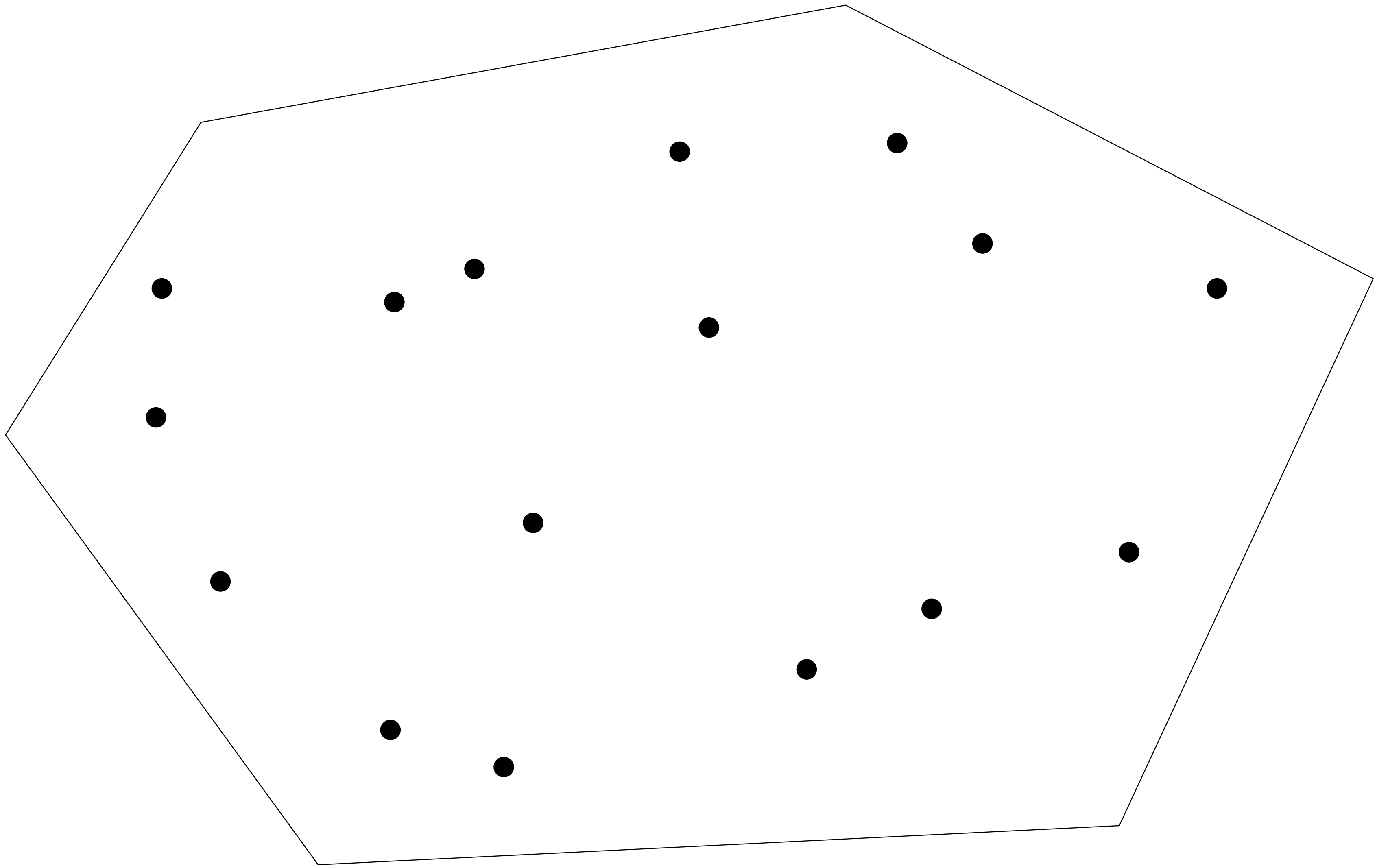
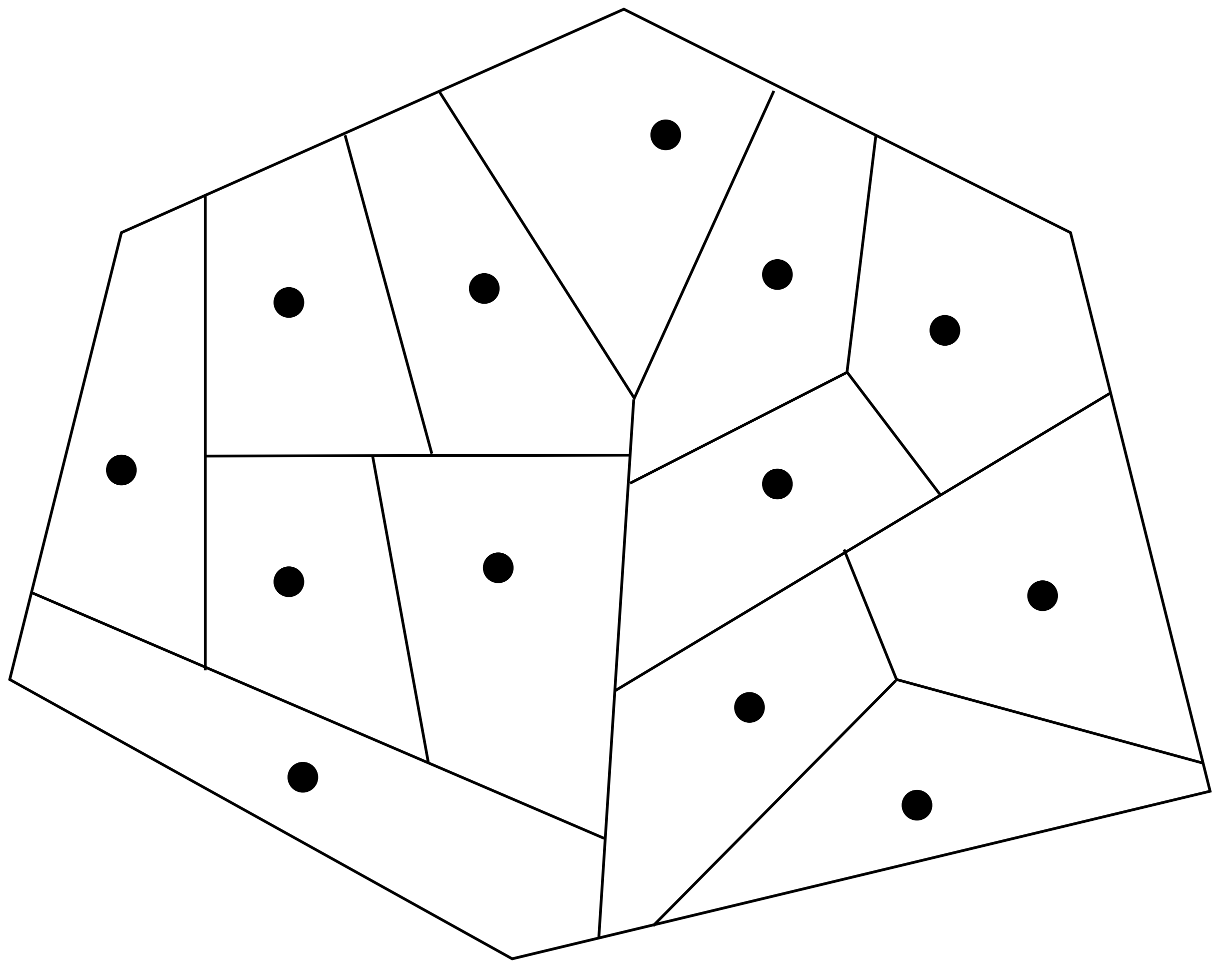
Finding Equitable Convex Partitions of Points in a Polygon Efficiently

John Gunnar Carlsson, Benjamin Armbruster, Yinyu Ye

The problem:



Find a partition of C into convex pieces of equal area, each containing one point.

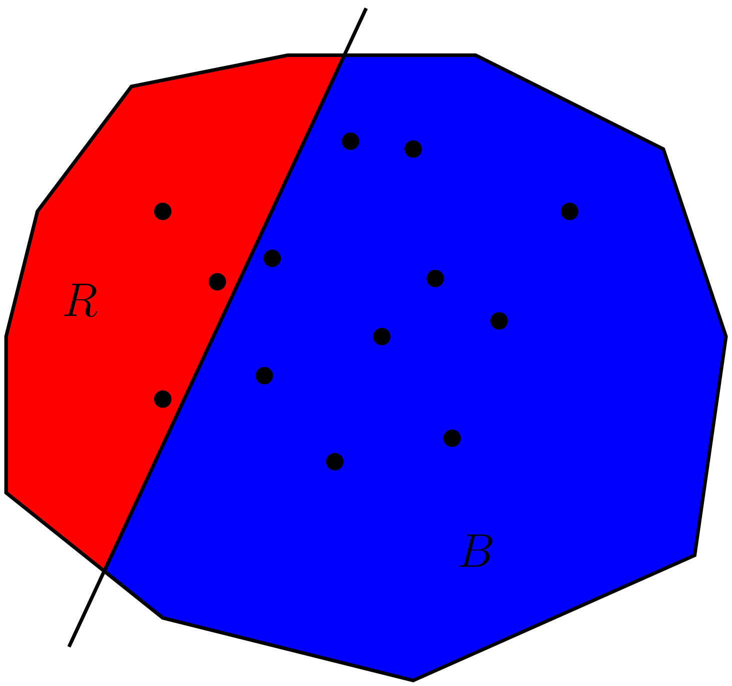


C: convex polygon with m vertices

P: set of n points in general position

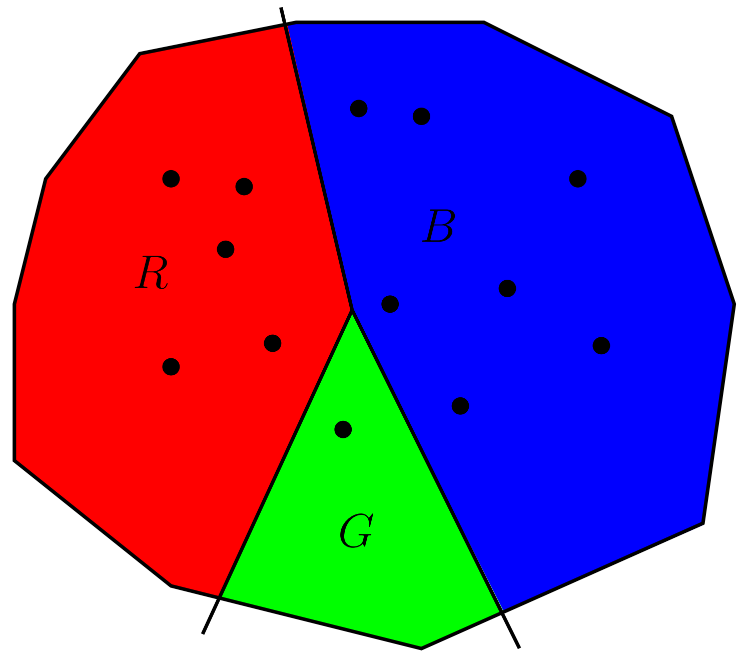
Claim: In $O(Nn \log N)$ we can find a partition, where $N = m + n$.

Equitable 2-partition



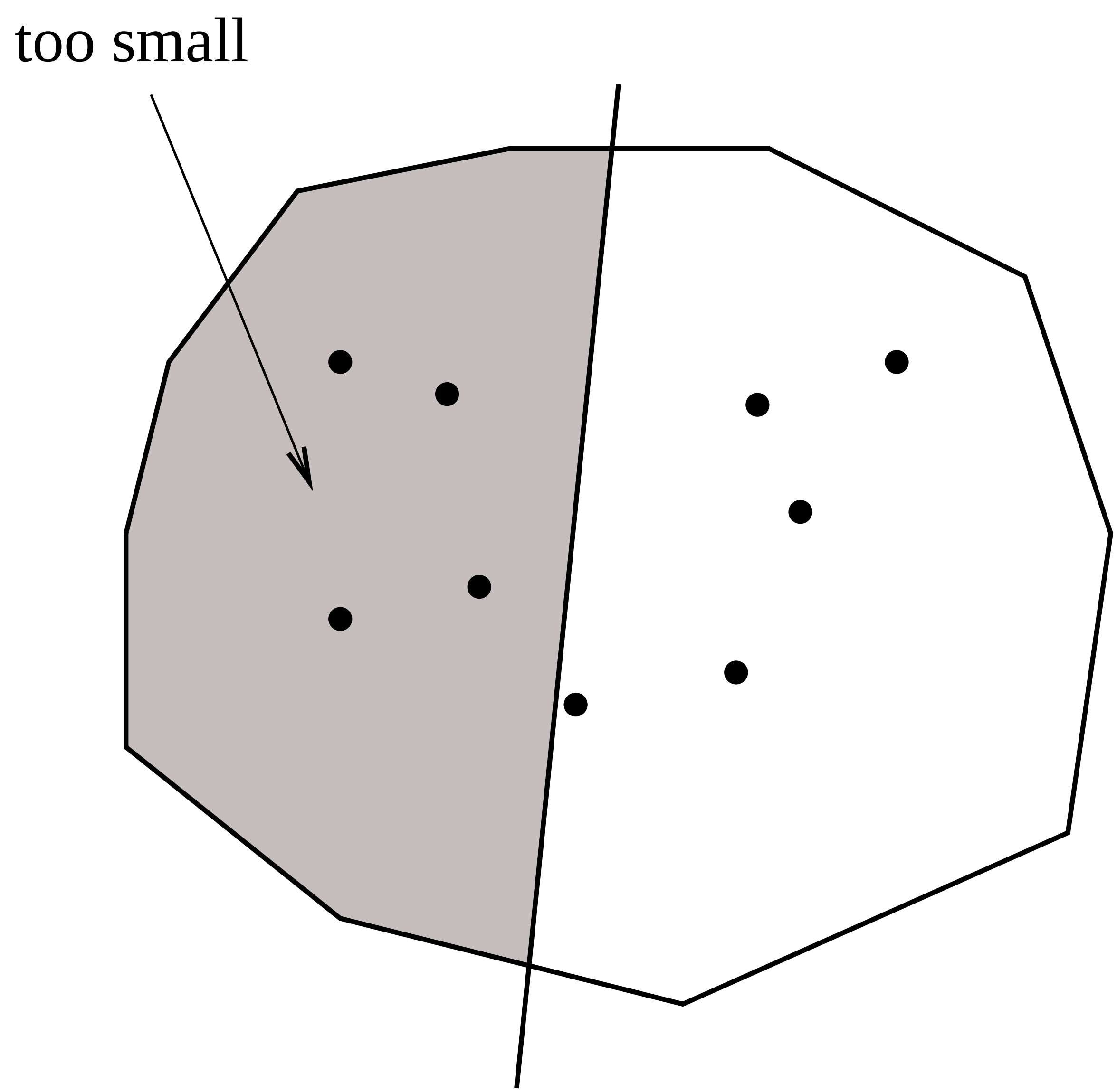
$$\frac{\text{Area}(R)}{\text{points in } R} = \frac{\text{Area}(B)}{\text{points in } B}$$

Equitable 3-partition

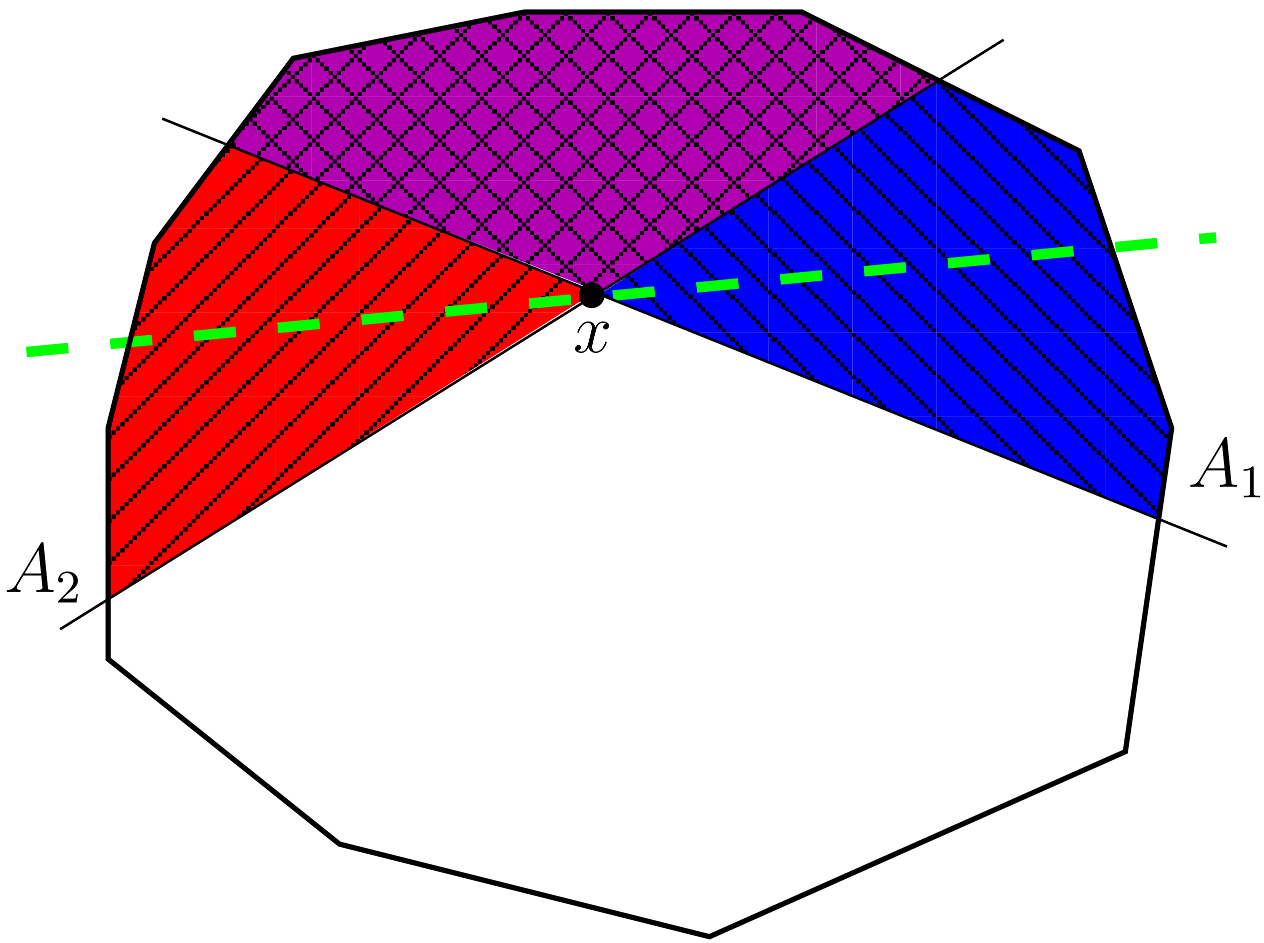


$$\frac{\text{Area}(R)}{\text{points in } R} = \frac{\text{Area}(B)}{\text{points in } B} = \frac{\text{Area}(G)}{\text{points in } G}$$

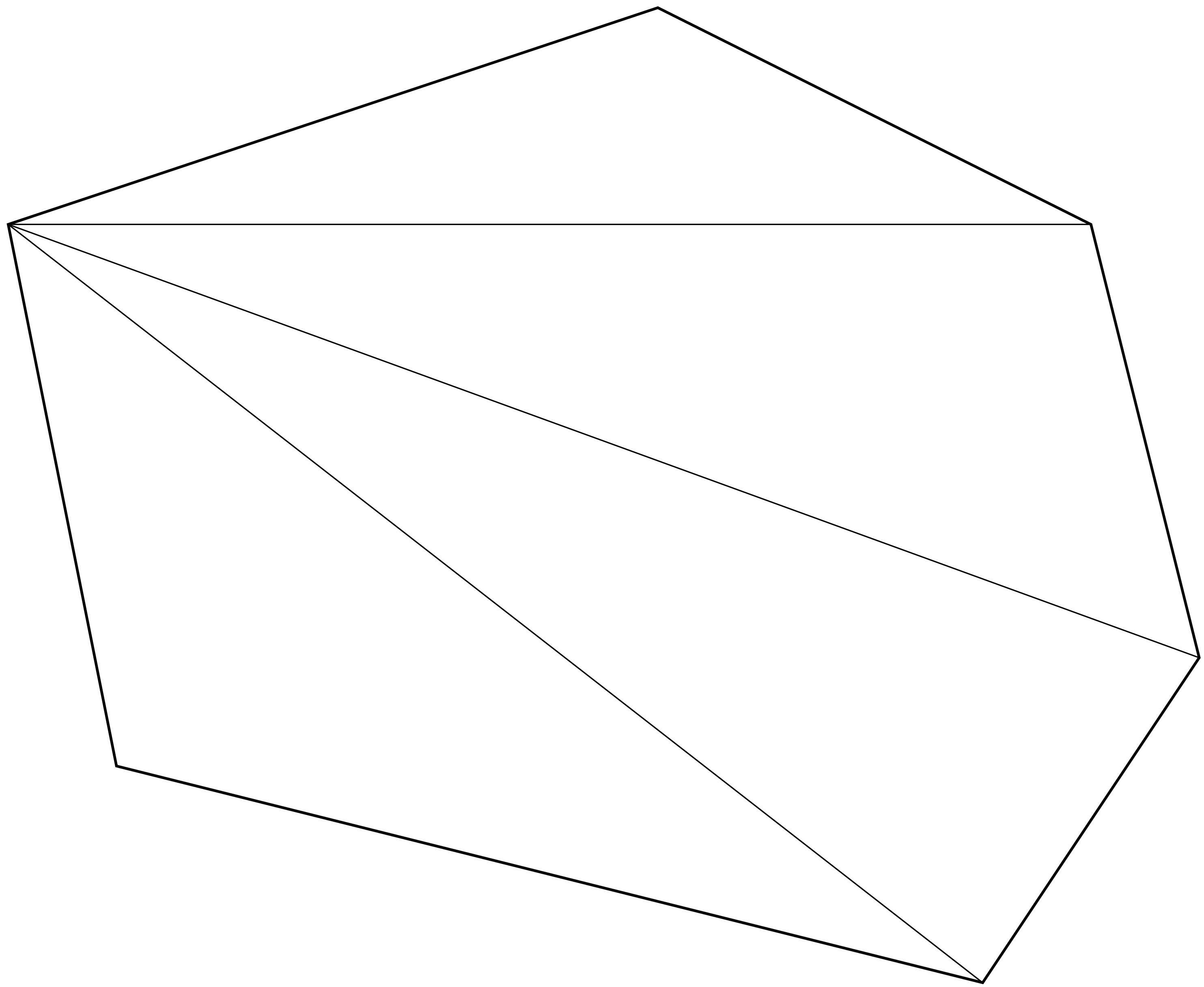
Helper Lemma 1: If R contains q points, $q \leq \lfloor n/2 \rfloor$, and area $(R) <$ area $(C) \cdot q/n$, then we can find an equitable 2-partition in $O(N \log N)$ time.



Helper Lemma 2: If two half-spaces through x cut-off areas A_1 and A_2, then for any $A \in [A_1, A_2]$, we can find a half-space in between cutting off area A in $O(m)$ time.

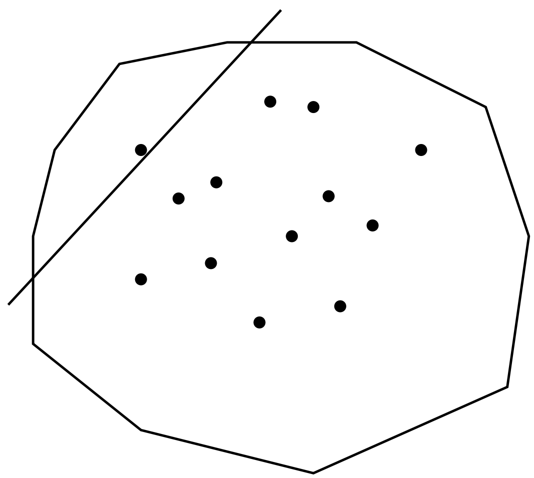


Helper Lemma 3: We can calculate the area of a convex polygon with k sides in $O(k)$ time.

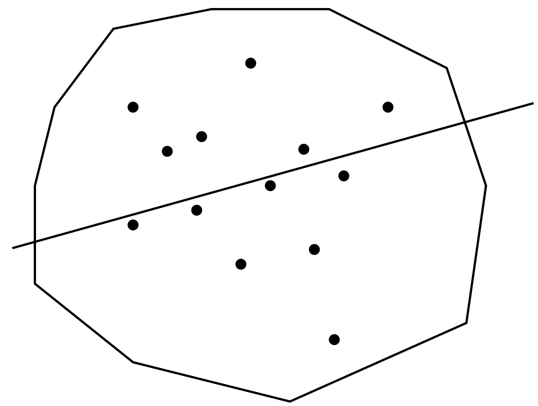


Main Claim:

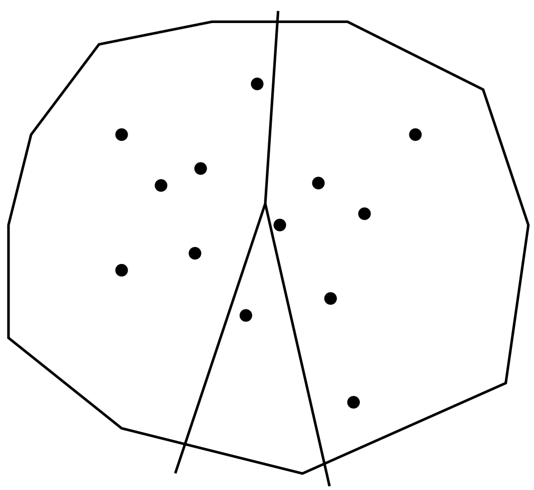
$n = 2q + 1$ points. In $O(N \log N)$ time, we find an equitable 2- or 3-partition:



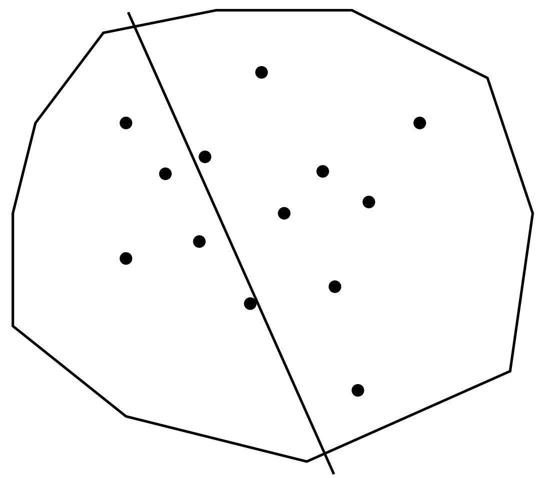
$\{1, 2q\}$ partition



$\{q, q+1\}$ partition



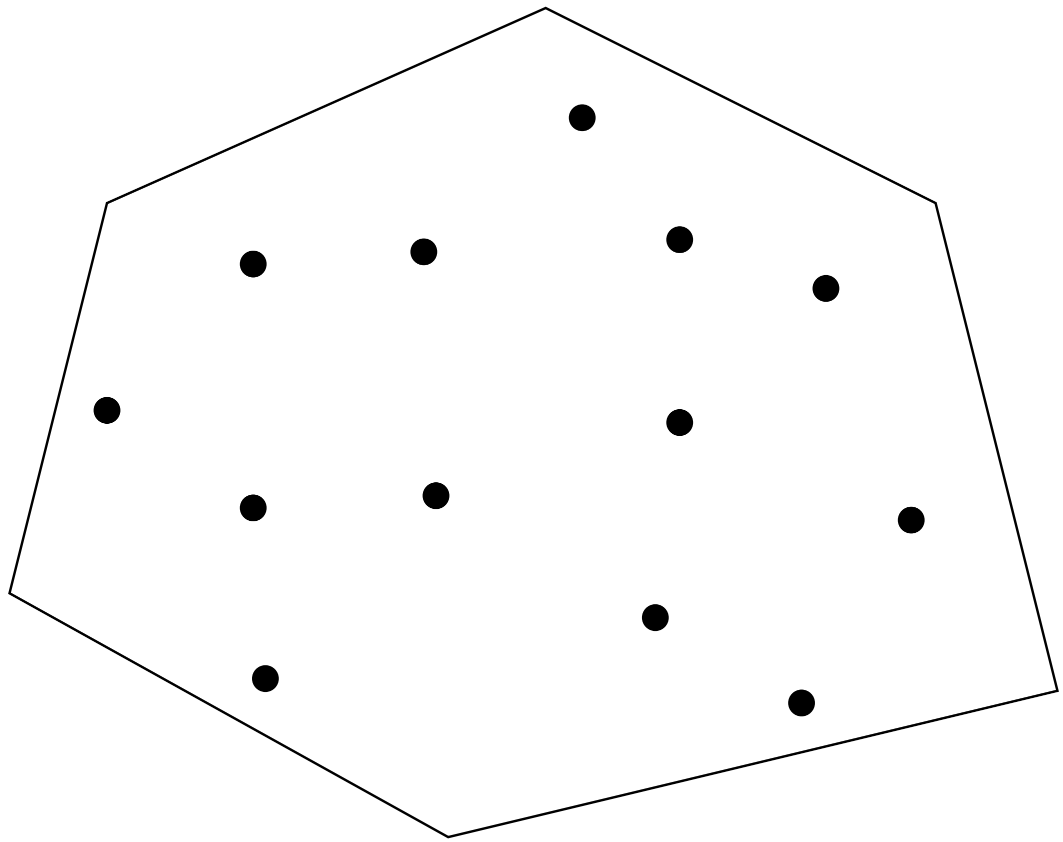
$\{q, q, 1\}$ partition



$\{k_1, k_2\}$ partition, with $k_1 + k_2 = n$

Region Partition Algorithm:

Find a 2- or 3-partition using Lemma 1. Solve smaller sub-problems recursively.



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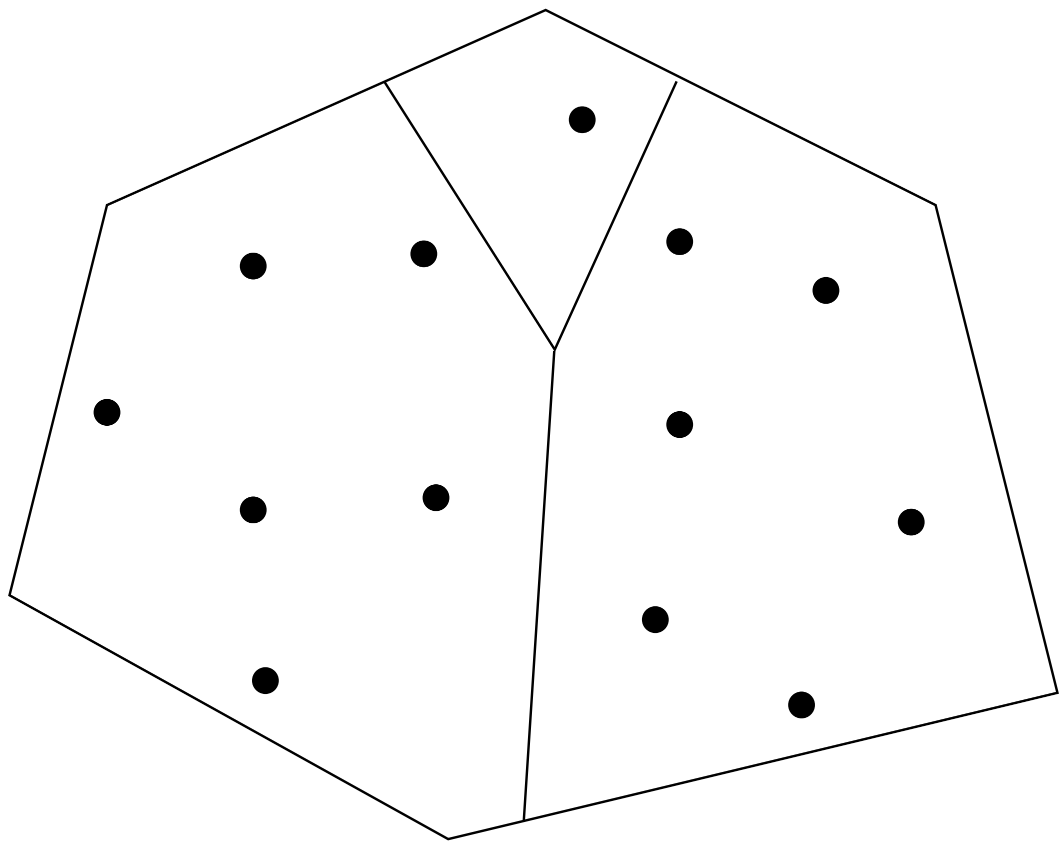

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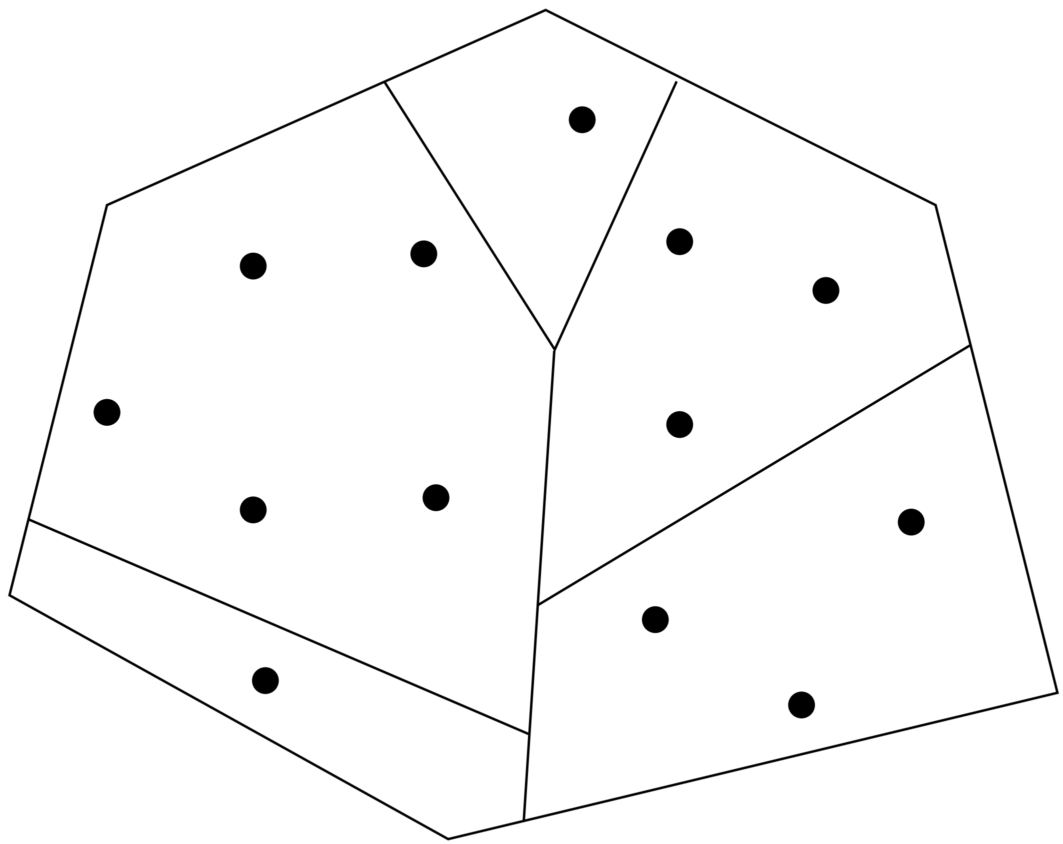

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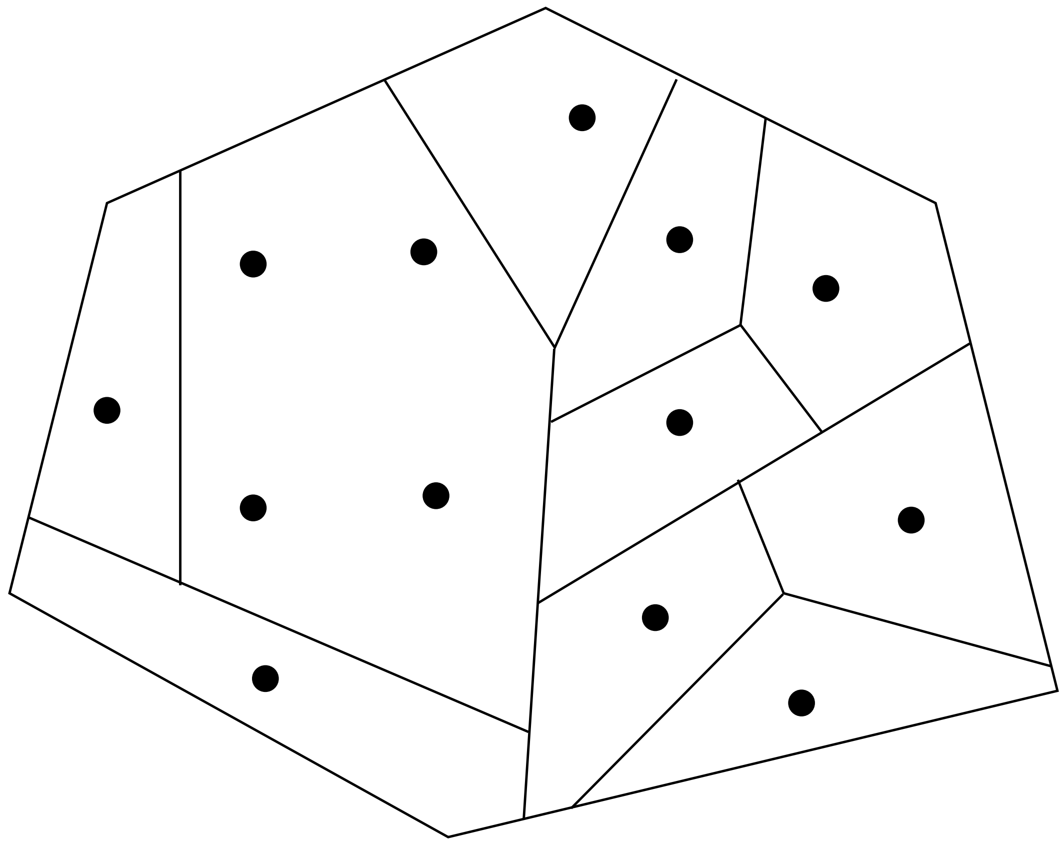

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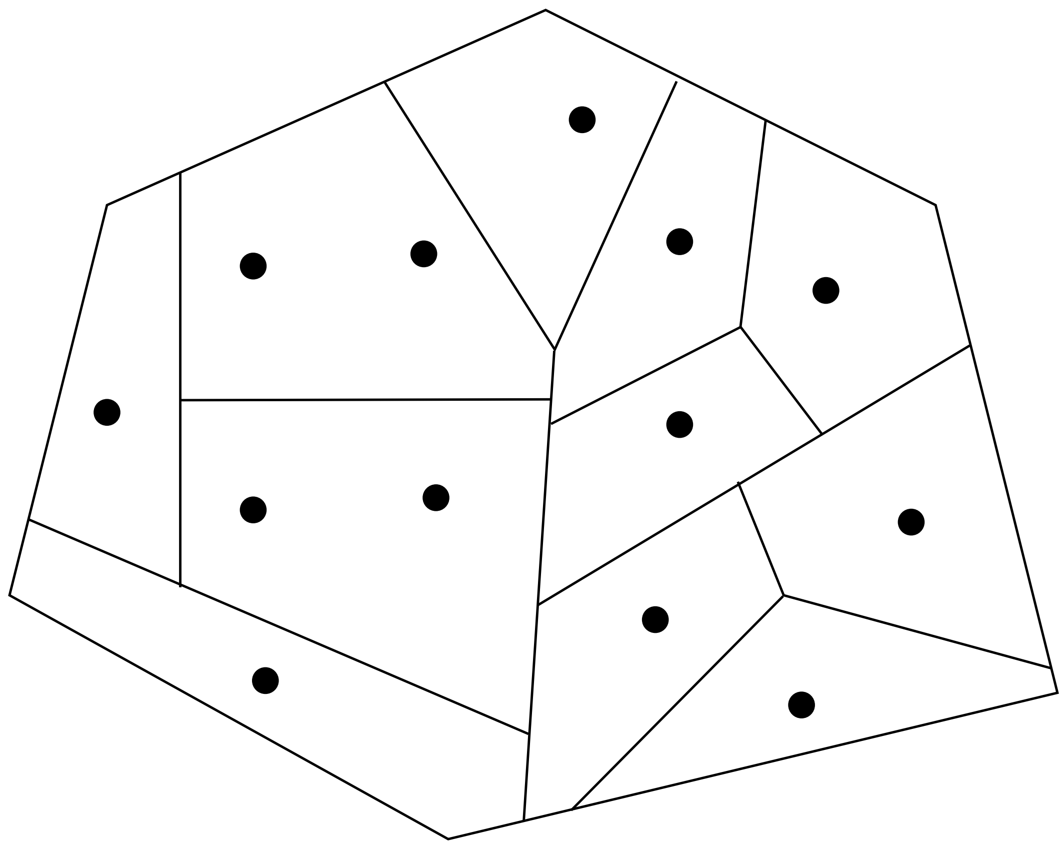
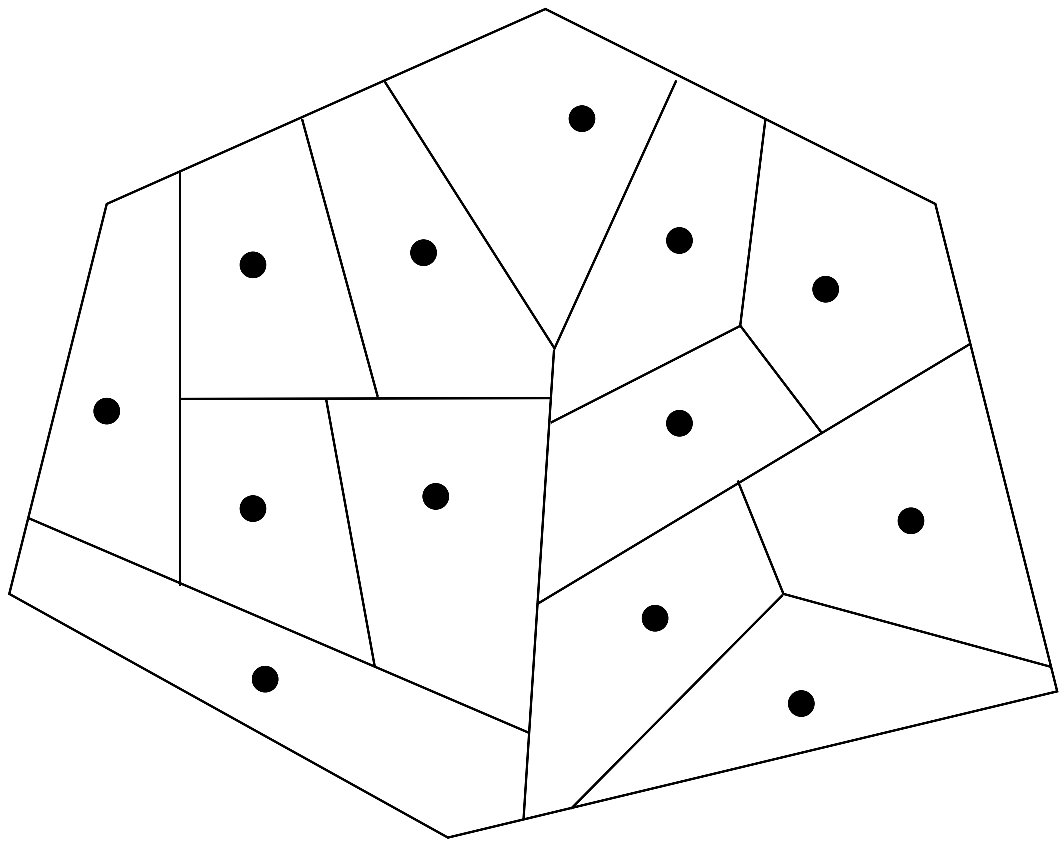
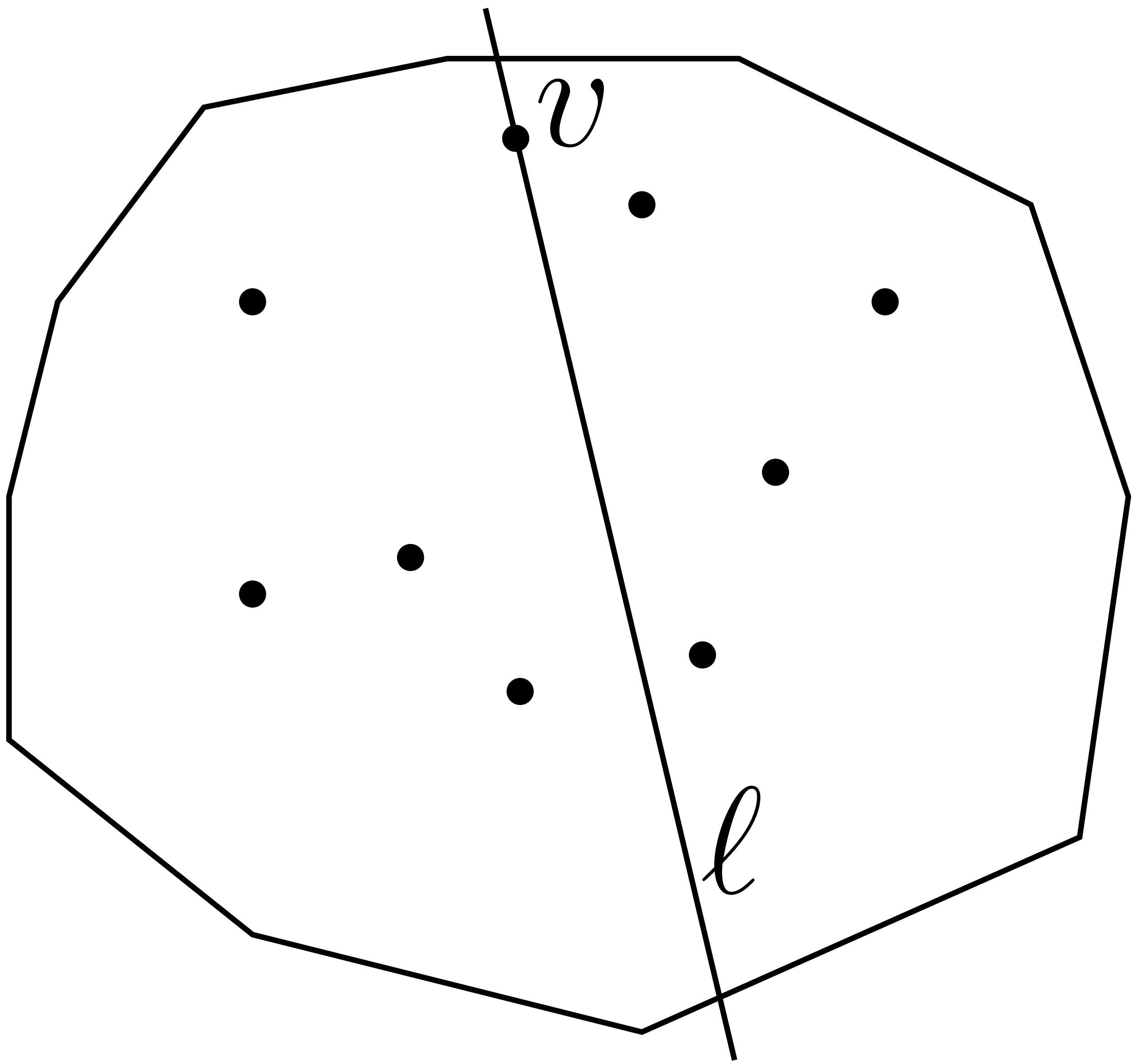
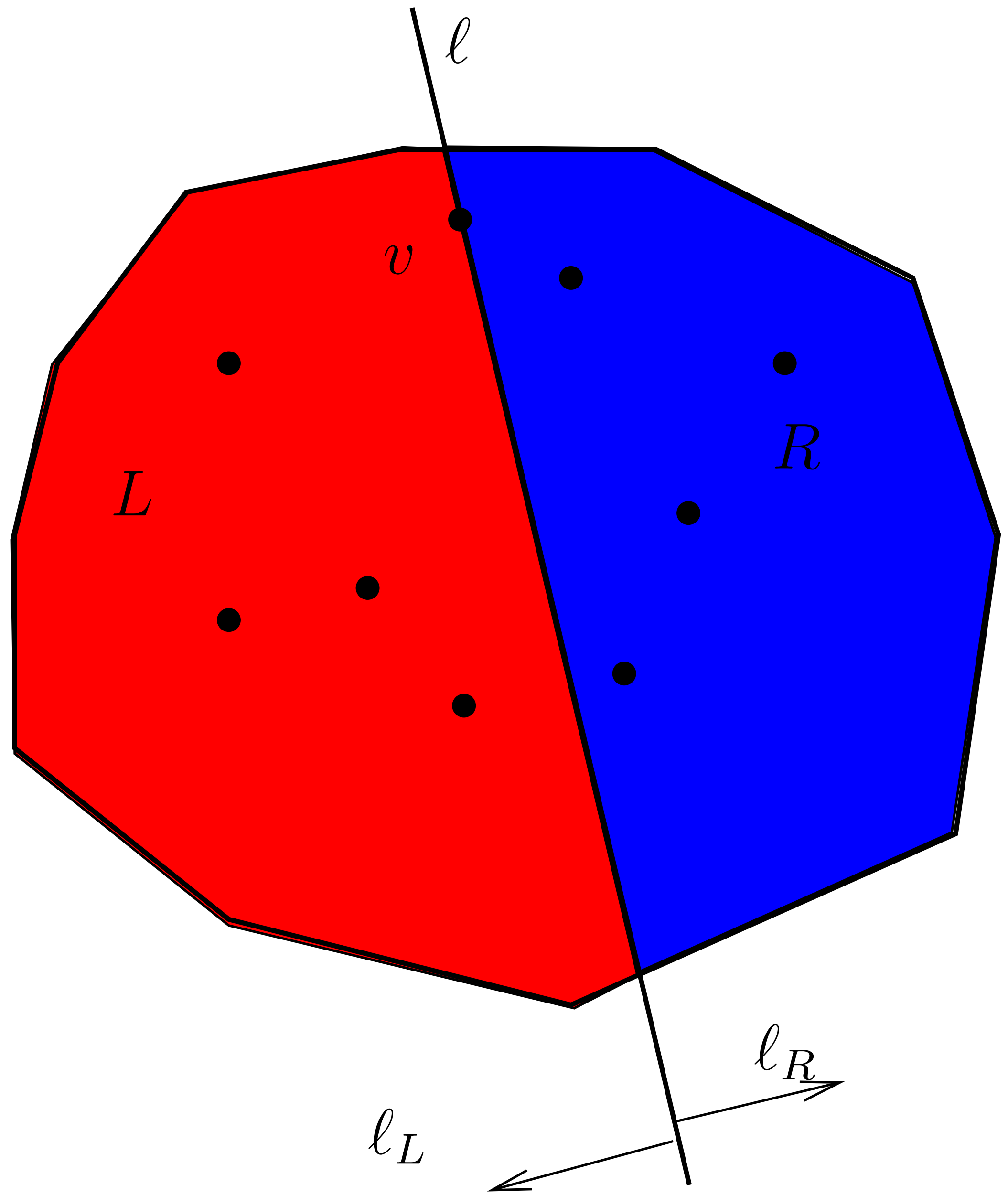


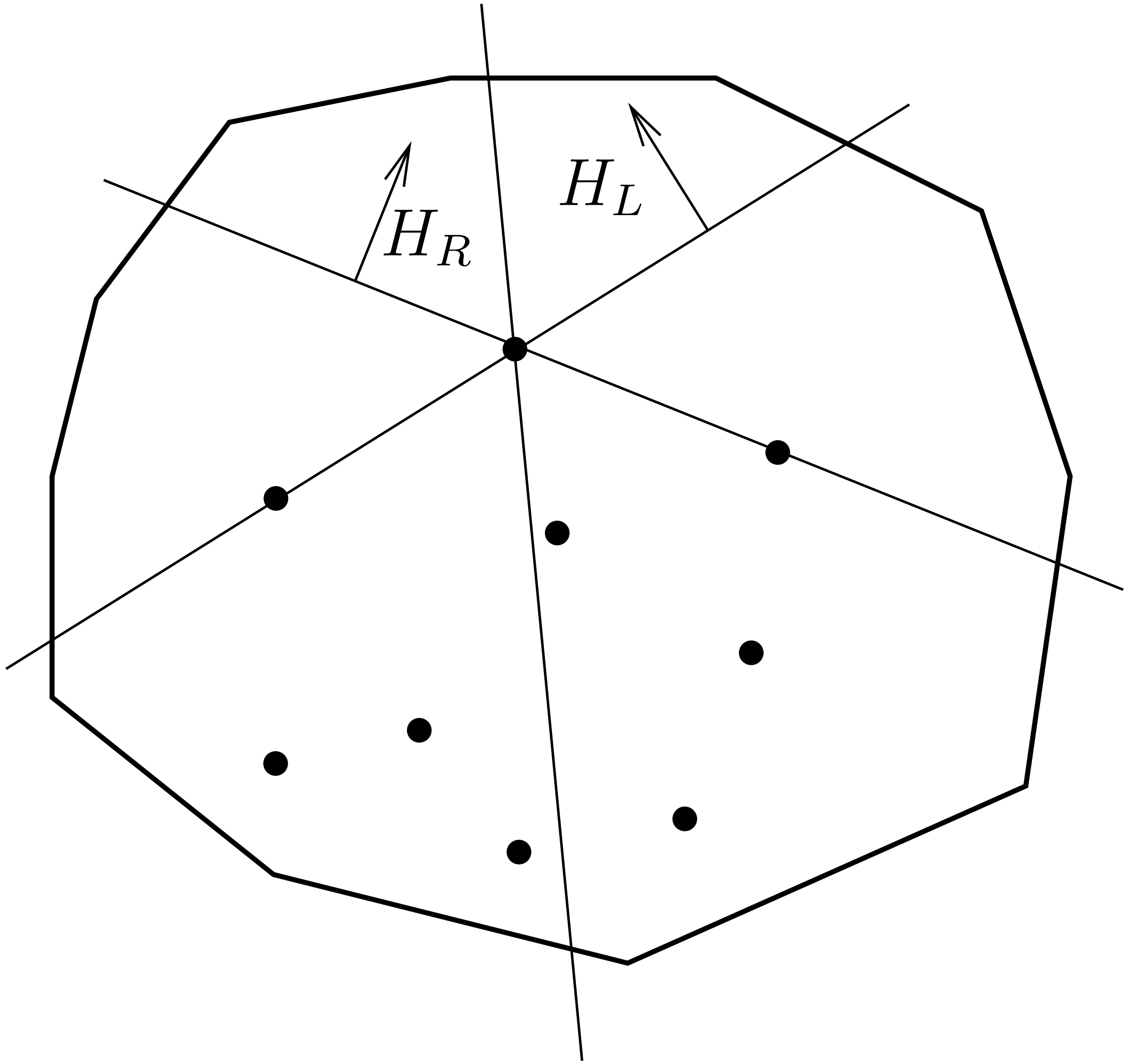


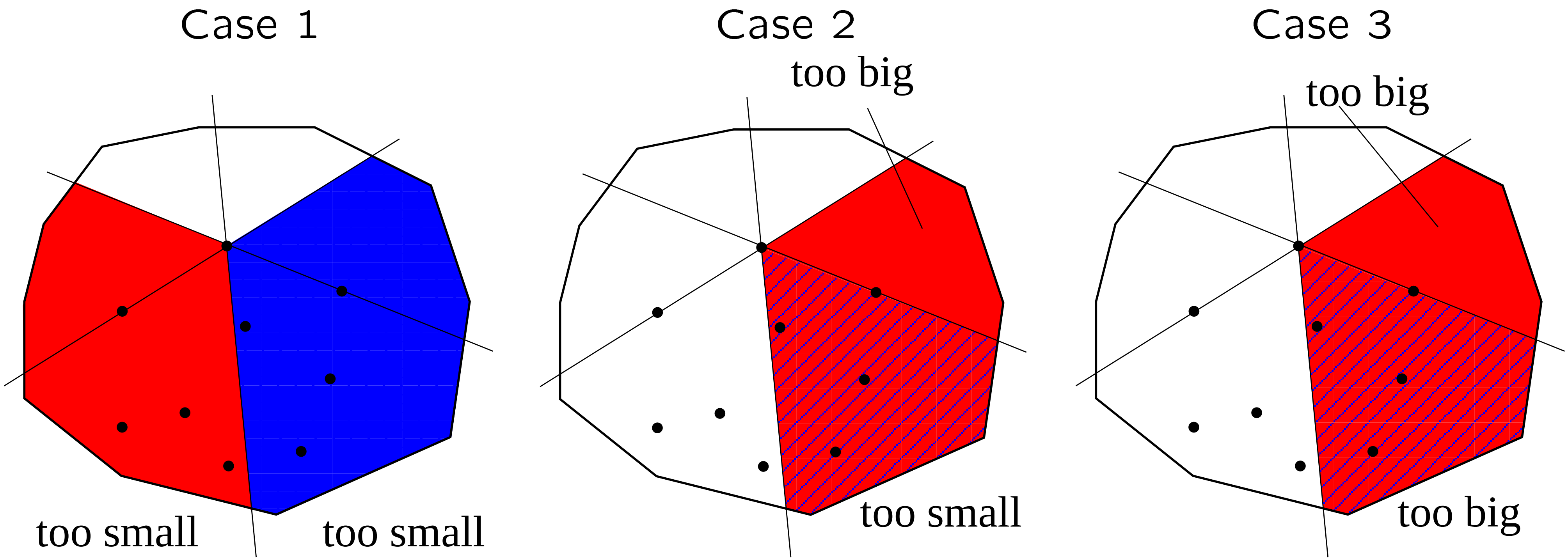
Proof outline of Main Claim



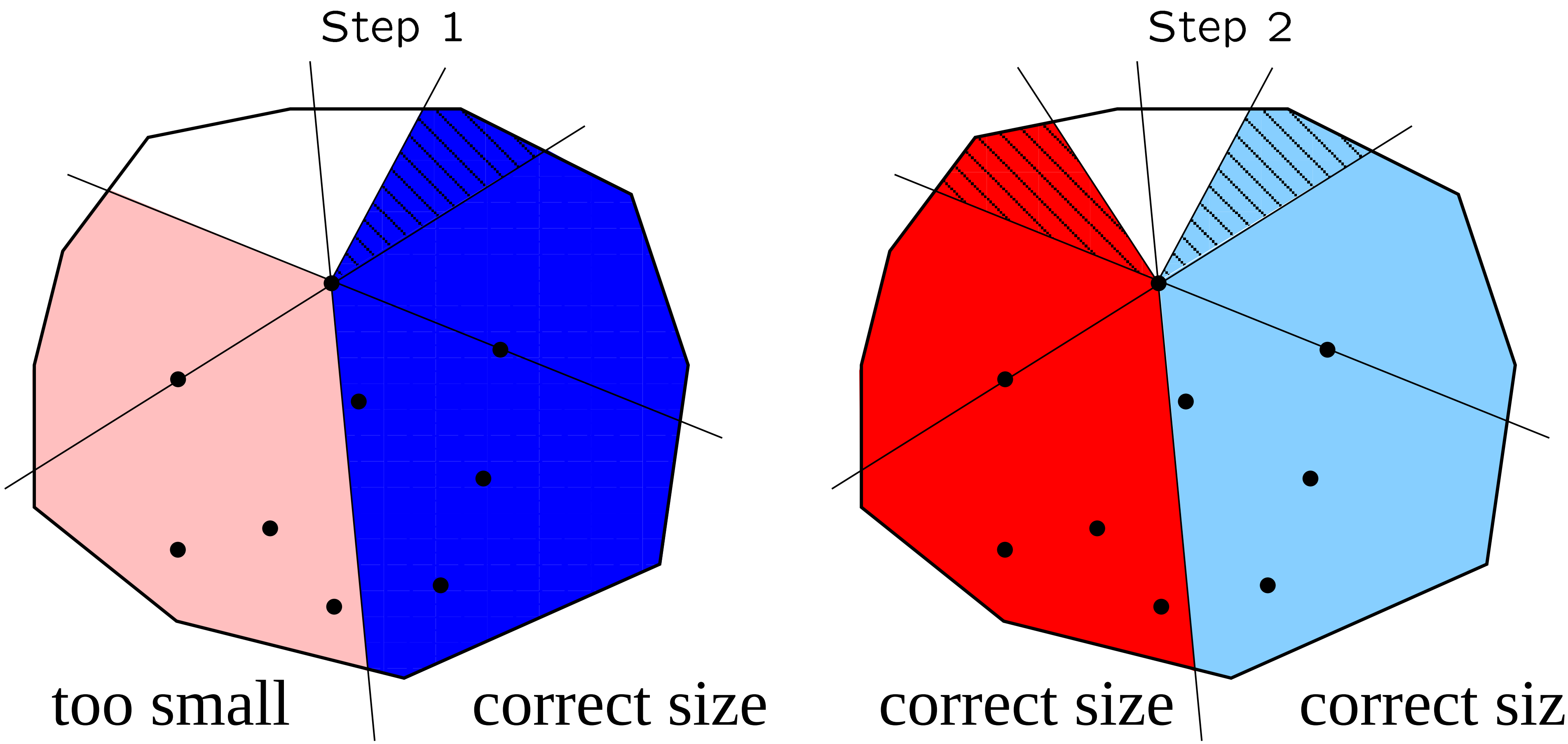
$area(L) \in area(C)[\frac{q}{n}, \frac{q+1}{n}]$ and $area(R) \in area(C)[\frac{q}{n}, \frac{q+1}{n}]$




H_R
H_L

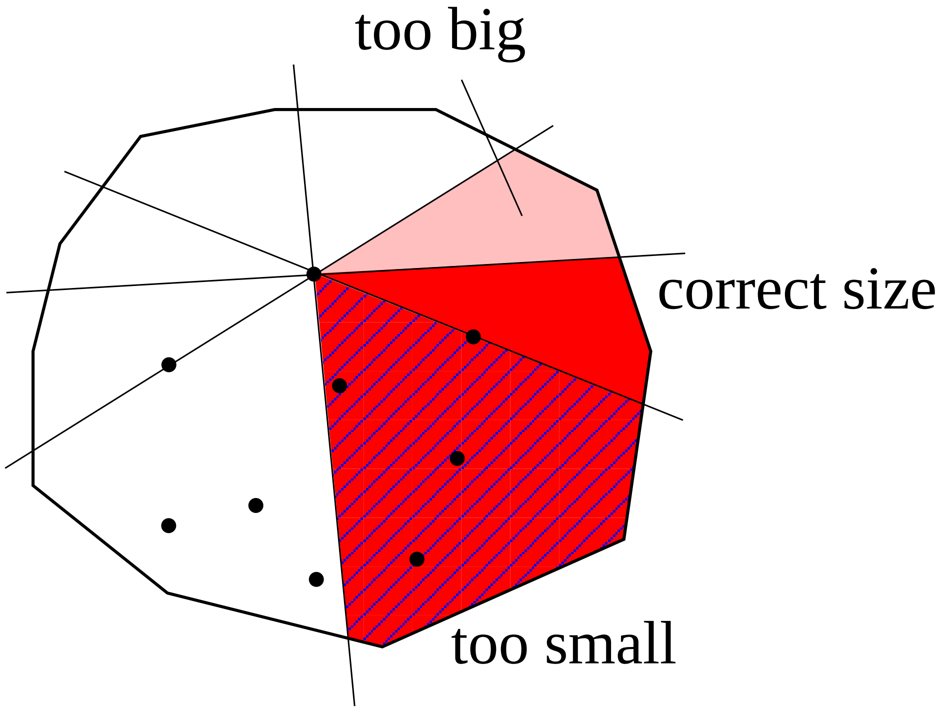

Case 1
too small
too small
Case 2
too big
too small
Case 3
too big
too big

Case 1

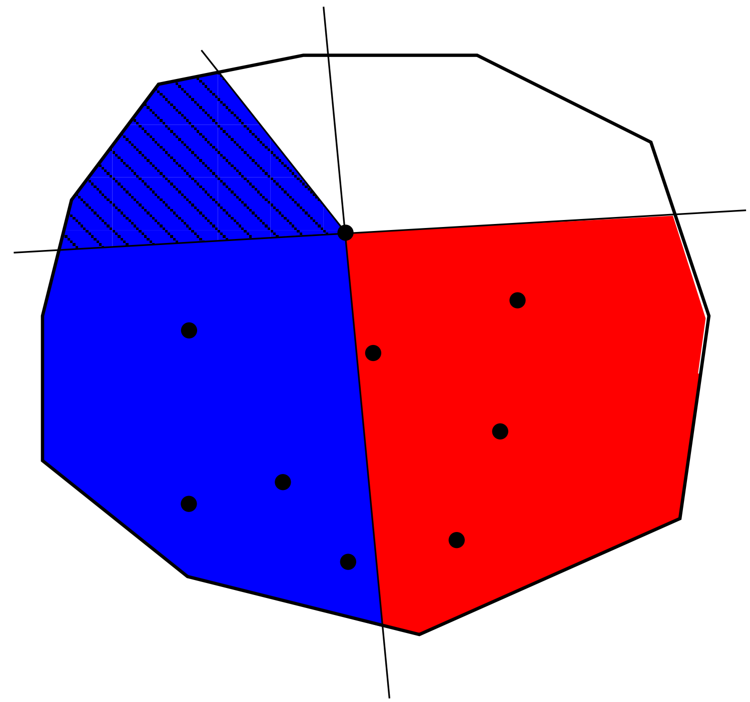

Step 1
Step 2
too small
correct size
correct size
correct size

Case 2

Step 1

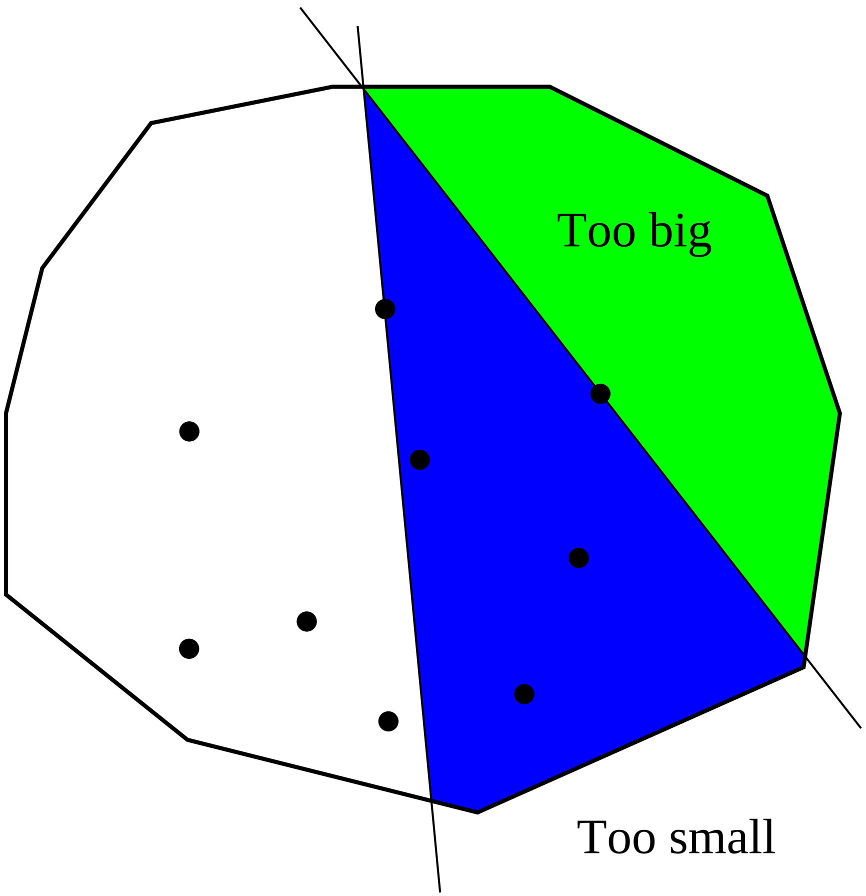


Step 2

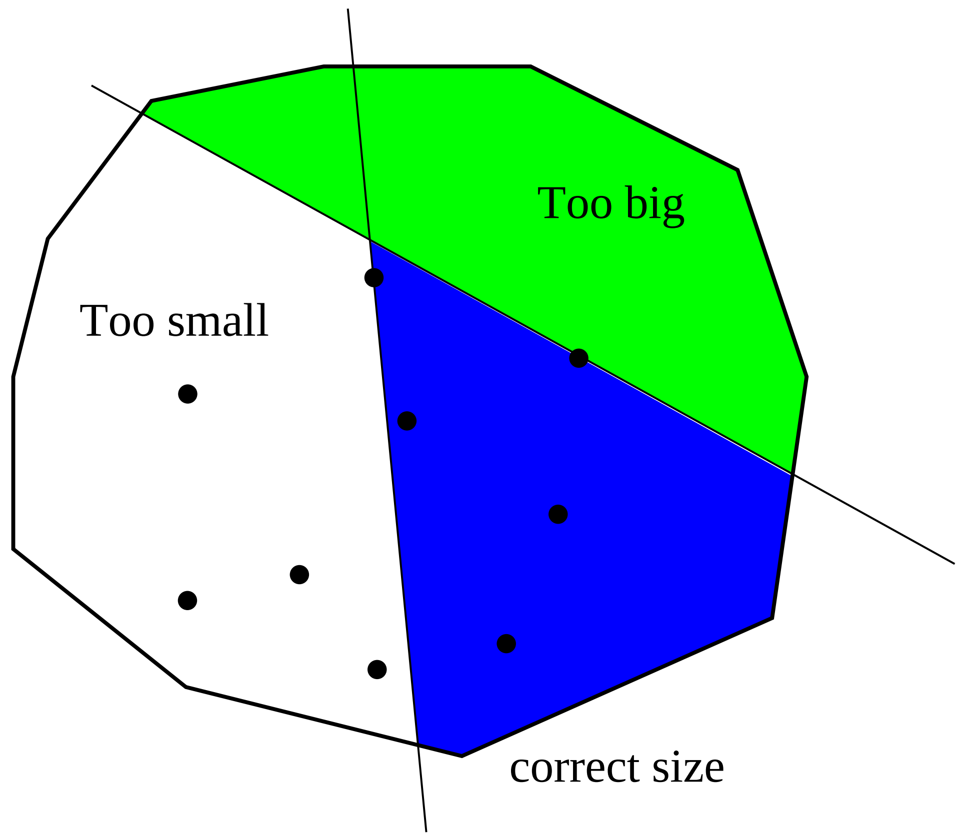


Case 3

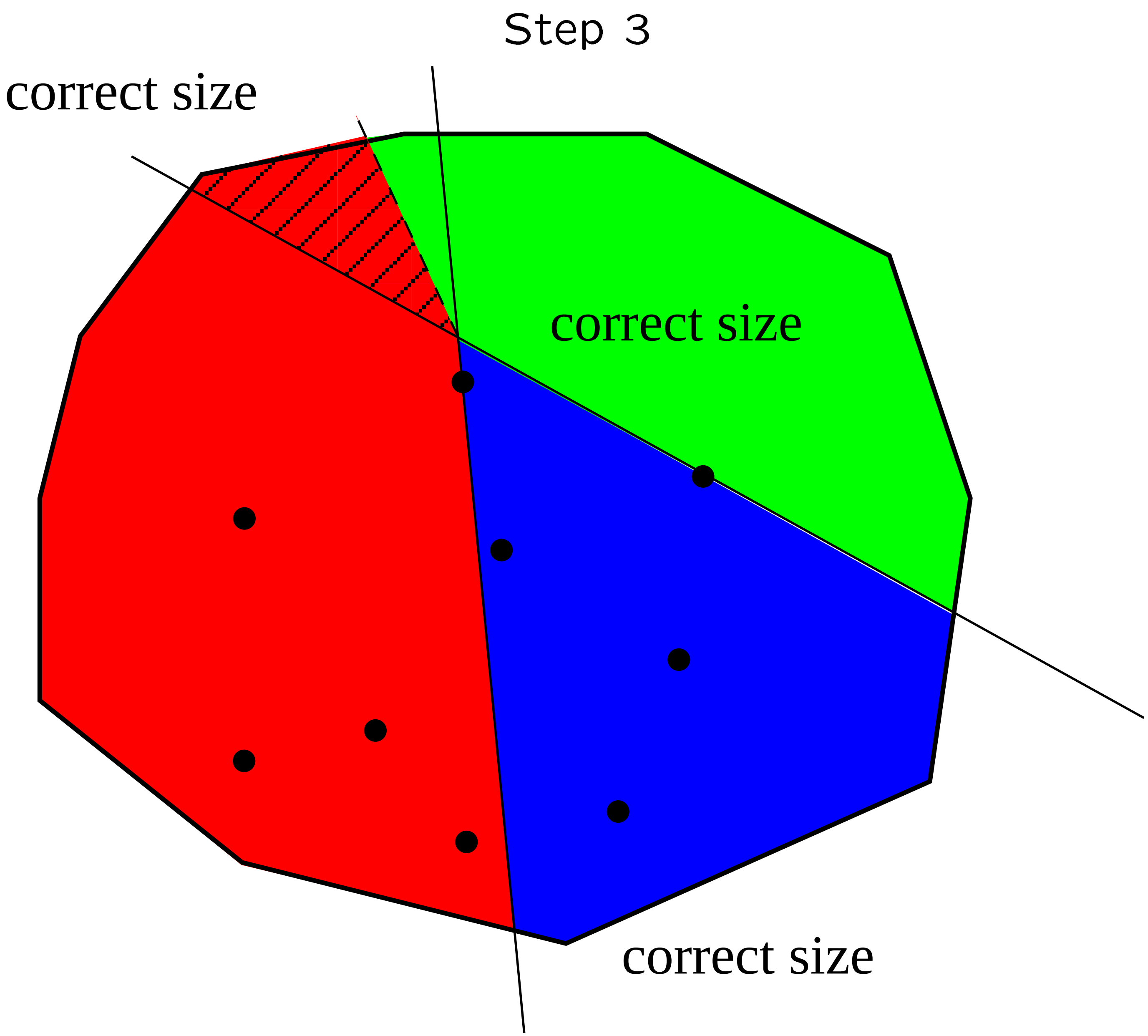
Step 1



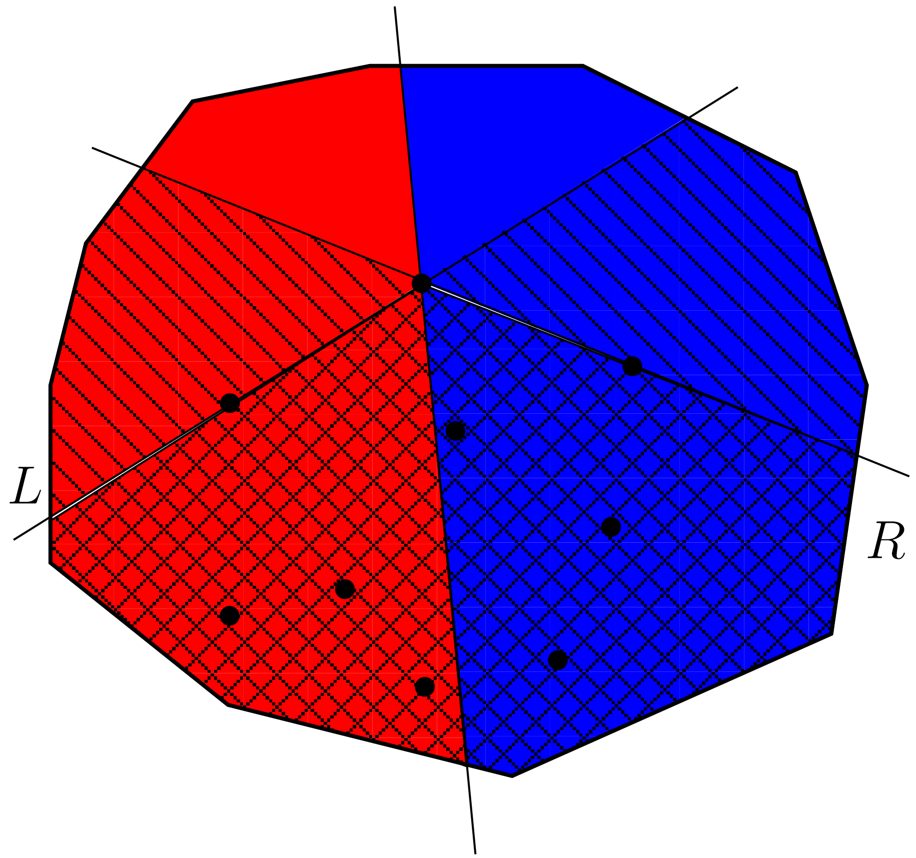
Step 2



Case 3, continued



Overview of Cases

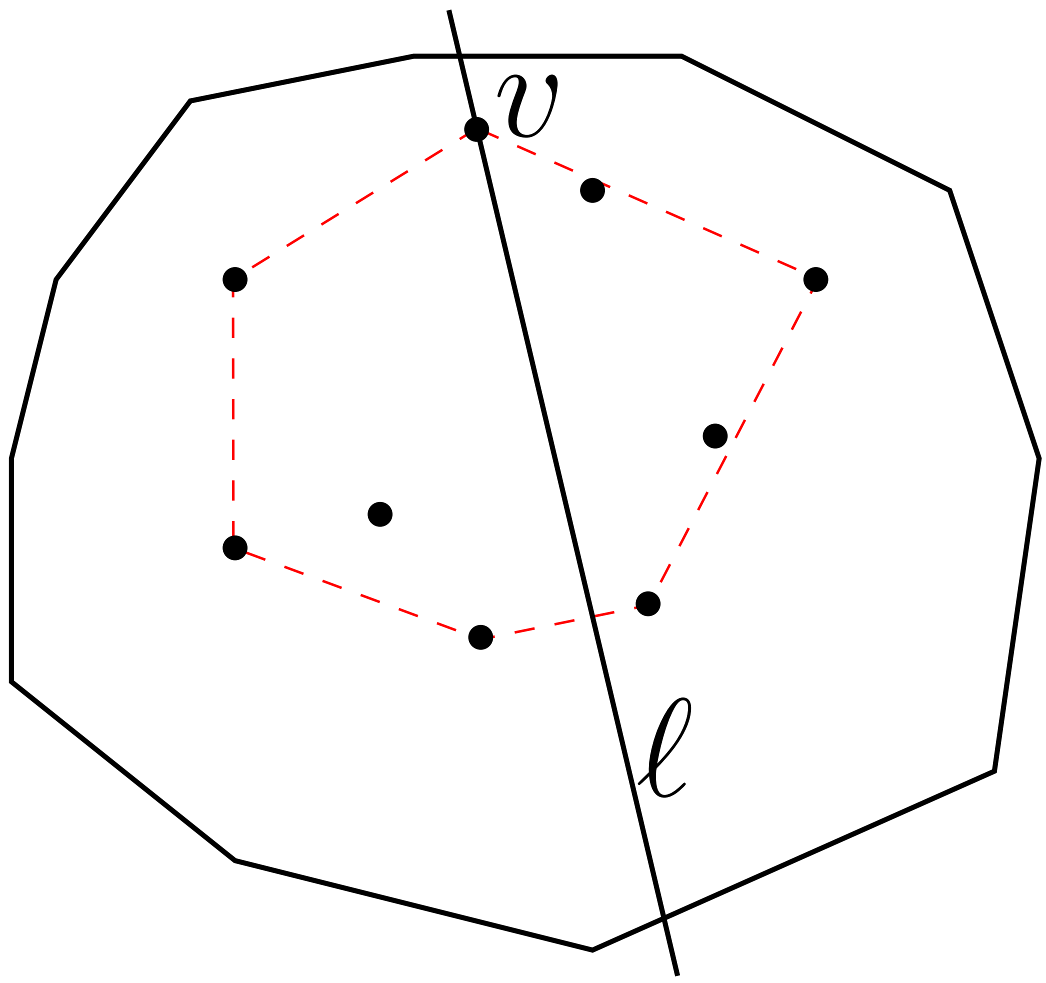


R

L		(−,−)	(−,+)	(+,+)
	(−,−)	1	2	3
	(−,+)	2	2	2,3
	(+,+)	3	2,3	3

More detailed proof of Main Claim

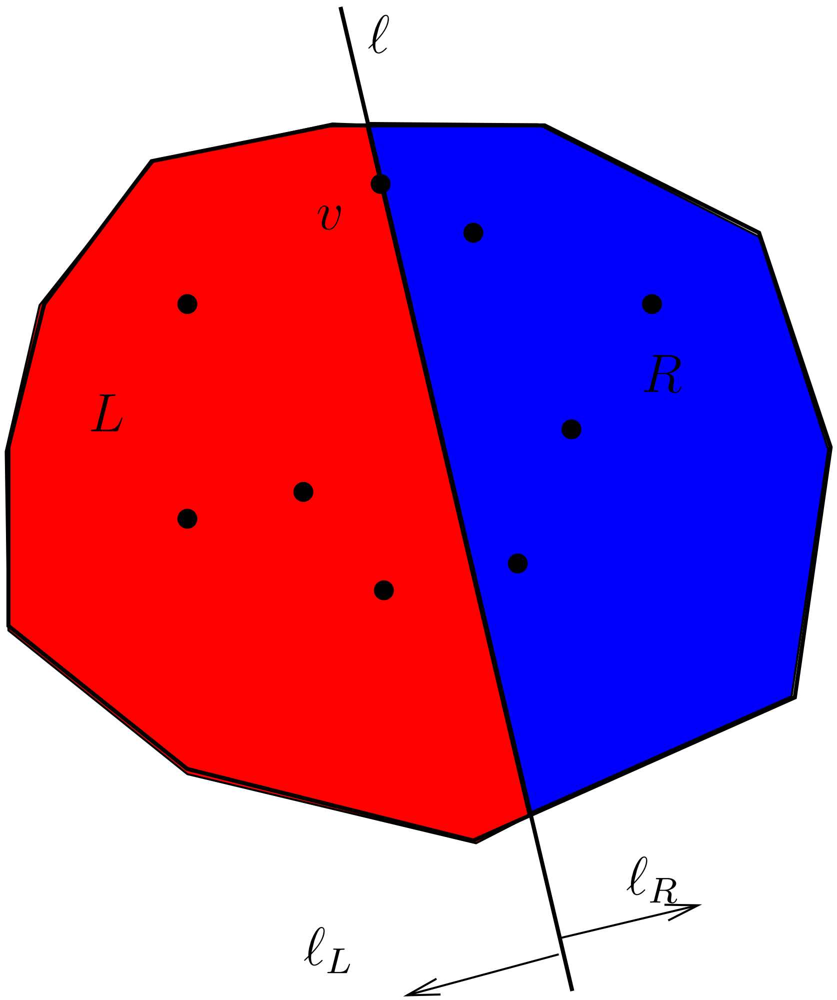
v: Vertex of conv(P); ℓ: line through v such that q points lie to the left and right of ℓ



ℓ_L, ℓ_R: half-spaces left and right of ℓ. Check that

$$\frac{q}{n} \cdot A \leq \text{area}\,(\ell_L \cap C)\,, \text{area}\,(\ell_R \cap C) \leq \frac{q+1}{n} \cdot A$$

Otherwise, Helper Lemma 1 gives equitable 2-partition.

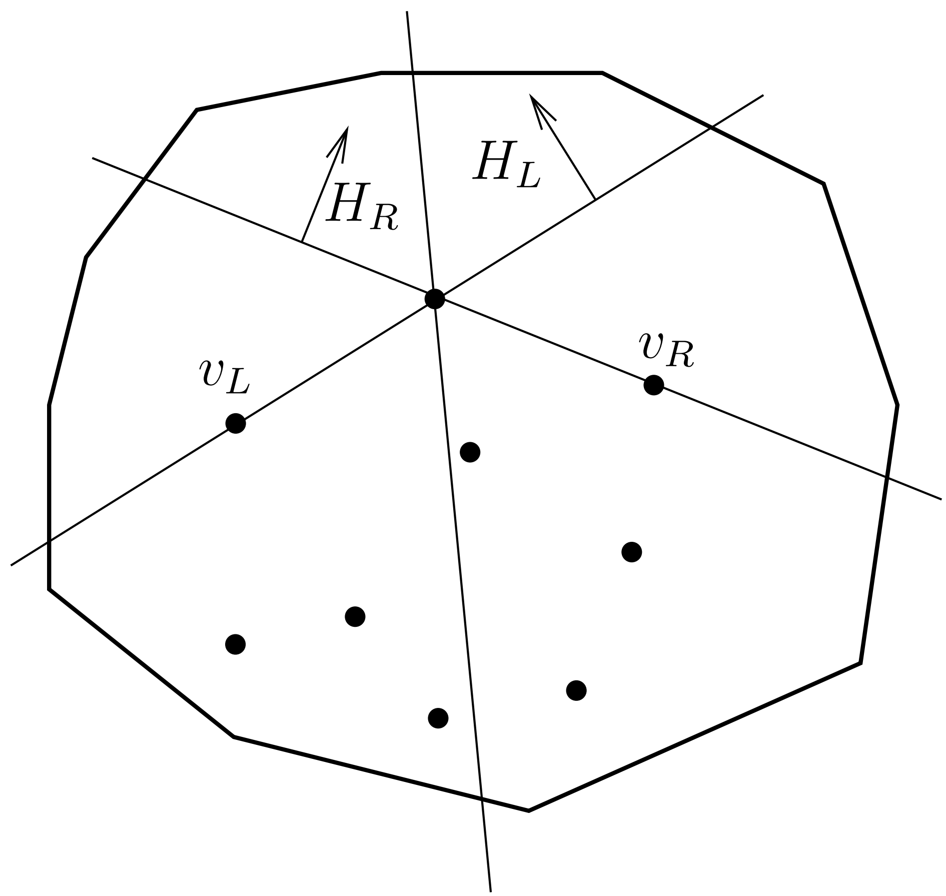


v_L, v_R : vertices of conv (P) left and right of v.

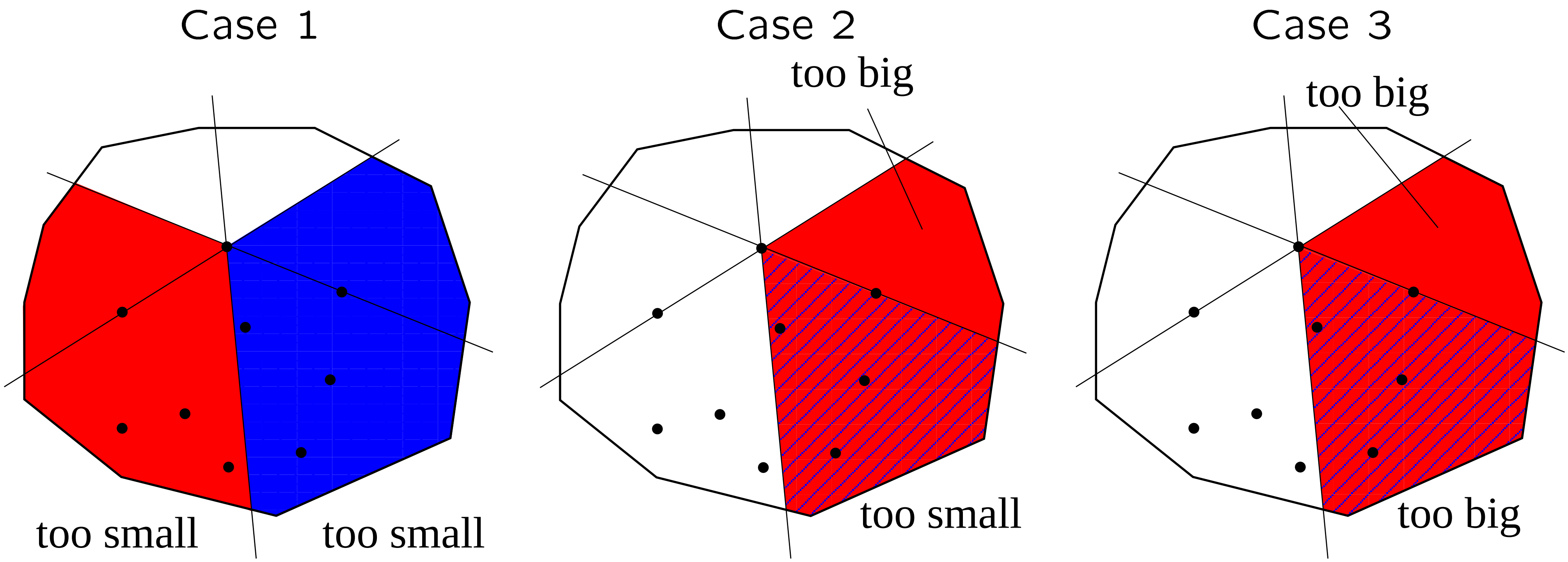
H_L, H_R: upward half-spaces cut off by vv_L and vv_R. Check that

$$\text{area}\,(H_L \cap C)\,, \text{area}\,(H_L \cap C) \geq A/n$$

Otherwise, Helper Lemma 1 gives equitable 2-partition

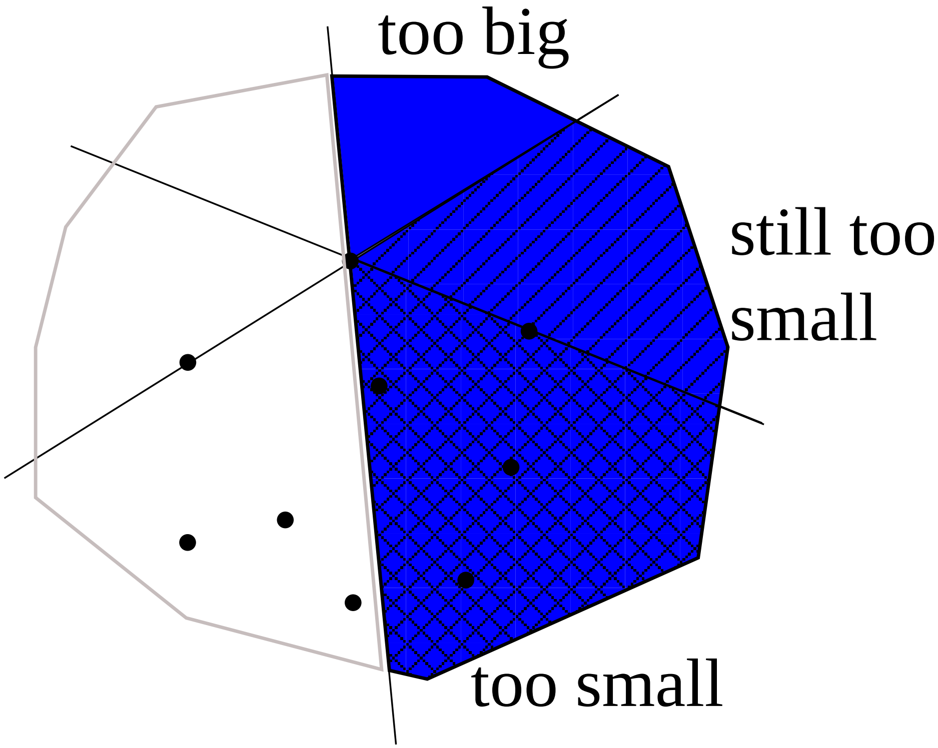


3 Cases: the red and blue regions will each contain q points.

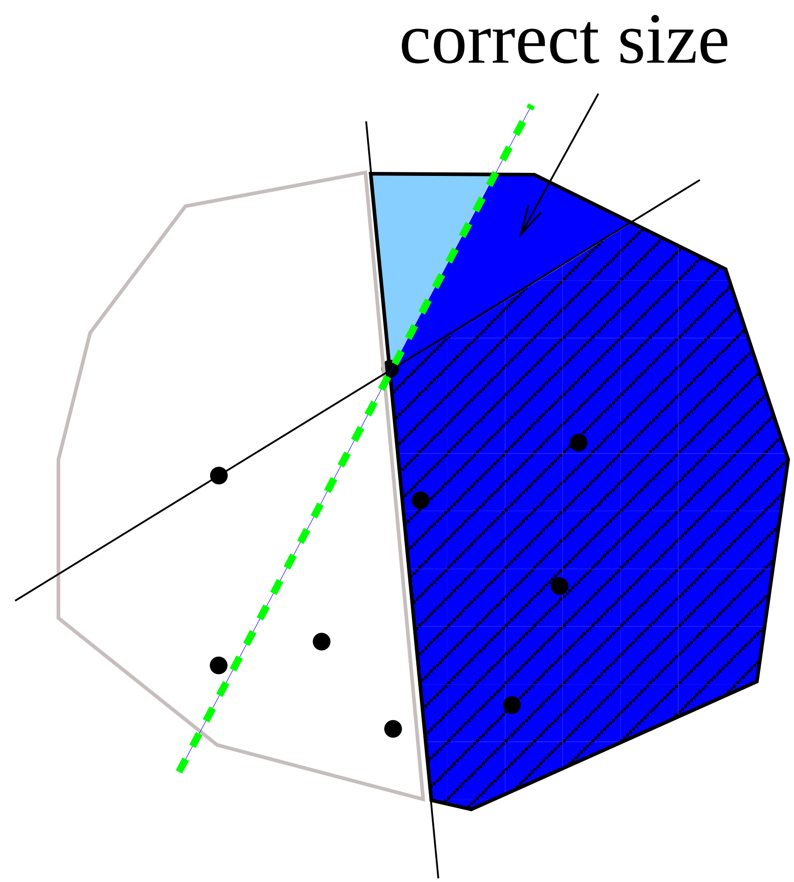


Case 1:

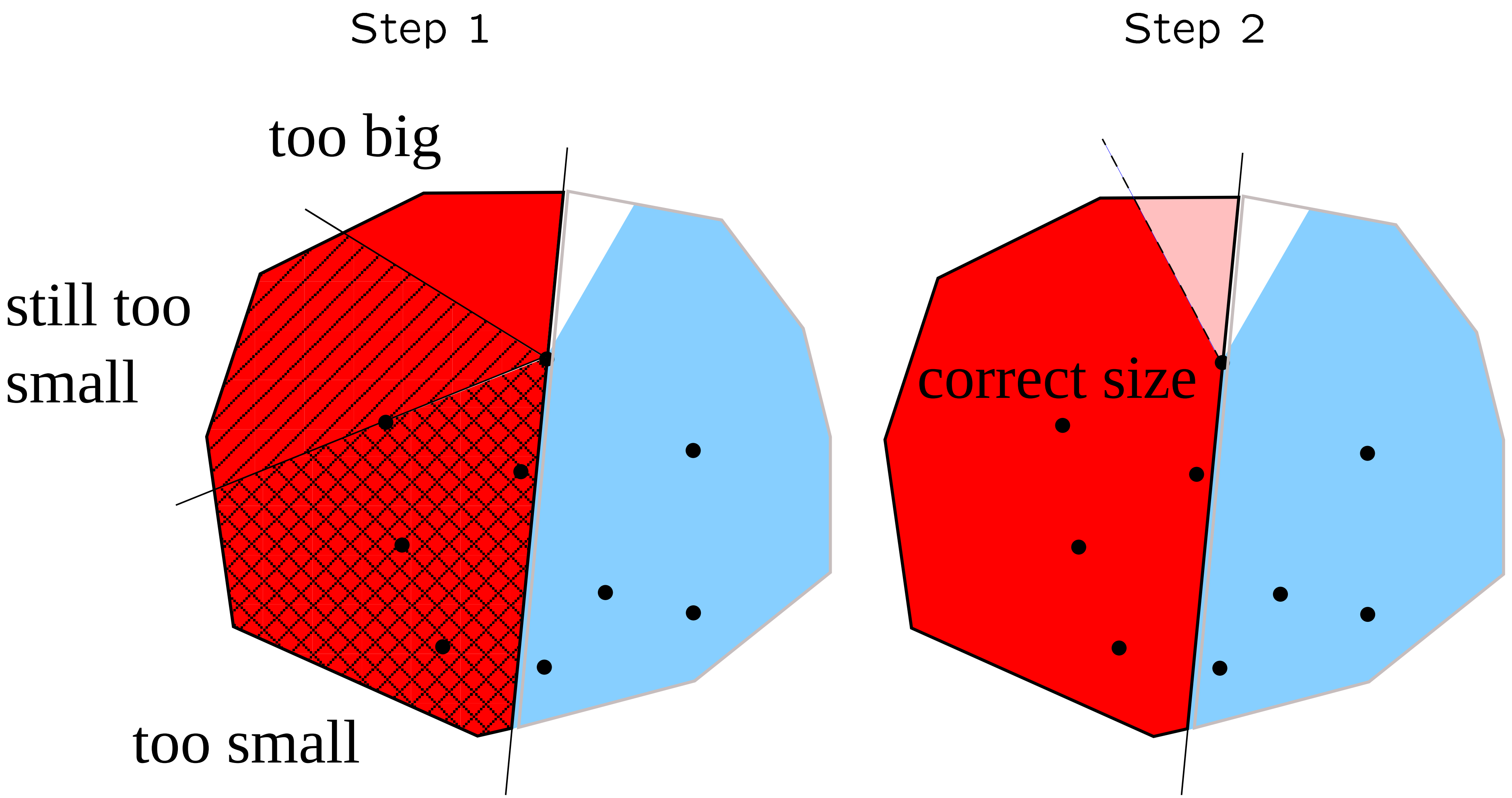
Step 1



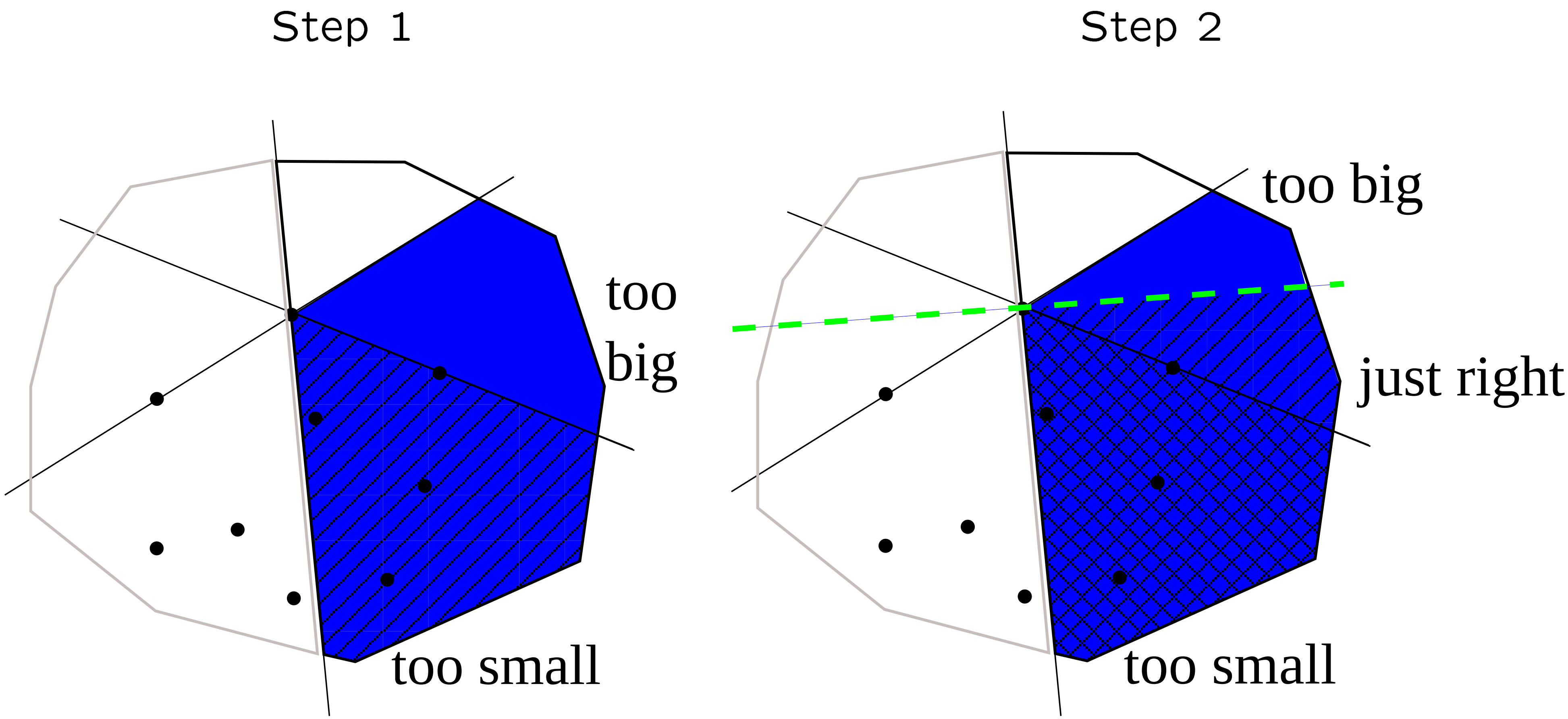
Step 2



Case 1, cont.:

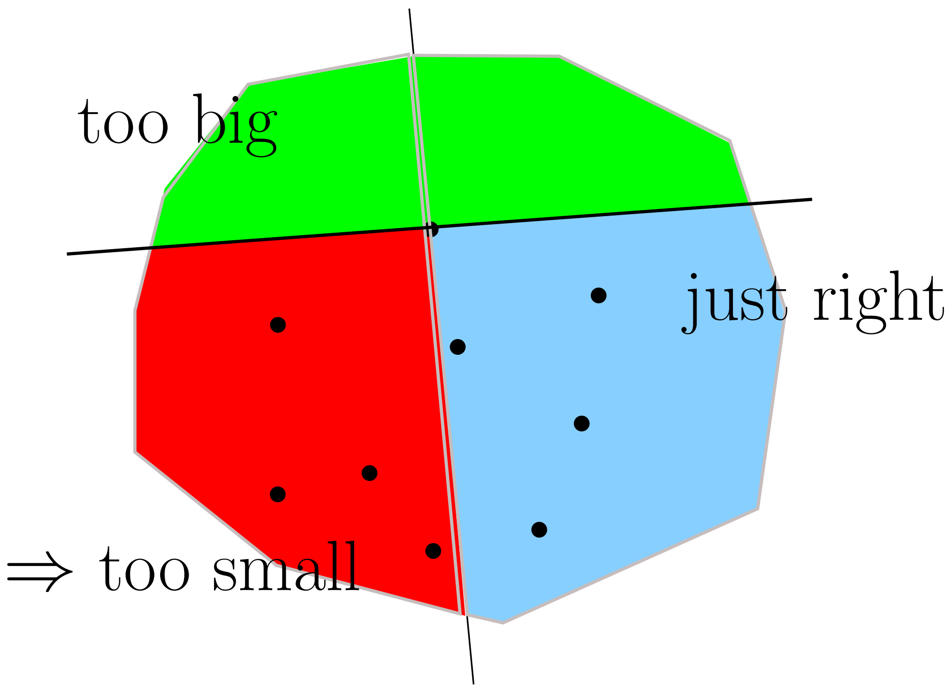


Case 2:

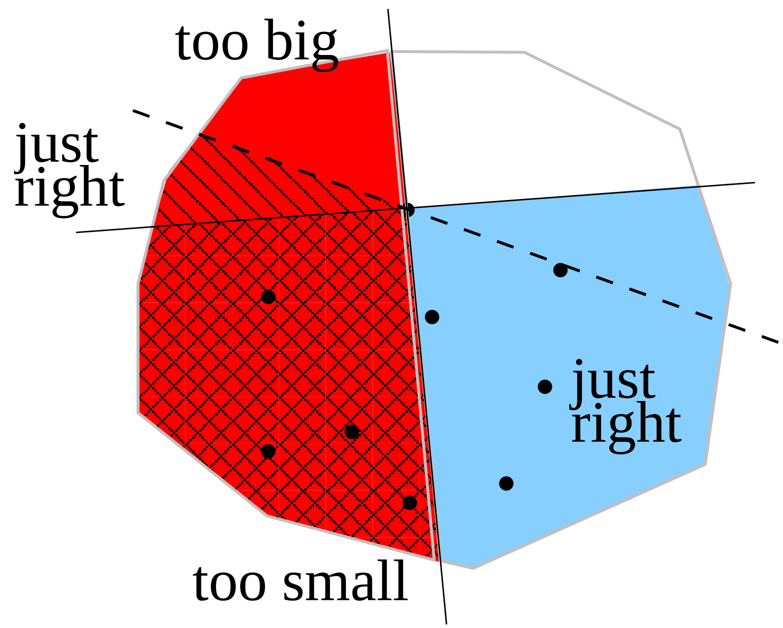


Case 2 cont:

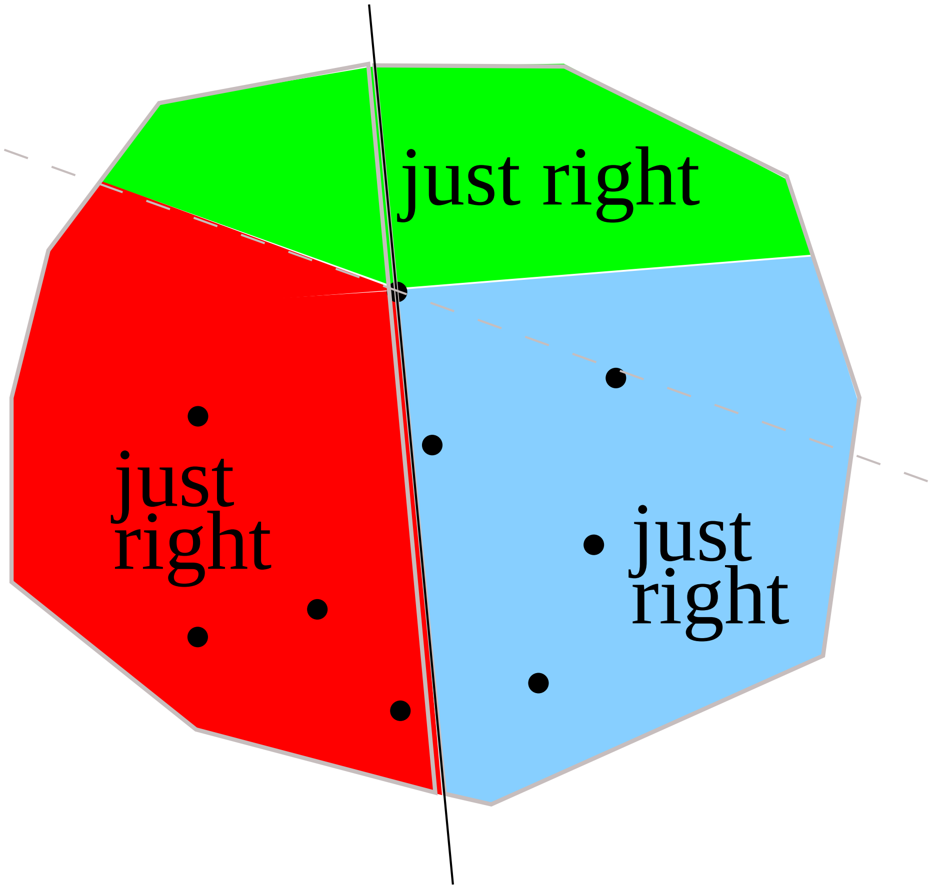
Step 1



Step 2

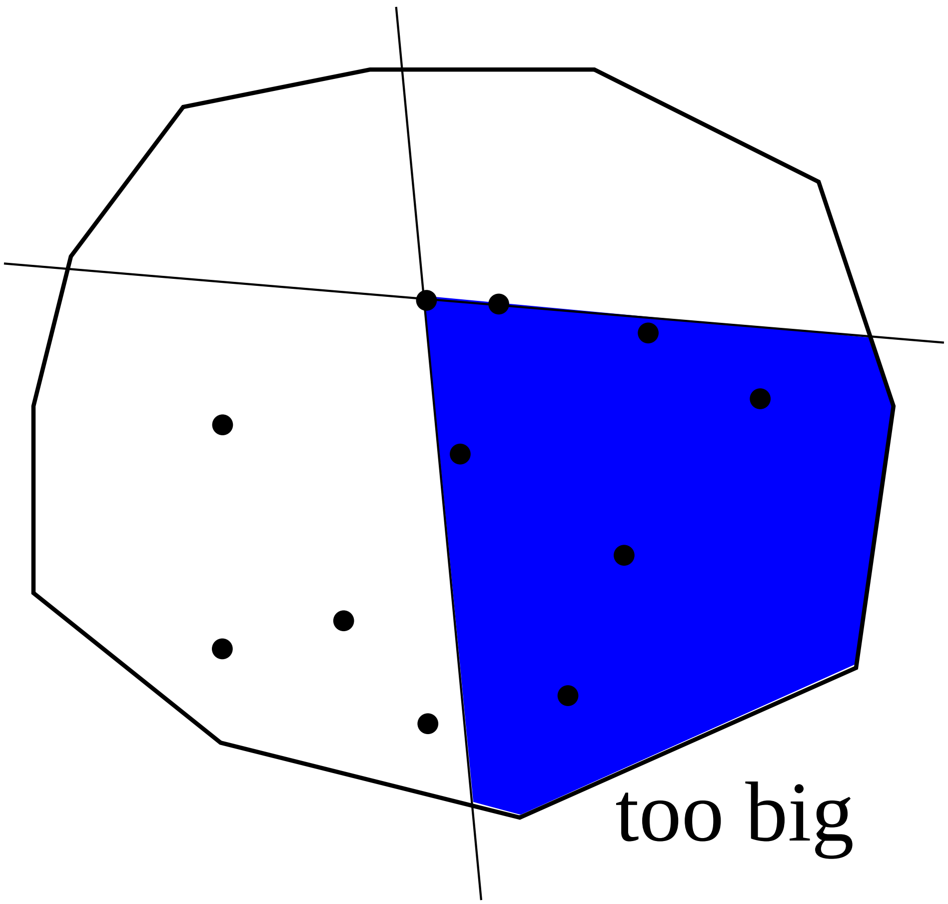


Case 2 result:

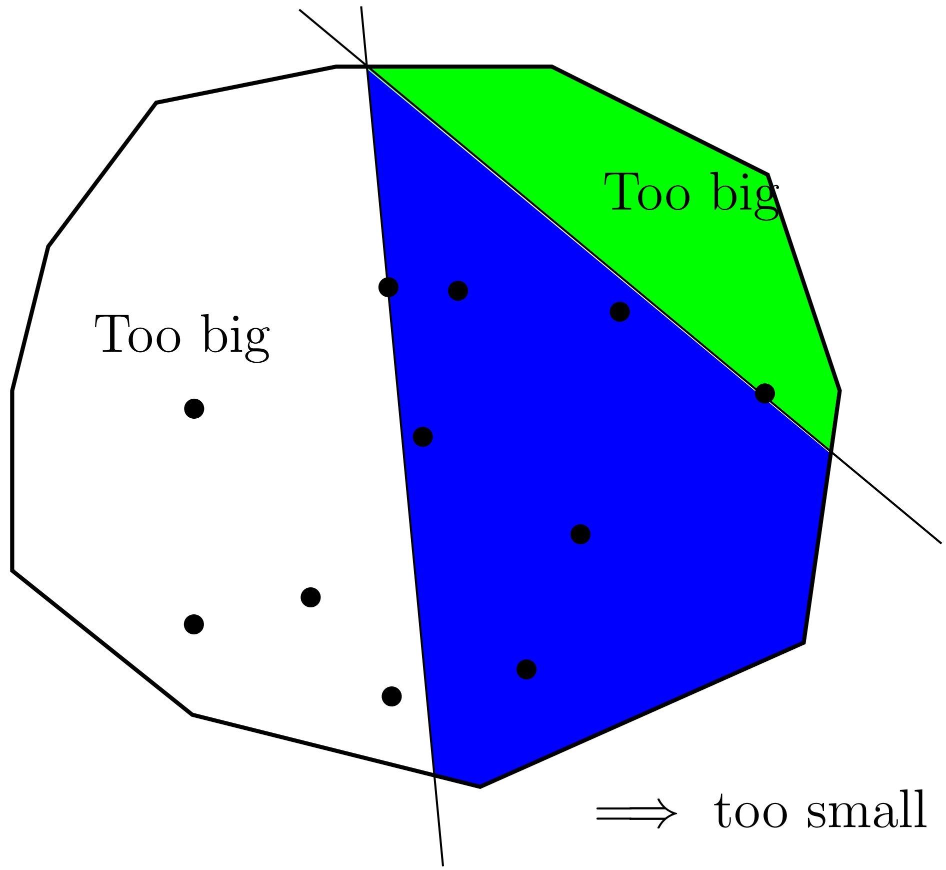


Case 3:

Step 1

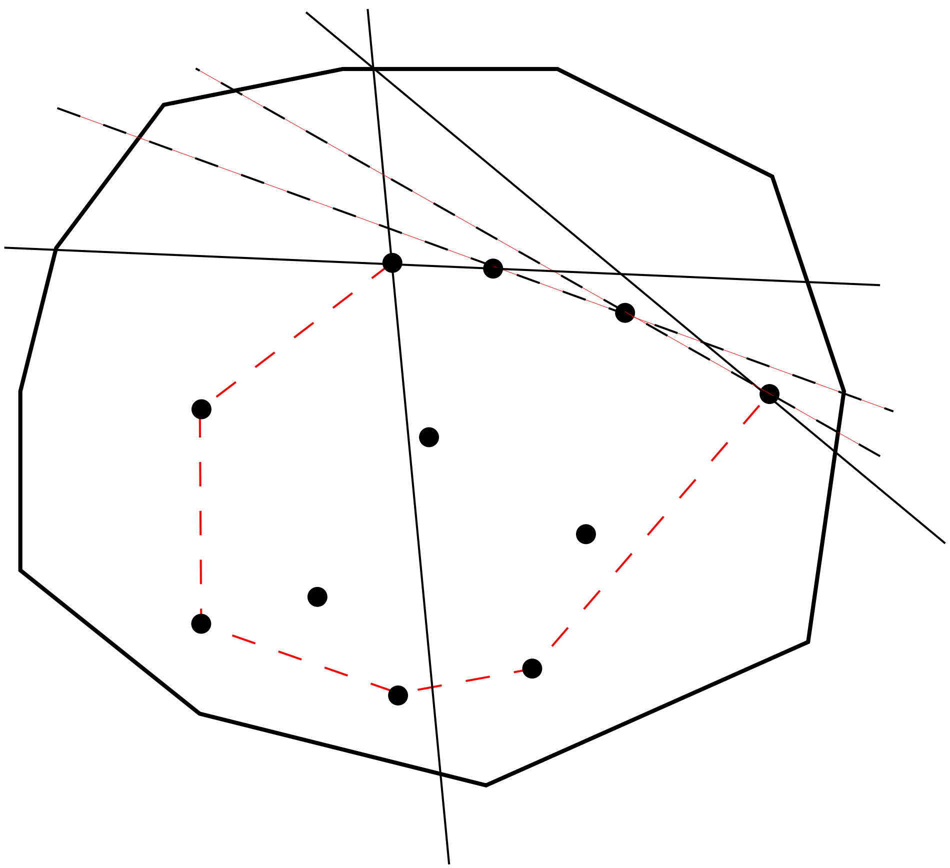


Step 2

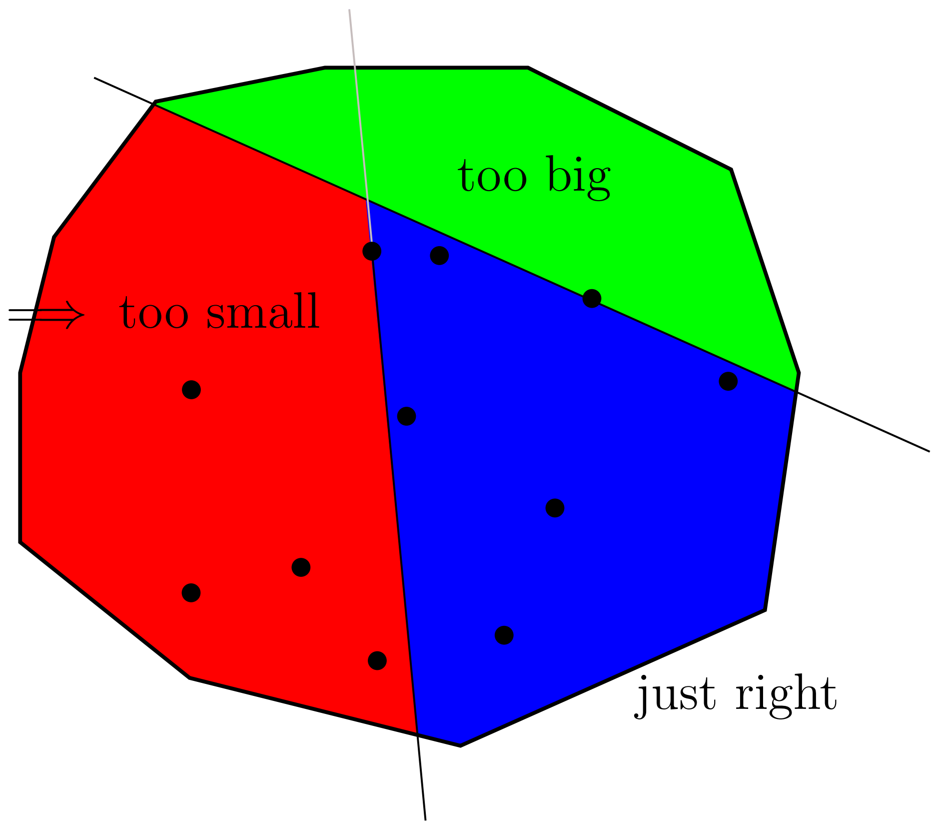


Case 3 cont:

Step 3

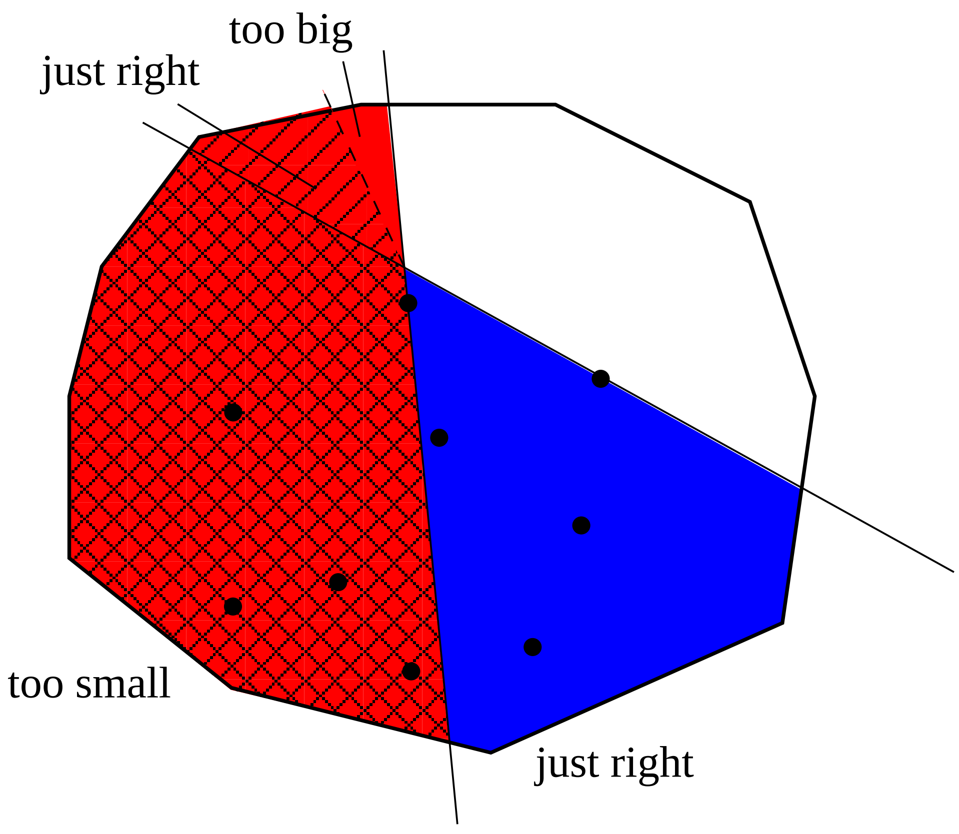


Step 4



Case 3 cont:

Step 4



Step 5