $j < m$,

$$\begin{aligned}\sum_{i\in N}(b'_{ij} - s'_{ij}) &= \sum_{i\in N}(\bar{b}_{ij} - \bar{b}_{im}) \\ &= \sum_{i\in N}\bar{b}_{ij} - \sum_{i\in N}\bar{b}_{im} \\ &= m\sum_{i\in N}\sum_{k\in M}\pi_k\bar{b}_{ik} - m\sum_{i\in N}\sum_{k\in M}\pi_k\bar{b}_{ik} \\ &= 0 \ .\end{aligned}$$

□

Thus, we have proven:

Theorem 1. *A vector* $\boldsymbol{\pi}$ *is a vector of equilibrium prices in a prediction market with buyers and sellers if and only if it is a vector of equilibrium prices in the buyers-only model.*

4.1 The Eisenberg-Gale concave maximization

The buyers equilibrium problem was essentially formulated by Eisenberg and Gale [2], who also analyzed it via the following optimization problem:

$$\begin{aligned}\text{Maximize}_{\bar{\boldsymbol{B}}} \quad & \sum_{i\in N} m_i \log\left(\sum_j p_{ij}\bar{b}_{ij}\right) \\ \text{subject to} \quad & \sum_{i\in N}\bar{b}_{ij} \le \sum_{i\in N} m_i \quad (j \in M) \\ & \bar{b}_{ij} \ge 0 \quad (i \in N,\ j \in M)\ .\end{aligned} \tag{10}$$

It follows that at an optimal solution $\bar{\boldsymbol{B}}$, there exist multipliers $\pi_1, \ldots, \pi_m$ such that

$$\begin{aligned}\pi_j &\ge \frac{m_i\, p_{ij}}{\sum_j p_{ij}\bar{b}_{ij}} \quad (i \in N,\ j \in M) \\ \bar{b}_{ij} > 0 \Rightarrow \pi_j &= \frac{m_i\, p_{ij}}{\sum_j p_{ij}\bar{b}_{ij}} \quad (i \in N,\ j \in M) \\ \sum_{i\in N}\bar{b}_{ij} &\le \sum_{i\in N} m_i \quad (j \in M) \\ \pi_j > 0 \Rightarrow \sum_{i\in N}\bar{b}_{ij} &= \sum_{i\in N} m_i \quad (j \in M)\end{aligned}$$

It follows that for every $j \in M$,

$$\frac{\sum_j p_{ij}\bar{b}_{ij}}{m_i} \geq \frac{p_{ij}}{\pi_j} \ ,$$

and for every $i \in N$ and $j \in M$, if $\bar{b}_{ij} > 0$, then

$$\frac{\sum_j p_{ij}\bar{b}_{ij}}{m_i} = \frac{p_{ij}}{\pi_j} \ .$$

This implies that for every $i \in N$, $(\bar{b}_{i1}, \ldots, \bar{b}_{im})$ is an optimal solution in buyer i's problem. Also, for every $i \in N$,

$$\frac{\sum_j p_{ij}\bar{b}_{ij}}{m_i} \sum_j \pi_j \bar{b}_{ij} = \sum_{j \in M} p_{ij}\bar{b}_{ij} \ ,$$

hence,

$$\sum_{j \in M} \pi_j \bar{b}_{ij} = m_i \ .$$

Furthermore,

$$\sum_{j \in M} \pi_j \sum_{i \in N} \bar{b}_{ij} = \sum_{j \in N} \pi_j \sum_{i \in N} m_i$$

and

$$\sum_{i \in N} \sum_{j \in M} \pi_j \bar{b}_{ij} = \sum_{i \in N} m_i \ ,$$

so

$$\sum_{j \in N} \pi_j = 1 \ .$$

The consequences of the Gale-Eisenberg characterization of buyers equilibrium with regard to the prediction-market equilibrium with buyers and sellers are the following:

- There exists a unique vector of equilibrium prices
- The equilibrium prices as well as equilibrium transaction can be found in polynomial time [1].

5 A market with a continuum of traders

In this section consider a market with a non-atomic continuum of traders. We work in the buyers-only setting. Each trader is represented by a probability distribution $\boldsymbol{p} = (p_1, \ldots, p_m)$ over the set $M = \{1, \ldots, m\}$ of outcomes. Let m be fixed. Thus, $\sum_{j=1}^{m} p_j = 1$ and $p_j \geq 0$, $j = 1, \ldots, m$. The set of all such vectors $\boldsymbol{p}$ is a simplex denoted by $\boldsymbol{\Delta}$. We consider a non-atomic measure $\mu : \boldsymbol{\Delta} \to \Re_+$, so that $\mu(\boldsymbol{p})$ is the density of money (i.e., budget) per unit volume at the point $\boldsymbol{p}$. Without loss of generality, assume

$$\int_{\boldsymbol{\Delta}} \mu(\boldsymbol{p})\, d\boldsymbol{p} = 1 \ . \tag{11}$$

An equilibrium is a pair $\langle \boldsymbol{\pi}, \boldsymbol{\beta} \rangle$, where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is a vector of prices and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ is a vector of measures $\beta_j : \boldsymbol{\Delta} \to \Re_+$, $j = 1, \ldots, m$, where $\beta_j(\boldsymbol{p})$ represents the density of the number of C_j-contracts bought per unit volume at $\boldsymbol{p}$; two conditions must be satisfied:

- The condition of individual optimality that appears in the finite model is generalized to the continuous model as follows. Consider a given vector of prices $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ and a vector of beliefs $\boldsymbol{p} = (p_1, \ldots, p_m)$. The optimization problem of traders in the neighborhood of $\boldsymbol{p}$ is to choose the values of $\beta_1(\boldsymbol{p}), \ldots, \beta_m(\boldsymbol{p})$ so as to solve the following minimization problem:

$$\begin{array}{lll} \text{Maximize} & \displaystyle\sum_{j \in M} (p_j - \pi_j)\, \beta_j(\boldsymbol{p}) & (12) \\ \text{subject to} & \displaystyle\sum_{j \in M} \pi_j\, \beta_j(\boldsymbol{p}) \le \mu(\boldsymbol{p}) & (13) \\ & \beta_j(\boldsymbol{p}) \ge 0 \quad (j \in M) \ . & \end{array}$$

  Here, the cost per unit volume is constrained by the budget per unit volume, and the objective is to maximize the net profit per unit volume.

- The balance constraint is the following:

$$\int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = 1 \quad (j \in M) \ . \tag{14}$$

5.1

The individual-optimality condition implies that in equilibrium $\langle \boldsymbol{\pi}, \boldsymbol{\beta} \rangle$,

$$\frac{p_j}{\pi_j} < \max_{k \in M} \frac{p_k}{\pi_k} \quad \Rightarrow \quad \beta_j(\boldsymbol{p}) = 0 \ . \tag{15}$$

For every $\mathbf{0} < \boldsymbol{\pi} \in \boldsymbol{\Delta}$ and $j \in M$, denote

$$\boldsymbol{\Delta}_j(\boldsymbol{\pi}) \equiv \left\{ \boldsymbol{p} \in \boldsymbol{\Delta} \ : \ (\forall k \in M) \left(k \ne j \Rightarrow \frac{p_j}{\pi_j} > \frac{p_k}{\pi_k} \right) \right\} \ . \tag{16}$$

The following theorem states that in equilibrium, for every $j \in M$, the fraction of the total budget that comes from the set $\boldsymbol{\Delta}_j(\boldsymbol{\pi})$ is equal to π_j.

Theorem 2. *If μ is non-atomic and $\langle \boldsymbol{\pi}, \boldsymbol{\beta} \rangle$ is an equilibrium, then for every for every $j \in M$,*

$$\int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \mu(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \ . \tag{17}$$

Proof. Denote

$$\boldsymbol{\Delta}_0(\boldsymbol{\pi}) = \bigcup_{j \in M} \boldsymbol{\Delta}_j(\boldsymbol{\pi}) \ .$$

Since the Lesbegue-measure of the set $\boldsymbol{\Delta} \setminus \boldsymbol{\Delta}_0(\boldsymbol{\pi})$ is zero, we have

$$\int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \int_{\boldsymbol{\Delta}_0(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} \ .$$

Therefore, by (14)–(16), for every $j \in M$,

$$1 = \int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \sum_{k \in M} \int_{\boldsymbol{\Delta}_k(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} \ . \tag{18}$$

On the other hand, by (13), in equilibrium,

$$\sum_{k \in M} \pi_k \beta_k(\boldsymbol{p}) = \mu(\boldsymbol{p}) \ . \tag{19}$$

It follows from (19) and (18) that

$$\int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \mu(\boldsymbol{p})\, d\boldsymbol{p} = \sum_{k \in M} \pi_k \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_k(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \int_{\boldsymbol{\Delta}_j(\boldsymbol{\pi})} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = \pi_j \ .$$

□

5.2 The Eisenberg-Gale program

The generalization of the Eisenberg-Gale optimization problem to the continuous model is the following:

$$\begin{aligned} \text{Maximize}_{\ \beta} \quad & \int_{\boldsymbol{\Delta}} \log\left(\sum_{j \in M} p_j\, \beta_j(\boldsymbol{p}) \right) \mu(\boldsymbol{p})\, d\boldsymbol{p} \\ \text{subject to} \quad & \int_{\boldsymbol{\Delta}} \beta_j(\boldsymbol{p})\, d\boldsymbol{p} = 1 \quad (j \in M) \\ & \beta_j(\boldsymbol{p}) \geq 0 \quad (j \in M) \ . \end{aligned}$$

It follows from this formulation that there exists a unique equilibrium price vectors.

5.3 A geometric illustration

We show that the subsets $\mathbf{\Delta}_j(\boldsymbol{\pi})$ have an intuitive geometric structure. For $\boldsymbol{\pi} > \mathbf{0}$, denote

$$\overline{\mathbf{\Delta}}_j(\boldsymbol{\pi}) \equiv \left\{ \boldsymbol{p} \in \mathbf{\Delta} \, : \, (\forall k \in M) \left(\frac{p_j}{\pi_j} \geq \frac{p_k}{\pi_k} \right) \right\} \; . \tag{20}$$

The subdivision

$$\mathbf{\Delta} = \overline{\mathbf{\Delta}}_1(\boldsymbol{\pi}) \cup \cdots \cup \overline{\mathbf{\Delta}}_1(\boldsymbol{\pi})$$

is generated by the hyperplanes

$$H_{j\ell} \equiv \left\{ \boldsymbol{p} \in \mathbf{\Delta} \, : \, \frac{p_j}{\pi_j} = \frac{p_j}{\pi_\ell} \right\} \; .$$

Obviously, for every j and ℓ in M,

$$\overline{\mathbf{\Delta}}_j(\boldsymbol{\pi}) \cap \overline{\mathbf{\Delta}}_\ell(\boldsymbol{\pi}) = \left\{ \boldsymbol{p} \in \mathbf{\Delta} \, : \, (\forall k \in M) \left(\frac{p_j}{\pi_j} = \frac{p_\ell}{\pi_\ell} \geq \frac{p_k}{\pi_k} \right) \right\} \; . \tag{21}$$

Also,

$$\bigcap_{j \in M} \overline{\mathbf{\Delta}}_j(\boldsymbol{\pi}) = \left\{ \boldsymbol{p} \in \mathbf{\Delta} \, : \, \frac{p_1}{\pi_1} = \frac{p_2}{\pi_2} = \cdots = \frac{p_m}{\pi_m} \right\} = \{\boldsymbol{\pi}\} \; .$$

Furthermore, for every $j \in M$, the unit vector $\boldsymbol{e}^j$ (where $e^j_j = 1$, and $e^j_k = 0$ for $k \neq j$) belongs to $\overline{\mathbf{\Delta}}_j(\boldsymbol{\pi})$. Also note that for $i = 1, \ldots, m$, the $(m-i)$-dimensional faces of the polyhedron $\overline{\mathbf{\Delta}}_j(\boldsymbol{\pi})$ are of the form:

$$\Phi_j(\ell_1, \ldots, \ell_i) \equiv \left\{ \boldsymbol{p} \in \mathbf{\Delta} \, : \, (\forall k \in M) \left(\frac{p_j}{\pi_j} = \frac{p_{\ell_1}}{\pi_{\ell_1}} = \cdots = \frac{p_{\ell_i}}{\pi_{\ell_i}} \geq \frac{p_k}{\pi_k} \right) \right\} \; ,$$

where $j, \ell_1, \ldots, \ell_i$ are pairwise distinct. The case of $m = 3$ is depicted in the figure below. Note that in this case the faces $\Phi_j(\ell)$ (where $\ell \neq j$) are the straight line segments, each of which connects a vertex with the respective opposite facet, passing through the point $\boldsymbol{\pi}$.

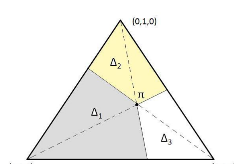

Figure 1: Geometric interpretation of equilibrium price

A Nonlinear utilities

In this section we prove the existence of an equilibrium when the utility functions of the players are not linear but concave.

Denote

$$\Pi = \{\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \ : \ (\forall i)(0 \le \pi_j \le 1)\} \ . \tag{22}$$

A.1 The domain E

Definition 4. *Denote by E the set of all triples $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ that satisfy*

$$\sum_i b_{ij} \quad = \quad \sum_i s_{ij} \quad (j \in M) \tag{23}$$

$$\sum_{k \in M} \pi_k \, b_{ik} - \sum_{k \in M} \pi_k \, s_{ik} + s_{ij} \quad \le \quad m_i \quad (i \in N, \ j \in M) \tag{24}$$

$$b_{ij}, \ s_{ij} \quad \ge \quad 0 \quad (i \in N, \ j \in M) \ . \tag{25}$$

Proposition 11. *The set E is compact.*

Proof. Obviously, E is closed. We now prove that it is also bounded. Suppose $(\boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S}) \in D$. Note that since for every $j \in M$,

$$\pi_j \sum_i b_{ij} = \pi_j \sum_i s_{ij}$$

it follows that

$$\sum_{i,j} \pi_j \, b_{ij} = \sum_{i,j} \pi_j \, s_{ij}$$

and hence by (24),

$$\sum_i \max_j \{s_{ij}\} \le \sum_i m_i \ .$$

Since all of the s_{ij}'s are nonnegative, this implies

$$s_{ij} \le \sum_i m_i \tag{26}$$

for all $i \in N$ and $j \in M$. Furthermore, for every $j \in M$,

$$\sum_i b_{ij} = \sum_i s_{ij} \le n \sum_i m_i$$

and hence

$$b_{ij} \le n \sum_i m_i \tag{27}$$

for all $i \in N$ and $j \in M$. □

A.2 Definition of the domain D

We now define a domain D by dropping the "balance" requirement (23) from E and adding instead the bounds (26)-(27) from the proof of Proposition 11.

Definition 5. *Denote by D the set of all triples $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$, such that*

$$\sum_k \pi_k \, b_{ik} - \sum_k \pi_k \, s_{ik} + s_{ij} \le m_i \quad (i \in N,\ j \in M)$$

$$0 \le b_{ij} \le n \sum_i m_i \quad (i \in N,\ j \in M)$$

$$0 \le s_{ij} \le \sum_i m_i \quad (i \in N,\ j \in M)$$

$$0 \le \pi_j \le 1 \quad (j \in M) \ .$$

For every $\boldsymbol{\pi} \in \Pi$ and $i \in N$, denote by $D_i(\boldsymbol{\pi})$ the set of all vectors

$$\boldsymbol{u}_i \equiv (b_{i1}, \ldots, b_{im}, s_{i1}, \ldots, s_{im})$$

that satisfy the constraints of D (see Definition 5). Note that for every fixed $\boldsymbol{\pi}$, the set $D_i(\boldsymbol{\pi})$ is a convex polyhedron. Consider the optimization problem of trader i as follows. For simplicity, denote also

$$\boldsymbol{u}'_i \equiv (b'_{i1}, \ldots, b'_{im}, s'_{i1}, \ldots, s'_{im}) \ .$$

Denote by $U_i(x)$ the risk-neutral utility function of trader i. Under trader i's belief with probability p_{ij} his final net wealth is equal to

$$m_i - \sum_k \pi_k (b_{ik} - s_{ik}) + (b_{ij} - s_{ij}) \ .$$

In the linear case his expected utility is:

$$m_i - \sum_k \pi_k (b_{ik} - s_{ik}) + \sum_j p_{ij} \left[(b_{ij} - s_{ij}) \right] = m_i + \sum_j (p_{ij} - \pi_j)(b_{ij} - s_{ij}) \ ,$$

and, in general, the expected utility is

$$\sum_j p_{ij} U_i \left[m_i - \sum_k \pi_k (b_{ik} - s_{ik}) + (b_{ij} - s_{ij}) \right] \ .$$

Thus, given $\boldsymbol{\pi}$, for each $i \in N$, denote by $P_i(\boldsymbol{\pi})$ the following concave maximization problem:

$$P_i(\boldsymbol{\pi}) \qquad \begin{array}{ll} \text{Maximize}_{\boldsymbol{u}_i} & \displaystyle\sum_j p_{ij} U_i \left[m_i - \sum_k \pi_k (b_{ik} - s_{ik}) + (b_{ij} - s_{ij}) \right] \\ \text{subject to} & \sum_k \pi_k \, (b_{ik} - s_{ik}) + s_{ij} \le m_i \quad (j \in M) \\ & 0 \le b_{ij} \le n \sum_i m_i \quad (j \in M) \\ & 0 \le s_{ij} \le \sum_i m_i \quad (j \in M) \ . \end{array}$$

Denote the objective function of $P_i(\boldsymbol{\pi})$ by f, i.e.,

$$f(\boldsymbol{u}_i) = f(\boldsymbol{u}_i; \boldsymbol{\pi}, i) = \sum_j p_{ij}\, U_i \left[m_i - \sum_k \pi_k (b_{ik} - s_{ik}) + (b_{ij} - s_{ij}) \right] \ .$$

Also, denote

$$g(\boldsymbol{u}_i) = g(\boldsymbol{u}_i; \boldsymbol{\pi}, i) = \max\left\{0,\ \textstyle\sum_k \pi_k\, b_{ik} - \sum_k \pi_k\, s_{ik} + \max_k\{s_{ik}\} - m_i \right\} \ .$$

Finally, for any real $C > 0$, denote

$$F_C(\boldsymbol{u}_i) = f(\boldsymbol{u}_i) - C \cdot g(\boldsymbol{u}_i) \ .$$

Note that if $\boldsymbol{u}_i$ is feasible in $P_i(\boldsymbol{\pi})$, then $F_C(\boldsymbol{u}_i) = f(\boldsymbol{u}_i)$. Consider the following concave maximization problem, which we denote by $\tilde{P}_i(\boldsymbol{\pi})$:

$$\tilde{P}_i(\boldsymbol{\pi}) \qquad \begin{array}{rl} \text{Maximize}_{\boldsymbol{u}_i} & F_C(\boldsymbol{u}_i) \\ \text{subject to} & 0 \le b_{ij} \le n \sum_i m_i \quad (j \in M) \\ & 0 \le s_{ij} \le \sum_i m_i \quad (j \in M) \ . \end{array}$$

Note that $\tilde{P}_i(\boldsymbol{\pi})$ is obtained from $P_i(\boldsymbol{\pi})$ by relaxing the budget constraints and adding a penalty for violating them. Denote by $\tilde{D}_i$ the feasible domain of $\tilde{P}_i(\boldsymbol{\pi})$, and note that $\tilde{D}_i$ is a convex bounded polyhedron, which is independent of $\boldsymbol{\pi}$. For every face Φ of $\tilde{D}_i$, denote by $\text{aff}(\Phi)$ the affine span of Φ. Denote by $\boldsymbol{v}(\Phi)$ a maximizer of $F_C(\boldsymbol{u}_i)$ over $\text{aff}(\Phi)$, and denote by V_i the set of all the $\boldsymbol{v}(\Phi)$ for all faces Φ of $\tilde{D}_i$. One of the members of V_i is a maximizer of $F_C(\boldsymbol{u}_i)$ over $\tilde{D}_i$.

Denote

$$\varepsilon = \varepsilon(\boldsymbol{\pi}) \equiv \min\{g(\boldsymbol{v}) \ : \ \boldsymbol{v} \in V_i,\ g(\boldsymbol{v}) > 0\} \ .$$

Let $\boldsymbol{v}^* \in V_i$ be a maximizer of $f(\boldsymbol{v})$ over $\tilde{D}_i$. Denote

$$C(\boldsymbol{\pi}) \equiv \frac{f(\boldsymbol{v}^*)}{\varepsilon(\boldsymbol{\pi})} \ .$$

It follows that if $\boldsymbol{w} \in V_i$ is not feasible in $P_i(\boldsymbol{\pi})$, then for every $C \ge C(\boldsymbol{\pi})$,

$$\begin{aligned} F_C(\boldsymbol{w}) &= f(\boldsymbol{w}) - C \cdot g(\boldsymbol{w}) \\ &\le f(\boldsymbol{v}^*) - C(\boldsymbol{\pi}) \cdot \varepsilon(\boldsymbol{p}) \\ &= f(\boldsymbol{v}^*) - f(\boldsymbol{v}^*) = 0 \ . \end{aligned}$$

Thus, with such a large C, an optimal solution of $\tilde{P}_i(\boldsymbol{\pi})$ must be feasible in $P_i(\boldsymbol{\pi})$. Now, recall that $\boldsymbol{\Pi}$ is compact, define

$$C^* \equiv \max\{C(\boldsymbol{\pi}) \ : \ \boldsymbol{\pi} \in \boldsymbol{\Pi}\} \ ,$$

and denote

$$F^*(\boldsymbol{v}) = F_{C^*}(\boldsymbol{v}) \ .$$

A.3 Definition of the domain H

We now define a domain H by dropping the budget constraints from D.

Definition 6. *Denote by H the set of all triples $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ such that*

$$0 \leq b_{ij} \leq n \sum_i m_i \quad (i \in N,\ j \in M)$$

$$0 \leq s_{ij} \leq \sum_i m_i \quad (i \in N,\ j \in M)\ .$$

$$0 \leq \pi_j \leq 1 \quad (j \in M)\ .$$

Obviously, H is box and hence homeomorphic to a ball.

A.4 Definition of a continuous mapping on H

We now define a continuous mapping $\Psi : H \to H$. The mapping Ψ maps a triple $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle \in H$ to a triple $\langle \boldsymbol{\pi}', \boldsymbol{B}', \boldsymbol{S}' \rangle \in H$ as explained below.

A.4.1 Definition of $\boldsymbol{\pi}'$

Given $\boldsymbol{B}$ and $\boldsymbol{S}$, for every $j \in M$, denote the "excess demand" by

$$e_j \equiv \sum_i b_{ij} - \sum_i s_{ij}$$

and set

$$\pi'_j = \max\{0, \min\{1, \pi_j + e_j\}\} = \begin{cases} \min\{1, \pi_j + e_j\} & \text{if} \quad e_j \geq 0 \\ \max\{0, \pi_j + e_j\} & \text{if} \quad e_j < 0 \end{cases} .$$

The following is obvious:

Proposition 12. *If $0 \leq \pi_j \leq 1$ for every $j \in M$, then*

(i) *for every $j \in M$, $0 \leq \pi'_j \leq 1$,*

(ii) *if for every $j \in M$, $\sum_{i \in N} b_{ij} = \sum_{i \in N} s_{ij}$, then $\boldsymbol{\pi}' = \boldsymbol{\pi}$.*

(iii) *If $\boldsymbol{\pi}' = \boldsymbol{\pi}$, then for every $j \in M$, (1) if $0 < \pi_j < 1$, then $e_j = 0$, (2) if $\pi_j = 1$, then $e_j \geq 0$, and (3) if $\pi_j = 0$ then $e_j \leq 0$.*

A.4.2 Definition of $(\boldsymbol{B}', \boldsymbol{S}')$

We now set the values of $(\boldsymbol{B}', \boldsymbol{S}')$. For every $i \in N$ and for every $\boldsymbol{v} \in V_i$, denote

$$\alpha_{\boldsymbol{v}}(\boldsymbol{\pi}) = \alpha_{\boldsymbol{v}}(\boldsymbol{\pi}; i) = \max\{0, F^*(\boldsymbol{v}) - F^*(\boldsymbol{u}_i)\} \ .$$

Note that $\alpha_{\boldsymbol{v}}(\boldsymbol{\pi})$ is a continuous function of $\boldsymbol{\pi}$. Next, given i, $\boldsymbol{\pi}$ and $\boldsymbol{u}_i$,the part of the image under Ψ that is denoted by $\boldsymbol{u}_i'$ is the vector defined as follows. Denote

$$\alpha^*(\boldsymbol{\pi}) = \sum_{\boldsymbol{v} \in V_i} \alpha_{\boldsymbol{v}}(\boldsymbol{\pi}) \ .$$

If $\alpha^*(\boldsymbol{\pi}) = 0$, define $\boldsymbol{u}_i' = \boldsymbol{u}_i$; otherwise, $\alpha^*(\boldsymbol{\pi}) > 0$, and we set

$$\boldsymbol{u}_i' = \boldsymbol{u}_i + \frac{\min\{\alpha^*(\boldsymbol{\pi}), 1\}}{\alpha^*(\boldsymbol{\pi})} \cdot \sum_{\boldsymbol{v} \in V_i} \alpha_{\boldsymbol{v}}(\boldsymbol{\pi})(\boldsymbol{v} - \boldsymbol{u}_i) \ .$$

Recall the abbreviated notation where the components of $\boldsymbol{u}_i'$ are the following:

$$\boldsymbol{u}_i' = (b_{i1}', \ldots, b_{im}', s_{i1}', \ldots, s_{im}') \ .$$

Proposition 13. *The vector $\boldsymbol{u}_i$ is a feasible solution of the problem $\tilde{P}_i(\boldsymbol{\pi})$.*

Proof. Note that if $\alpha^*(\boldsymbol{\pi}) \neq 0$, then by convexity, the vector

$$\bar{\boldsymbol{u}}_i \equiv \boldsymbol{u}_i + \frac{1}{\alpha^*(\boldsymbol{\pi})} \cdot \sum_{\boldsymbol{v} \in V_i} \alpha_{\boldsymbol{v}}(\boldsymbol{\pi})(\boldsymbol{v} - \boldsymbol{u}_i) = \frac{1}{\alpha^*(\boldsymbol{\pi})} \cdot \sum_{\boldsymbol{v} \in V_i} \alpha_{\boldsymbol{v}}(\boldsymbol{\pi})\, \boldsymbol{v}$$

is a feasible solution. Therefore, by convexity, $\boldsymbol{u}_i'$ is also feasible. □

Proposition 14. *The mapping from $(\boldsymbol{\pi}, \boldsymbol{u}_i)$ to $\boldsymbol{u}_i'$ is continuous.*

Proof. Because of the factor $\min\{\alpha^*, 1\}$, the mapping from $(\boldsymbol{\pi}, \boldsymbol{u}_i)$ to $\boldsymbol{u}_i'$ is continuous even at points where $\alpha^*(\boldsymbol{\pi}) = 0$. □

Proposition 15. *$\boldsymbol{u}_i' = \boldsymbol{u}_i$ if and only if $\boldsymbol{u}_i$ is an optimal solution of $\tilde{P}_i(\boldsymbol{\pi})$.*

Proof. $\alpha^*(\boldsymbol{\pi}) > 0$ if and only if $F^*(\boldsymbol{u}_i') > F^*(\boldsymbol{u}_i)$. Also, $\alpha_{\boldsymbol{v}}(\boldsymbol{\pi}) > 0$ if and only if $F^*(\boldsymbol{v}) > F^*(\boldsymbol{u}_i)$. It follows that $\alpha^*(\boldsymbol{\pi}) = 0$ if and only if $\boldsymbol{u}_i$ is an optimal solution of $\tilde{P}_i(\boldsymbol{\pi})$. Also, $\alpha^*(\boldsymbol{\pi}) = 0$ if and only if $\boldsymbol{u}_i' = \boldsymbol{u}_i$. This implies the claim. □

Theorem 3. *There exists an equilibrium.*

Proof. Because Ψ is continuous over the polyhedron H, it follows from Brouwer's fixed-point theorem that Ψ has a fixed point. Thus, suppose $\boldsymbol{\pi}' = \boldsymbol{\pi}$, $\boldsymbol{B}' = \boldsymbol{B}$, and $\boldsymbol{S}' = \boldsymbol{S}$, i.e., $\boldsymbol{u}_i' = \boldsymbol{u}_i$ for every $i \in N$.

First, if for every $j \in M$, $0 < \pi_j < 1$, then for every $j \in M$, $\sum_i b_{ij} = \sum_i s_{ij}$. Thus, in this case, the triple $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ satisfies all the conditions that define the domain E. Furthermore, because the balance constraints are satisfied, it follows that the bounds (26)-(27) from the proof of Proposition 11 hold for every point in D. Hence, for every $i \in N$, the vector $\boldsymbol{u}_i$ is optimal not only in $P_i(\boldsymbol{\pi})$ but also in the respective optimization problem of player i as required in an equilibrium per Definition 1, where these additional bounds are not imposed.

In general, we now show that if the balance constraints are not satisfied for some outcomes $j \in M$ such that $\pi_j = 0$ or $\pi_j = 1$, then $\boldsymbol{B}$ and $\boldsymbol{S}$ can be modified into certain $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{S}}$, so that the resulting triple $\langle \boldsymbol{\pi}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{S}} \rangle$ is an equilibrium. This is shown as follows.

For every $j \in M$ such that $\sum_i b_{ij} = \sum_i s_{ij}$, we set $\tilde{b}_{ij} = b_{ij}$ and $\tilde{s}_{ij} = s_{ij}$ for every $i \in N$.

Consider an outcome $j \in M$ such that $\pi_j = 0$ and $\sum_i b_{ij} < \sum_i s_{ij}$. Thus, there exists an i such that $s_{ij} > 0$. Because of the optimality of $\boldsymbol{u}_i$ from the point of view i, it follows that $p_{ij} = 0$, and trader i is actually indifferent with regard to the value of s_{ij}. Therefore, s_{ij} could be decreased by a little bit without compromising optimality. Denote by R the set of traders i such that $s_{ij} > 0$. Let $a_i \geq 0$, $i \in R$, be any numbers such that $\sum_{i \in R_i} a_i = \sum_{i \in N} (s_{ij} - b_{ij})$ and $a_i \leq s_{ij}$. If $R = \{i_1, \ldots, i_r\}$, then such a_is can be found, for example, by setting $a_{i_k} = s_{i_k, j}$ for $k = 1, \ldots, p-1$,

$$a_{i_p} = \sum_{i \in N} (s_{ij} - b_{ij}) - \sum_{k=1}^{p-1} s_{i_k, j}$$

and $a_{i_k} = 0$ for $k > p$. We can modify the s_{ij}s of $i \in R$ into

$$\tilde{s}_{ij} = s_{ij} - a_i \qquad (i \in R) \ ,$$

set $\tilde{s}_{ij} = 0$ for i such that $s_{ij} = 0$, and set $\tilde{b}_{ij} = b_{ij}$ for all i, so that $\sum_i \tilde{b}_{ij} = \sum_i \tilde{s}_{ij}$.

Analogously, consider an outcome j such that $\pi_j = 1$ and $\sum_i b_{ij} > \sum_i s_{ij}$. Thus, there exists an i such that $b_{ij} > 0$. Because of the optimality of $\boldsymbol{u}_i$ from the point of view i, it follows that $p_{ij} = 1$, and trader i is actually indifferent with regard to the value of b_{ij}. Therefore, b_{ij} could be decreased by a little bit without compromising optimality. Like the previous case, we can find $\tilde{b}_{ij}$s and $\tilde{b}_{ij}$s such that $\sum_i \tilde{b}_{ij} = \sum_i \tilde{s}_{ij}$. Obviously, the above-described modifications do not violate the constraints of the individual optimization problems and the modified amounts remain optimal from the points of view of the individual traders. Therefore, $\langle \boldsymbol{\pi}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{S}} \rangle$ is an equilibrium. □

Proposition 16. *If $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is a vector of equilibrium prices, then there exist $\boldsymbol{B}$ and $\boldsymbol{S}$ such that $\langle \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{S} \rangle$ is an equilibrium, and for every $i \in M$*

and $j \in M$,

$$\sum_k \pi_k(b_{ik} - s_{ik}) + s_{ij} = m_i \ .$$

Proof. As in the proof of Proposition 5, each trader can satisfy the equality by increasing the trading with himself, without compromising optimality and without violating the balance condition of the equilibrium. □

As noted in Section 3, equilibrium prices sum to 1 even when the utility functions are nonlinear. This implies that Proposition 8 also generalizes as follows.

Proposition 17. *If* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ *is a vector of equilibrium prices, then there exists a matrix* $\bar{\boldsymbol{B}} = (\bar{b}_{ij})$ $(i \in N,\ j \in M)$ *such that*

(*i*) *for every* $i \in N$*, given* $\boldsymbol{\pi}$*, the maximum utility that trader* i *can achieve by buying and selling is equal to*

$$\sum_{j \in M} p_{ij} U_i \left[m_i - \sum_{k \in M} \pi_k \bar{b}_{ik} + \bar{b}_{ij} \right] \ ,$$

(*ii*) *for every* $i \in N$,

$$\sum_{j \in M} \pi_j \bar{b}_{ij} = m_i \ ,$$

and

(*iii*) *for every* $j \in M$,

$$\sum_{i \in N} \bar{b}_{ij} = \sum_{i \in N} m_i \ .$$

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