Volatility Term Structure in the Q-Alpha-Sigma Model

Matt Dixon, Paul Oreto, David Starr, and Chen Zheng

Outline

Introduction

- Implied volatility surface/Q-alpha-sigma model

Statistical Overview

GARCH analysis

PCA Analysis

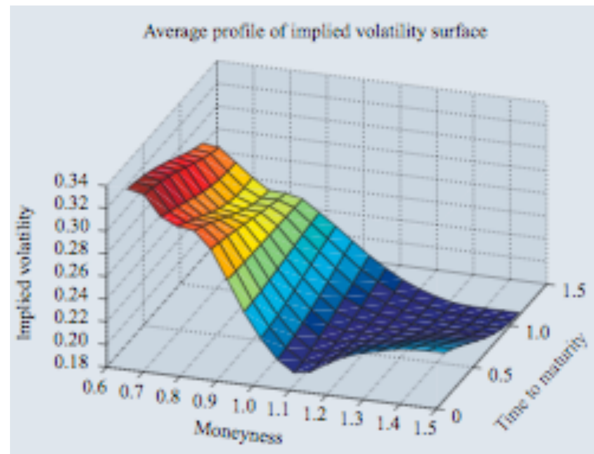
Vega hedging?

Conclusion

Implied Volatility Surface

Black-Scholes assumes constant volatility

Observed: volatility surface



Surface fluctuations:
How to model? Hedge?

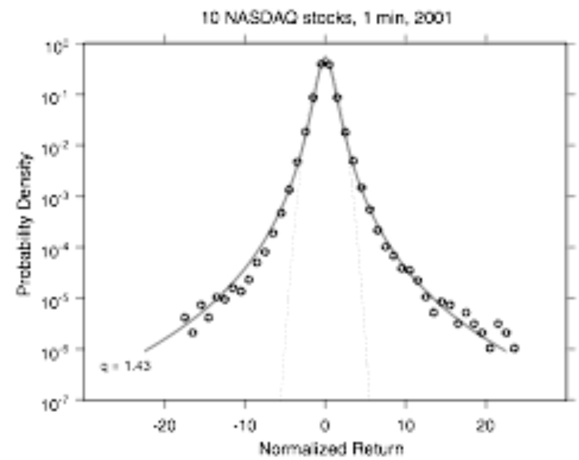
$$\text{vega} = \frac{dC}{d\sigma}$$

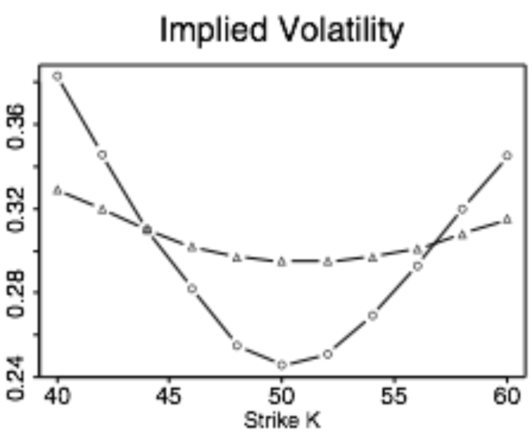
Q-Alpha-Sigma Model

(Borland, PRL 2002)

New model for underlying: not GBM

Captures fat tails of stock return:



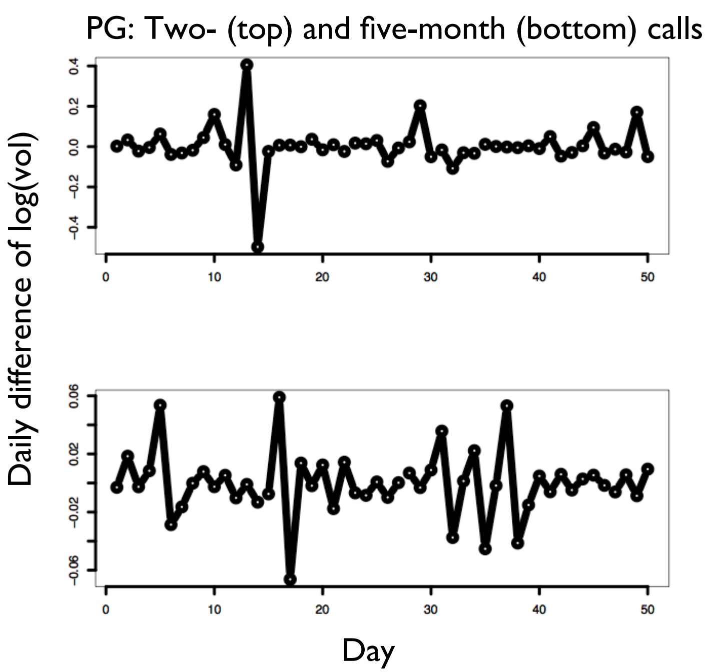


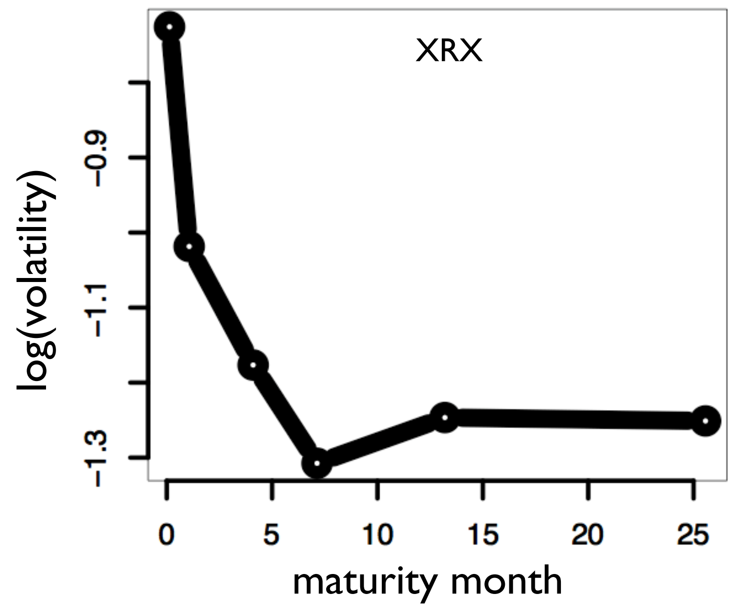
Successfully approximates the smile.

Term Structure

Volatility surface reduced
to term structure:

(Use logarithms of volatilities)





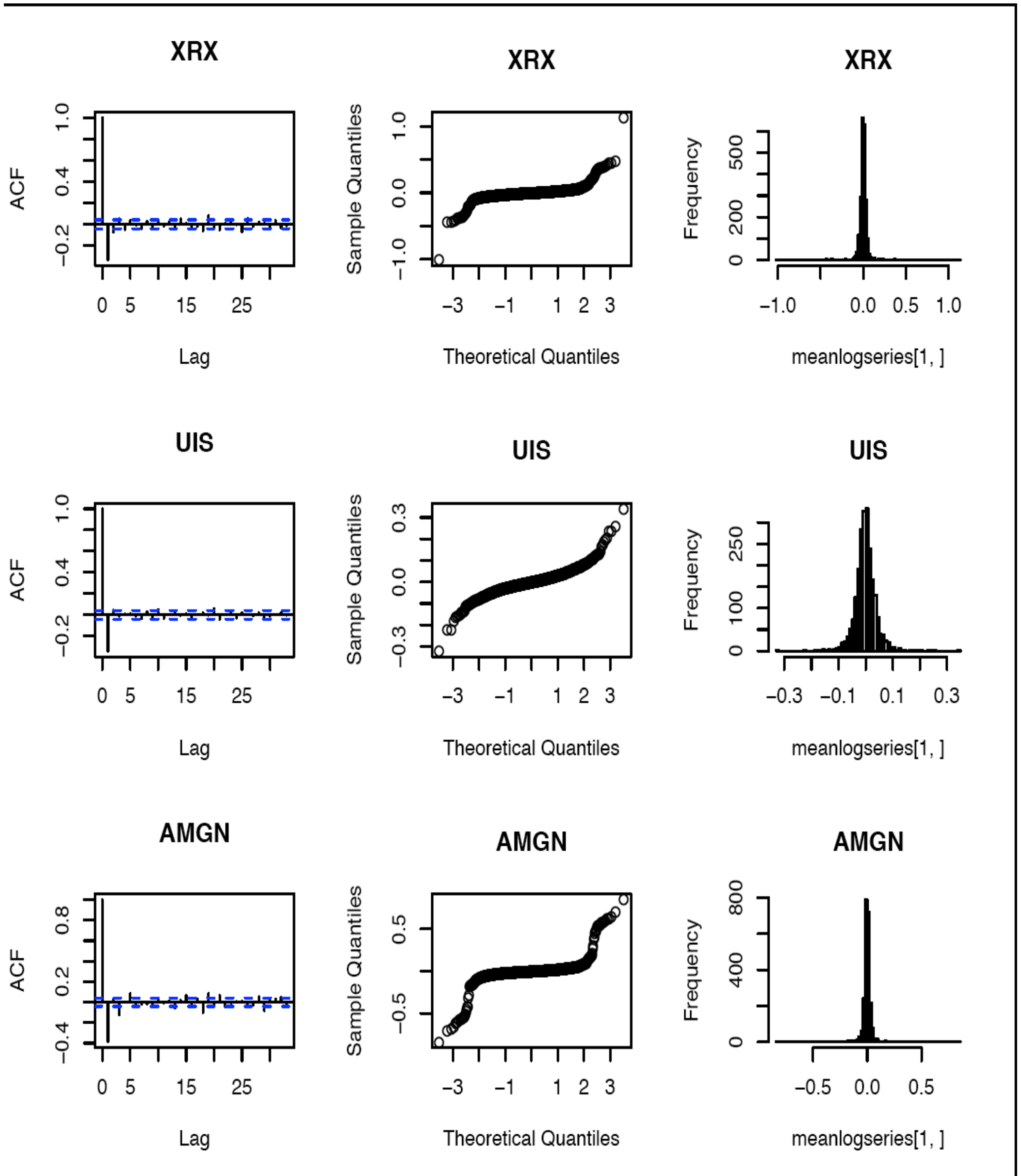
Correlation across maturities?

Statistical Properties of the Data

We can test the time series of fluctuations for

- •Repeating patterns:
- •Normal distribution:
  - –Needed for PCA
  - –Does log improve normality?

- ACF
  Box-Ljung Test
- Qq-plot
  Shapiro-Wilk
  Histogram



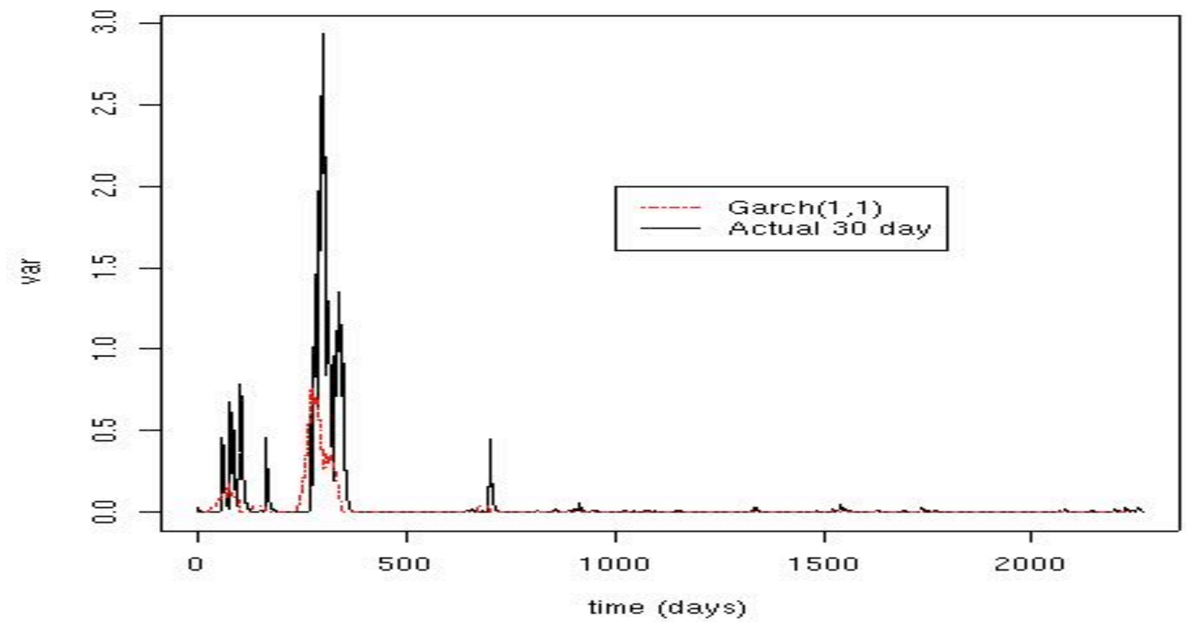
There is little autocorrelation.

The time series show normality near the center but the fat-tail shape of the histogram indicates some non-normality.

The GARCH Implied Volatility Model

- Assumes stationarity in the implied volatility time series
- Exhibits observed heteroskedasticity (vol of vol)
- Decomposes dynamics into those attributed to parallel shift and change of slope
- $\sigma_k(\tau) = \underline{\sigma}_k + (\tau - T/2)\Delta \sigma_k(\tau)$
- **Avellaneda, Marco and Zhu, Yingzi, "An E-ARCH Model for the Term Structure of Implied Volatility of FX Options", 1997**

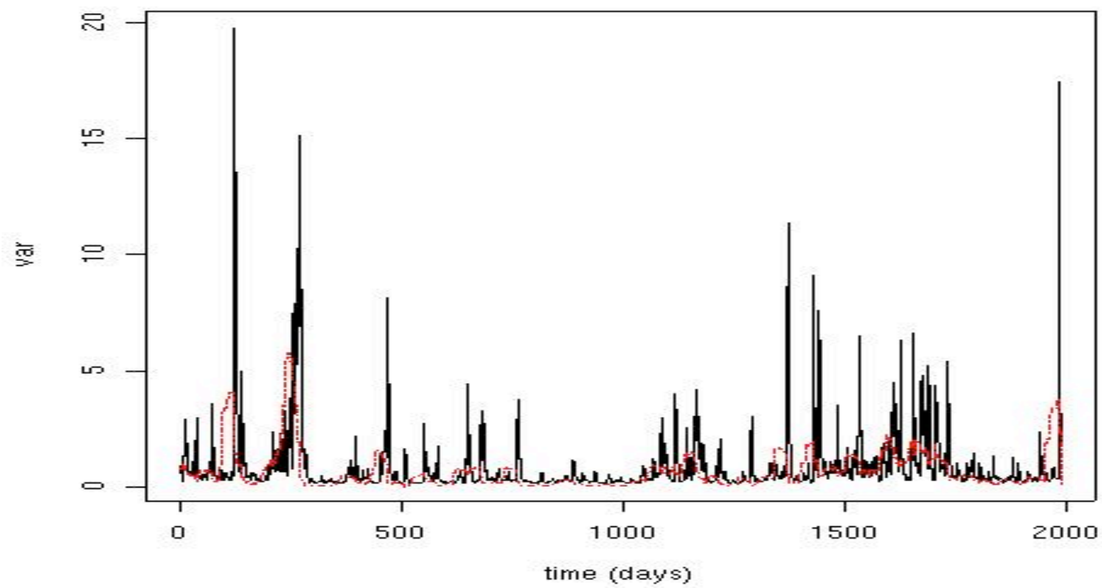
Variance of Mean Term-Structure



	Estimate	Std. Error
μ	-3.599e-04	9.201e-04
α_0	8.505e-05	2.276e-05
α_1	4.984e-01	4.378e-02
β_1	8.069e-01	6.485e-03

- $v_k := \mathrm{var}\,[x_k := \ln(\underline{\sigma}_{k+1} / \underline{\sigma}_k)]$,
- $x_{\underline{k}} = \mu + \varepsilon_k$, $v_k = \alpha_0 + \alpha_1 \varepsilon_k^2 + \beta_1 v_{k-1}$

Variance of Slope of Term-Structure

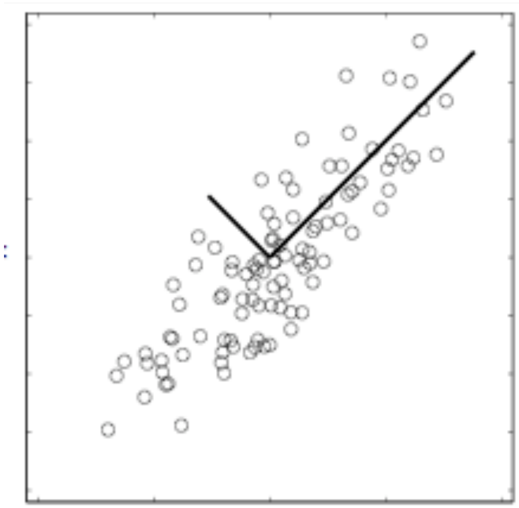


	Estimate	Std. Error
μ_Δ	-0.003558	0.011759
$\alpha_{\Delta,0}$	0.057078	0.008537
$\alpha_{\Delta,1}$	0.354185	0.045164
$\beta_{\Delta,1}$	0.651447	0.031985

- $w_k = \text{var}\,[y_k := \ln(\Delta\sigma_{k+1} / \Delta\sigma_k)]$
- $y_k = \mu_\Delta + \varepsilon_{\Delta,k}$, $w_k = \alpha_{\Delta,0} + \alpha_{\Delta,1}\varepsilon_{\Delta,k}^2 + \beta_{\Delta,1} w_{k-1}$

Principal Component Analysis

Finds uncorrelated axes of variation
(eigenvectors of covariance matrix)



$$\Sigma_{ij} = \mathbb{E}[(X_i - \overline{X_i})(X_j - \overline{X_j})]$$

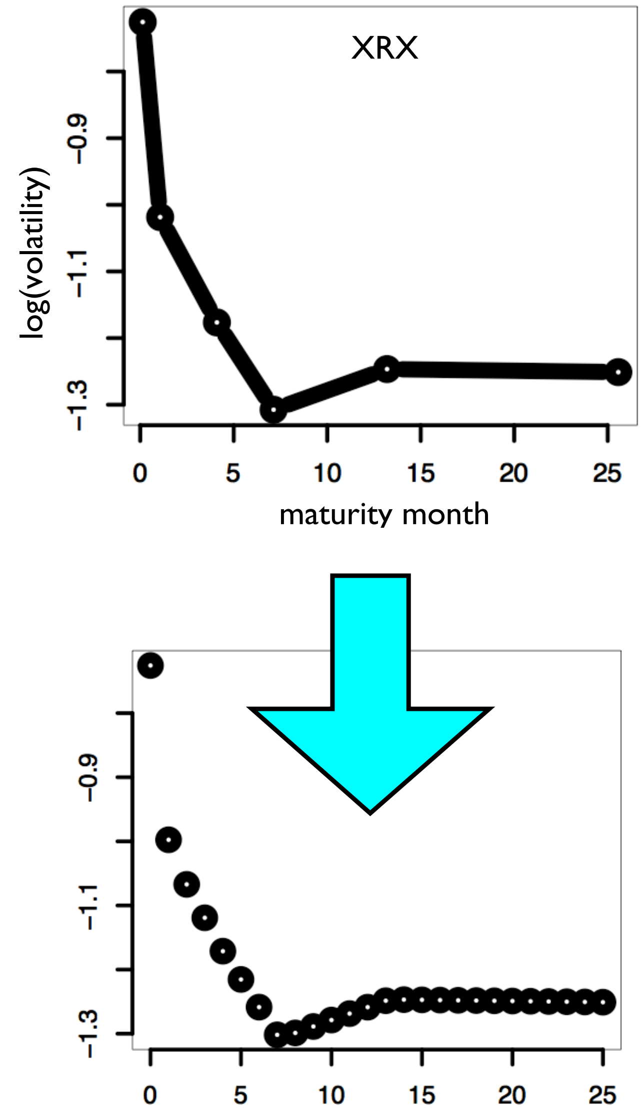
For us: determines dominant deformations

PCA: Implementation

Interpolate term structure curve from observed maturities and vols:

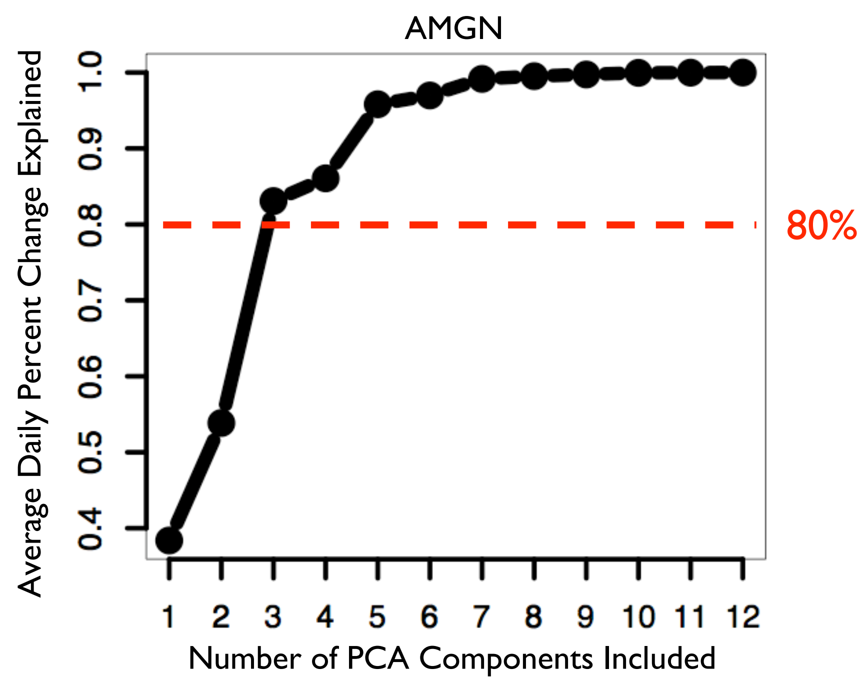
Sample curves at each month

Study daily displacement of sample points



Reducing Dimensionality

How much change is captured by the most dominant eigenvectors?



The first three capture 80% of the change.

Vega Hedging: Principles

How do you hedge against fluctuations
of the volatility surface?

Q-alpha-sigma and PCA can help:
they reduce the dimensionality of fluctuations.

Instead of hedging every strike and maturity (~30 options),
you only hedge the dominant PCA components
(in maturity space) (~3 such).

Vega Hedging: Practicalities

1. Compute the exposure of each option to the dominant PCA eigenvector.

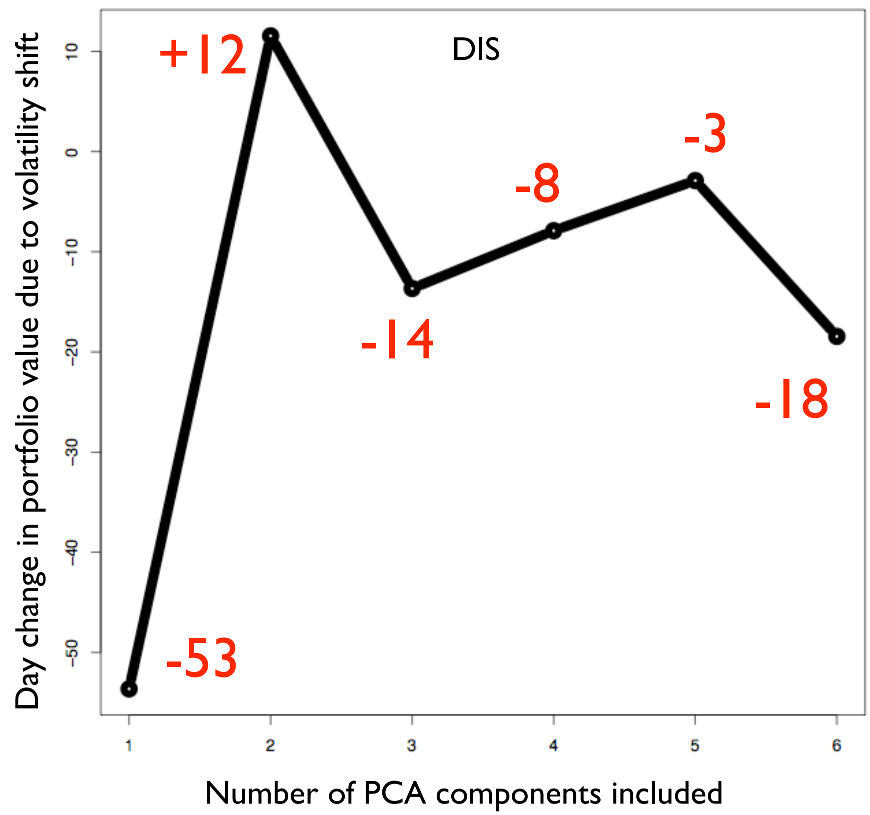
2. Compute your portfolio exposure, using the options you hold.

3. Buy options to cancel this exposure (as cheaply as possible)

Large Vega Fluctuations

Portfolio exposure does not smoothly asymptote

Difficult to hedge: must include many eigenvectors



Potential Solution: Perform PCA on vega-convoluted surface?

Conclusions

Modeled volatility surface dynamics using GARCH

Performed PCA analysis of volatility surface fluctuations

Attempted simple vega-hedging strategy

Future Directions

Understand dynamics better

Study convolution of shifts and vega

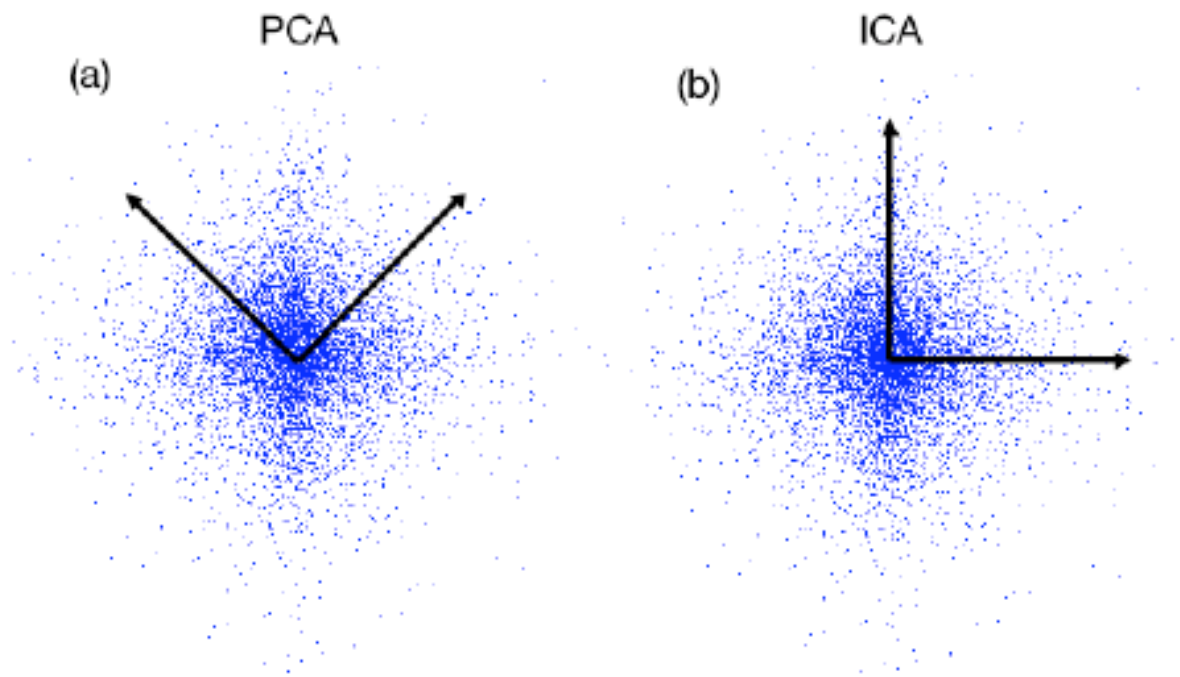
ICA: an alternative way?

Acknowledgement

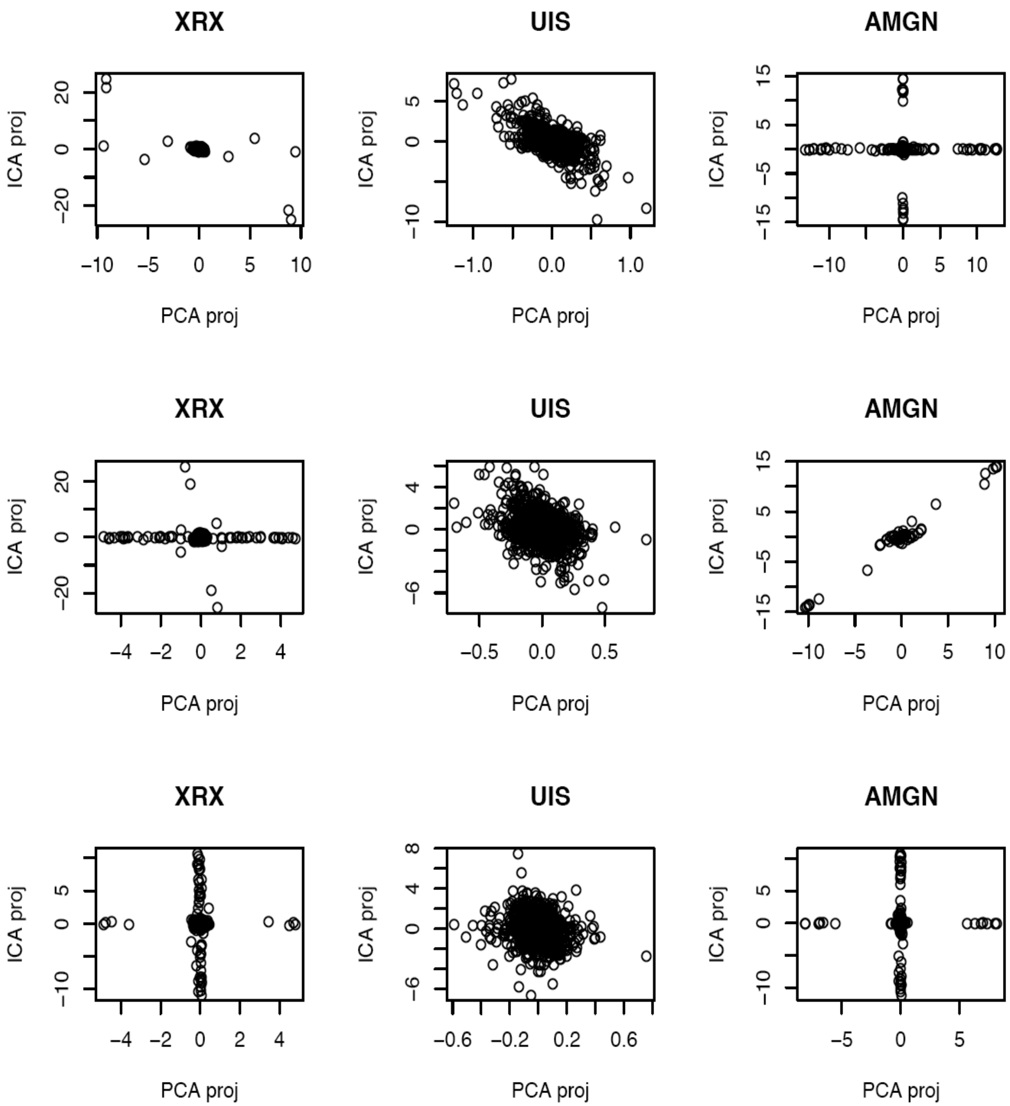
- Thanks to Kay Giesecke, Benjamin Armbruster and
- EvA: Christian Silva, Lisa Borland and Jeremy Evnine.

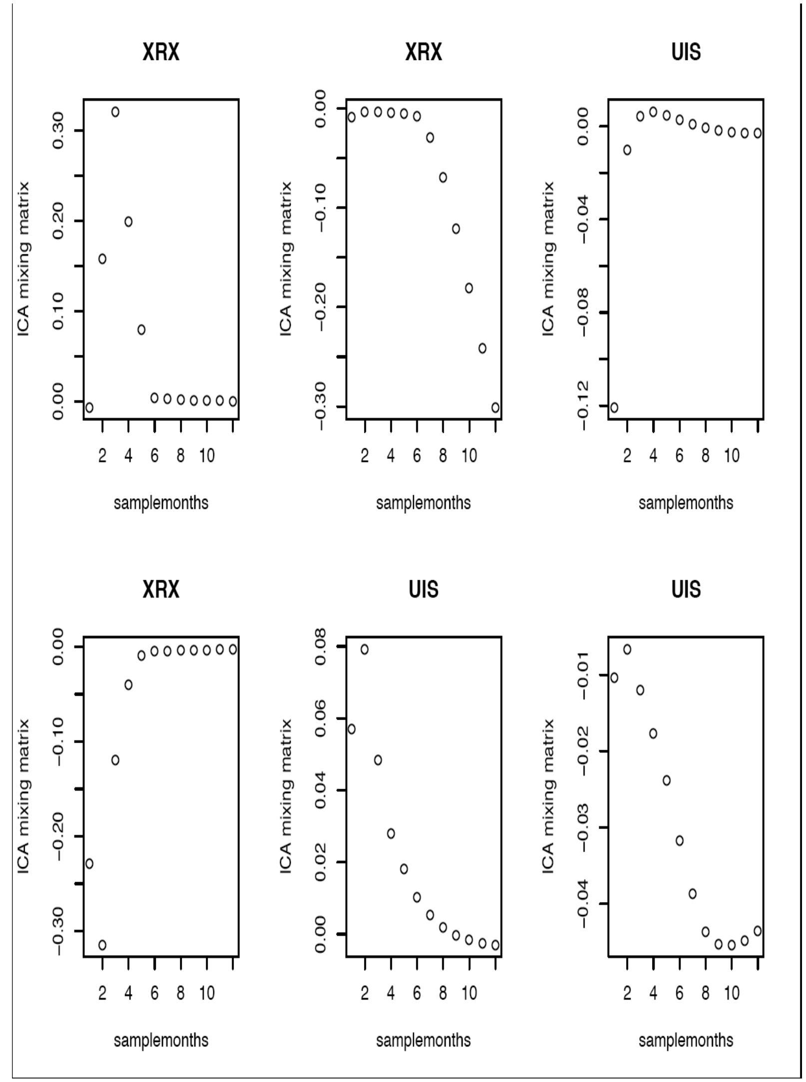
Appendix

ICA - Alternative Way?



- To find a transformation of the data in which the components are statistically as independent from each other as possible





- ICA proj condense to PCA proj.
- ICA vectors still preserve the sharp peaks.

Box-Ljung & Shapiro-Wilk Test

- XRX
  - BL: X-squared = 384.0319, df = 1, p-value < 2.2e-16
  - SW: W = 0.3153, p-value < 2.2e-16
- UIS
  - X-squared = 425.4982, df = 1, p-value < 2.2e-16
  - W = 0.7293, p-value < 2.2e-16
- AMGN
  - X-squared = 144.5561, df = 1, p-value < 2.2e-16
  - W = 0.816, p-value < 2.2e-16
- DIS
  - X-squared = 303.0574, df = 1, p-value < 2.2e-16
  - W = 0.8836, p-value < 2.2e-16
- PG
  - X-squared = 351.1064, df = 1, p-value < 2.2e-16
  - W = 0.7998, p-value < 2.2e-16