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Estimation of a parametric function associated with the lognormal distribution[1]

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Abstract Estimation of the mean of the lognormal distribution has received much attention in the literature beginning with Finney (1941). The problem is of significant practical importance because of the ubiquitous use of log-transformation. In this paper we consider estimation of a parametric function associated with the lognormal distribution of which the mean, median and moments are special cases. We generalize various estimators from the literature for the mean to this parametric function and propose a new simple estimator. We present the estimators in a unified framework and use this framework to derive asymptotic expressions for their biases and mean square errors (MSEs). Next we make asymptotic and small sample comparisons via simulations between them in terms of their MSEs. Our proposed estimator outperforms many of the previously proposed estimators. A numerical example is given to illustrate the various estimators.

Keywords Asymptotics, Bias, Logarithmic transformation, Mean square error.

1 Introduction

Logarithmic transformation is the most commonly used transformation in practice to symmetrize and normalize right-skewed data. It is also used to linearize multiplicative and power relationships in regression. The implicit assumption made is that the response variable follows the lognormal distribution, so that after log-transformation it is normally distributed.

How to estimate the mean of the lognormal distribution is an old problem dating back to Finney (1941), who considered it for the independent and identically distributed (i.i.d.) setting; also see Oldham (1965). It arises more commonly in regression settings where it has been studied in the econometric literature starting with Goldberger (1968), who considered it for estimating the Cobb-Douglas production function. There has been much later work reviewed in the book by Crow and Shimizu (1988). More recent papers are by Shen et al. (2006) and Shen and Zhu (2008).

Let $Z_1, \ldots, Z_n$ be i.i.d. random variables each having a lognormal distribution with mean η and variance τ^2. Then $Y_1 = \ln Z_1, \ldots, Y_n = \ln Z_n$ are i.i.d. normal with mean μ and variance σ^2 where η and τ^2 are related to μ and σ^2 as follows:

$$\eta = \exp(\mu + \sigma^2/2) \quad \text{and} \quad \tau^2 = \eta^2 \left[\exp(\sigma^2) - 1\right].$$

Let $\overline{Z}$ and T^2 denote the sample mean and variance of the Z_i's, and $\overline{Y}$ and S^2 denote the sample mean and variance of the Y_i's. Although $\overline{Z}$ is an unbiased estimator of η, it is not efficient besides being sensitive to

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extreme observations, which are common with lognormal data because of their long right hand tail. On the other hand, the naive estimator $\exp(\overline{Y})$ is biased since $E[\exp(\overline{Y})] = \exp(\mu + \sigma^2/2n) < \eta$ for $n > 1$. Finney (1941) derived an efficient unbiased estimator of η using the sufficient statistics $\overline{Y}$ and S^2 which is in the form of an infinite series in S^2.

In this paper we consider a more general parametric function of μ and σ^2 than η. We review various estimators proposed in the literature for the mean by first generalizing them to this parametric function and present them in a unified framework. We also propose a simple estimator which has a smaller MSE than most of the other competing estimators as shown by asymptotic analyses and simulations for small samples.

The paper is organized as follows. In Section 2 we state the problem. Section 3 gives a review of competing estimators including our proposed estimator. Section 4 gives asymptotic MSE comparisons between the competing estimators. Section 5 gives simulation comparisons. Section 6 gives a numerical example. The paper ends with some concluding remarks in Section 7. Mathematical derivations of the asymptotic MSE expressions for selected estimators are given in the appendix.

2 Problem Statement

Consider a general parametric function:

$$\theta = \exp(a\mu + b\sigma^2/2), \tag{1}$$

where a and b are known constants. If θ is the kth moment of the lognormal distribution then $a = k$ and $b = k^2$ for $k = 1, 2, \ldots$. We shall study in detail the cases $k = 1$ corresponding to the mean η, $k = 2$ corresponding to the second moment and $a = 1, b = 0$ corresponding to the median $\exp(\mu)$.

After log-transformation of the data, suppose we have available an estimator $\widehat{\mu}$ of μ such that $\widehat{\mu} \sim N(\mu, d^2\sigma^2)$ where $d^2 > 0$ is a known constant. We also have available an estimator S^2 of σ^2 with m degrees of freedom (d.f.) which is distributed as $\sigma^2\chi^2_m/m$ independent of $\widehat{\mu}$. In the i.i.d. case, $\widehat{\mu} = \overline{Y}$, $d^2 = 1/n$ and S^2 is the sample variance with $m = n - 1$ d.f.

We shall consider estimators of the form

$$\widehat{\theta} = \exp(a\widehat{\mu})f(S^2), \tag{2}$$

where $f(\cdot)$ is a nonnegative function. If $\widehat{\theta}$ is desired to be an unbiased estimator then we must have

$$E(\widehat{\theta}) = E[\exp(a\widehat{\mu})]E[f(S^2)] = \exp(a\mu + a^2d^2\sigma^2/2)E[f(S^2)] = \theta,$$

from which it follows that $f(S^2)$ must satisfy

$$E[f(S^2)] = \exp\{(b - a^2d^2)\sigma^2/2\}. \tag{3}$$

We can think of $f(S^2)$ as a bias correction factor. Several estimators discussed below are of the form:

$$\widehat{\theta} = \exp(a\widehat{\mu})g(cS^2/2). \tag{4}$$

In other words, $f(S^2) = g(cS^2/2)$ where $g(\cdot)$ is a nonnegative function and $c > 0$ is a constant.

3 Review of Estimators

3.1 Estimators of the Form (4)

Simple Adjustment Estimator: A simple choice for $g(cS^2/2)$ is $\exp(c_0 S^2/2)$ where $c_0 = b - a^2 d^2$. We denote the corresponding estimator by

$$\widehat{\theta}_0 = \exp(a\widehat{\mu}) \exp(c_0 S^2/2). \tag{5}$$

It has been used by many authors including Meulenberg (1965) and Kennedy (1981) in econometrics.

Finney's Estimator: Finney (1941) derived the formula

$$E\left[S^{2p}\exp(cS^2)\right] = \frac{\Gamma(m/2+p)}{\Gamma(m/2)}\left(\frac{2\sigma^2}{m}\right)^p \left(1 - \frac{2c\sigma^2}{m}\right)^{-m/2-p}, \tag{6}$$

where $c > 0$ is a constant. Next he introduced a function

$$\begin{aligned} g_1(t) &= 1 + \frac{t}{1!} + \frac{m}{(m+2)}\frac{v^2}{2!} + \frac{m^2}{(m+2)(m+4)}\frac{t^3}{3!} + \cdots \\ &= 1 + \frac{t}{1!} + \frac{\nu}{(\nu+1)}\frac{v^2}{2!} + \frac{\nu^2}{(\nu+1)(\nu+2)}\frac{t^3}{3!} + \cdots, \end{aligned} \tag{7}$$

where $\nu = m/2$. Using result (6), he showed that

$$E[g_1(cS^2/2)] = \exp(c\sigma^2/2) \tag{8}$$

from which follows that his estimator

$$\widehat{\theta}_{\mathrm{F}} = \exp(a\widehat{\mu}) g_1(c_0 S^2/2). \tag{9}$$

is unbiased. Note that if $m \to \infty$ then $g_1(t) \to \exp(t)$ and so $g_1(c_0 S^2/2) \to \exp(c_0 S^2/2)$ and hence $\widehat{\theta}_{\mathrm{F}} \to \widehat{\theta}_0$ almost surely.

Shen (1998) showed that Finney's estimator is the minimum variance unbiased estimator (MVUE) which follows from the Lehmann-Scheffé theorem since it is a function of the complete sufficient statistic $(\widehat{\mu}, S^2)$. Shen also derived Finney's estimator using the Rao-Blackwell theorem.

Evans and Shaban's Estimator: While Finney's estimator is MVUE, Evans and Shaban (1974, 1976) derived an approximately minimum MSE estimator:

$$\widehat{\theta}_{\mathrm{ES}} = \exp(a\widehat{\mu}) g_1(c_1 S^2/2), \tag{10}$$

where $c_1 = b - 3a^2 d^2$. Replacing c_0 in Finney's estimator by c_1 introduces a small bias in exchange of reduction in variance with overall reduction in MSE.

Rukhin's Estimator: Rukhin (1986) modified Evans and Shaban's estimator for further improved MSE performance by replacing c_1 by $c_2 = c\nu/(\nu+1)$:

$$\widehat{\theta}_{\mathrm{R}} = \exp(a\widehat{\mu}) g_1\left(c_2 S^2/2\right). \tag{11}$$

Thus $\widehat{\theta}_{\mathrm{F}}, \widehat{\theta}_{\mathrm{ES}}$ and $\widehat{\theta}_{\mathrm{R}}$ all use the same $g_1(cS^2/2)$ function for different choices of c.

Our Proposed Estimator: Our proposed estimator is of the form $\widehat{\theta}_0$ defined in (5) except that instead of setting $c = c_0$ we choose c to give an unbiased estimator of θ. Putting $p = 0$ in (6) (which gives the moment generating function of $S^2 \sim \sigma^2\chi^2_m/m$), we get

$$E\left[\exp(cS^2/2)\right] = \left(1 - \frac{c\sigma^2}{m}\right)^{-m/2}.$$

Equating this to $\exp(c_0\sigma^2/2)$ to satisfy (3), solving for c and replacing σ^2 in the solution by S^2 we get

$$c = \frac{m}{S^2}\left[1 - \exp\left\{\frac{-c_0S^2}{m}\right\}\right]. \tag{12}$$

Although c is not a constant as required in (4), by choosing a suitable $g(\cdot)$ function we can express our proposed estimator in the form (4) as follows:

$$\begin{aligned}
\exp(cS^2/2) &= \exp\left\{\frac{m}{S^2}\left[1 - \exp\left\{\frac{-c_0S^2}{m}\right\}\right]\frac{S^2}{2}\right\} \\
&= \exp\left\{-\nu\left[\exp\left(-\frac{c_0S^2}{2\nu}\right) - 1\right]\right\} \\
&= \exp\left\{-\nu\left[\exp\left(-\frac{t}{\nu}\right) - 1\right]\right\} \\
&= g_2(t),
\end{aligned} \tag{13}$$

where $t = c_0S^2/2$. Thus our proposed estimator is

$$\widehat{\theta}_{\mathrm{GT}} = \exp(a\widehat{\mu})g_2(c_0S^2/2). \tag{14}$$

By expanding the exponential function in (12) for m large, we obtain the following first-order approximation:

$$c \approx \frac{m}{S^2}\left[\frac{c_0S^2}{m}\right] = c_0;$$

thus $\widehat{\theta}_{\mathrm{GT}} \to \widehat{\theta}_0$ as $m \to \infty$ almost surely.

Just as the MSE of Finney's estimator was improved by Evans and Shaban by replacing c_0 by c_1 and further by Rukhin by replacing c_1 by $c_2 = c_1\nu/(\nu+1)$, we can expect to improve the MSE of our proposed estimator by replacing c_0 by c_2. We have verified this by asymptotic analyses of their MSEs. The details are available from the authors. We will denote our original proposed estimator by $\widehat{\theta}_{\mathrm{GT}(c_0)}$ and this improved version of it by $\widehat{\theta}_{\mathrm{GT}(c_2)}$.

3.2 Estimators of More General Form (2)

Shen et al. (2006) proposed the minimum MSE estimator for the i.i.d. setting in which case $d^2 = 1/n$. Later, Shen and Zhu (2008) extended this minimum MSE estimator to the regression setting under the assumption that $d^2 = O(1/n)$; they also derived the minimum bias estimator.

The regression setting is as follows. Let $\boldsymbol{Z} = (Z_1, \ldots, Z_n)'$ be a vector of independent lognormally distributed responses and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$ be the corresponding vector of normally distributed logtransformed responses. Let $\boldsymbol{X}$ be an $n \times (p+1)$ matrix of predictors and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)'$ be a $(p+1) \times 1$ vector of unknown regression coefficients. Assume the general linear model $E(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\beta}$ under the usual normality, independence and homoscedasticity assumptions. Let $\widehat{\boldsymbol{\beta}} = (\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_p)'$ be the least squares (LS) estimator of $\boldsymbol{\beta}$. Suppose we want to estimate $E(Y|\boldsymbol{x}) = \mu = \boldsymbol{x}'\boldsymbol{\beta}$ for some specified predictor vector $\boldsymbol{x} = (1, x_1, \ldots, x_p)'$. The LS estimator $\widehat{\mu} = \boldsymbol{x}'\widehat{\boldsymbol{\beta}}$ is normally distributed with mean μ and variance $d^2\sigma^2$ where $d^2 = \boldsymbol{x}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{x} = O(1/n)$. Further let SSE be the error sum of squares and $S^2 = \text{SSE}/[n-(p+1)] \sim \sigma^2\chi^2_m/m$ where $m = n-(p+1)$. Then the Shen et al. estimator of $\eta = E(Z|\boldsymbol{x}) = E[\exp(Y|\boldsymbol{x})]$ is given by

$$\widehat{\eta}(k) = \exp(\widehat{\mu}) \exp\left(\frac{\text{SSE}}{2(n-k)}\right) = \exp(\widehat{\mu}) \exp\left(\frac{n-(p+1)}{n-k}\frac{S^2}{2}\right), \tag{15}$$

where k is a coefficient to be determined. Shen and Zhu (2008) determined k to approximately minimize the bias or MSE of the estimator; the resulting k is a function of S^2. Hence $[n-(p+1)]/(n-k)$ is not a constant as required in (4).

Shen and Zhu (2008) showed that the bias of $\widehat{\eta}(k)$ up to the order $O(1/n)$ is given by

$$\text{Bias}[\widehat{\eta}(k)] = \eta\left[nd^2 + k - (p+1) + \frac{\sigma^2}{2}\right]\frac{\sigma^2}{n} + o\left(\frac{1}{n}\right) \tag{16}$$

and the MSE of $\widehat{\eta}(k)$ up to the order $O(1/n^2)$ is given by

$$\begin{aligned}\text{MSE}[\widehat{\eta}(k)] = \eta^2 \Bigg[&1 + \frac{\sigma^2}{2} + \frac{\sigma^2}{4n}(k^2 + 6nd^2 - 2(p-1) + 3\sigma^2) - 1 + p^2 \\ &-6nd^2(p+1) + 7n^2d^2 + (1 - 3p + 7nd^2)\sigma^2 + \frac{7\sigma^2}{4}\Bigg] + o\left(\frac{1}{n^2}\right).\end{aligned} \tag{17}$$

If k is chosen to minimize $\text{Bias}[\widehat{\eta}(k)]$ and the unknown σ^2 in the minimizing value of k is replaced by S^2 then we get Shen and Zhu's (2008) minimum bias estimator:

$$\widehat{\eta}_{\text{MB}} = \exp(\widehat{\mu}) \exp\left(\frac{m\text{SSE}}{2m[n-(p+1)+nd^2] + \text{SSE}}\right).$$

If k is chosen to minimize $\text{MSE}[\widehat{\eta}(k)]$ and the unknown σ^2 in the minimizing value of k is replaced by S^2 then we get Shen and Zhu's (2008) minimum MSE estimator:

$$\widehat{\eta}_{\text{MM}} = \exp(\widehat{\mu}) \exp\left(\frac{m\text{SSE}}{2m[n-(p-1)+3nd^2] + 3\text{SSE}}\right).$$

We now extend these estimators to our more general setting. We replace the restriction $d^2 = O(1/n)$ by $d^2 = O(1/\nu)$. Let

$$\widehat{\theta}(k) = \exp(a\widehat{\mu}) \exp\left(\frac{b\nu}{\nu - k}\frac{S^2}{2}\right). \tag{18}$$

The bias of $\widehat{\theta}(k)$ up to the order $O(1/\nu)$ equals

$$\text{Bias}[\widehat{\theta}(k)] = \frac{\theta\sigma^2}{2\nu}\left[\nu a^2 d^2 + \frac{b^2\sigma^2}{4} + bk\right] + o\left(\frac{1}{\nu}\right) \tag{19}$$

and the MSE of $\widehat{\theta}(k)$ up to the order $O(1/\nu^2)$ equals

$$\mathrm{MSE}[\widehat{\theta}(k)] = \frac{\theta^2\sigma^2}{\nu}\left[\nu a^2 d^2 + \frac{b^2\sigma^2}{4}\right.$$
$$\left. + \frac{b^2\sigma^2}{4\nu}\left\{k^2 + \left(2 + \frac{3}{2}b\sigma^2 + \frac{6\nu a^2 d^2}{b}\right)k + b\sigma^2 + \frac{7\nu^2 a^4 d^2}{b^2} + \frac{7}{16}b^2\sigma^4 + \frac{7}{2}\nu a^2 d^2\sigma^2\right\}\right] + o\left(\frac{1}{\nu^2}\right). \quad (20)$$

The asymptotic expansions (16) and (17) derived by Shen and Zhu (2008) for the bias and MSE of $\widehat{\eta}(k)$ are slightly different from the ones given above because Shen and Zhu considered the expansions in terms of $1/n$ whereas the above expansions are in terms of $1/\nu$. This results in $\widehat{\eta}_{\mathrm{MB}}$ and $\widehat{\eta}_{\mathrm{MM}}$ not being the exact special cases of their generalized versions given below.

Shen and Zhu's Minimum Bias Estimator: The bias expression (19) is minimized by choosing

$$k = -\frac{\nu a^2 d^2}{b} - \frac{b\sigma^2}{4}. \quad (21)$$

Substituting this value of k in (18) and replacing σ^2 by S^2 gives Shen and Zhu (2008) generalized minimum bias estimator:

$$\widehat{\theta}_{\mathrm{MB}} = \exp(a\widehat{\mu})\exp\left(\frac{2b^2\nu S^2}{4\nu\,(b + a^2d^2) + b^2S^2}\right). \quad (22)$$

Shen and Zhu's Minimum MSE Estimator: The MSE expression (20) is minimized by choosing

$$k = -1 - \frac{3b\sigma^2}{4} - \frac{3\nu a^2 d^2}{b}. \quad (23)$$

Substituting this value of k in (18) and replacing σ^2 by S^2 gives Shen and Zhu (2008) generalized minimum MSE estimator:

$$\widehat{\theta}_{\mathrm{MM}} = \exp(a\widehat{\mu})\exp\left(\frac{2\nu b^2 S^2}{4\nu\,(b + 3a^2d^2) + 4b + 3b^2S^2}\right). \quad (24)$$

4 Asymptotic MSE Comparisons Between Estimators

Derivations of the asymptotic expansions of the MSEs of the estimators under the assumption d^2 is fixed are given in the appendix. We derived similar expansions under the assumption $d^2 = O(1/\nu)$ but they were found to be not very accurate and so we have chosen not to report them. Therefore no asymptotic MSE expressions are provided for $\widehat{\theta}_{\mathrm{MB}}$ and $\widehat{\theta}_{\mathrm{MM}}$, which are derived under that assumption.

4.1 Comparison Between Unbiased Estimators $\widehat{\theta}_{\mathbf{F}}$ and $\widehat{\theta}_{\mathbf{GT}(c_0)}$

We first compare $\widehat{\theta}_{\mathrm{F}}$ with $\widehat{\theta}_{\mathrm{GT}(c_0)}$ in terms of their asymptotic MSEs because both are designed to be unbiased estimators of θ (of course, $\widehat{\theta}_{\mathrm{GT}(c_0)}$ is unbiased only in large samples since we have replaced σ^2 by S^2 in (12)). The formulae for their asymptotic MSEs up to the order $O(1/\nu^2)$ are as follows, where for convenience we have put $\delta = a^2d^2\sigma^2 > 0$:

$$\mathrm{MSE}(\widehat{\theta}_{\mathrm{F}}) = \theta^2 e^{\delta}\left[\left(1 - e^{-\delta}\right) + \frac{c_0^2\sigma^4}{4}\frac{1}{\nu} + \frac{c_0^4\sigma^8}{32}\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \quad (25)$$

and

$$\operatorname{MSE}(\widehat{\theta}_{\mathrm{GT}(c_0)}) = \theta^2 e^{\delta}\left[\left(1-e^{-\delta}\right)+\frac{c_0^2\sigma^4}{4}\frac{1}{\nu} + \left\{\frac{c_0^4\sigma^8}{32}-\left(1-e^{-\delta}\right)\left(\frac{c_0\sigma^4}{4}+\frac{c_0^3\sigma^6}{8}\right)\right\}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]. \tag{26}$$

Then we have the following theorem.

Theorem 1. *Up to the order $O(1/\nu^2)$, $MSE(\widehat{\theta}_{GT(c_0)}) < MSE(\widehat{\theta}_F)$ iff $(b-a^2d^2)\sigma^2 > -2$.*

Proof. The above inequality holds iff

$$-\left(1-e^{-\delta}\right)\left(\frac{c_0^2\sigma^4}{4}+\frac{c_0^3\sigma^6}{8}\right)<0 \Longleftrightarrow \frac{c_0^2\sigma^4}{4}+\frac{c_0^3\sigma^6}{8}>0 \Longleftrightarrow c_0\sigma^2>-2.$$

This completes the proof of the theorem. □

If $b \ge a^2d^2$ then the inequality $c_0\sigma^2 > -2$ is obviously satisfied. This is the case in many applications since d^2 is small. For example, for estimating the mean η in the i.i.d. case, we have $a = 1$, $b = 1$ and $d^2 = 1/n$ so that $b \ge a^2d^2$ for all $n \ge 1$. Thus our proposed estimator has a smaller MSE than Finney's classical estimator up to order $O(1/\nu^2)$ in most cases of practical interest.

4.2 Comparison Between Biased Estimators

We now make comparisons between selected biased estimators which are designed to minimize MSE. We do not include all estimators discussed above in order to keep the comparisons to a manageable level. Toward this end, we chose Rukhin's estimator $\widehat{\theta}_{\mathrm{R}}$ from the class of estimators $\widehat{\theta} = \exp(a\widehat{\mu})g_1(cS^2/2)$. Similarly we chose the improved version $\widehat{\theta}_{\mathrm{GT}(c_2)}$ of our proposed estimator. Finally, we chose Shen and Zhu's estimator $\widehat{\theta}_{\mathrm{MM}}$ defined in (24) as it minimizes MSE. We also included in our comparison $\widehat{\theta}_0$ defined in (5) as a benchmark for comparison to show how much improvement is obtained by other estimators.

The MSEs of $\widehat{\theta}_0$, $\widehat{\theta}_{\mathrm{R}}$ and $\widehat{\theta}_{\mathrm{GT}(c_2)}$ up to the order $1/\nu^2$ under the assumption that d^2 is fixed are given by the following expressions:

$$\operatorname{MSE}(\widehat{\theta}_0) = \theta^2 e^{\delta}\left[\left(1-e^{-\delta}\right)+\left(1-\frac{e^{-\delta}}{2}\right)\frac{c_0^2\sigma^4}{2}\frac{1}{\nu} + \left\{\left(1-\frac{e^{-\delta}}{4}\right)\frac{c_0^3\sigma^6}{3}+\left(1-\frac{e^{-\delta}}{8}\right)\frac{c_0^4\sigma^8}{8}\right\}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right], \tag{27}$$

$$\operatorname{MSE}(\widehat{\theta}_{\mathrm{R}}) = \theta^2 e^{-\delta}\left[\left(e^{\delta}-1\right)+\frac{c_1^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c_1^2\sigma^4}{4}-\frac{c_1^3\sigma^6}{4}+\frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right], \tag{28}$$

$$\operatorname{MSE}(\widehat{\theta}_{\mathrm{GT}(c_2)}) = \theta^2 e^{-\delta}\left[\left(e^{\delta}-1\right)+\frac{c_1^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c_1^2\sigma^4}{4}-\frac{c_1^3\sigma^6}{4}+\frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]. \tag{29}$$

The corresponding asymptotic expression for the MSE of $\widehat{\theta}_{\mathrm{MM}}$ is not available since it assumes that $d^2 = O(1/\nu)$. These MSEs are compared in the following theorem.

Theorem 2. *Up to the order $O(1/\nu^2)$ we have the following inequalities among the asymptotic MSEs of the estimators:*

$$MSE(\widehat{\theta}_0) \ge MSE(\widehat{\theta}_R) = MSE(\widehat{\theta}_{GT(c_2)}) \tag{30}$$

if $b - 3a^2d^2 \geq 0$.

Proof. To show that $\text{MSE}(\widehat{\theta}_0) \geq \text{MSE}(\widehat{\theta}_{\text{R}})$, first note that the leading term of (27) is $\geq$ the leading term of (28) since

$$e^{\delta}\left(1 - e^{-\delta}\right) \geq e^{-\delta}\left(e^{\delta} - 1\right) \quad \Longleftrightarrow \quad e^{\delta} + e^{-\delta} \geq 2,$$

with equality holding iff $\delta = 0$. Similarly, the first-order term of (27) is $\geq$ the first-order term of (28) since

$$e^{\delta}\left(1 - \frac{e^{-\delta}}{2}\right) \geq \frac{e^{-\delta}}{2} \quad \Longleftrightarrow \quad \left(e^{\delta} - 1\right)\left(2e^{\delta} + 1\right) \geq 0,$$

for all $\delta \geq 0$ and

$$\frac{c_0^2\sigma^4}{2} \geq \frac{c_1^2\sigma^4}{2} \quad \Longleftrightarrow \quad c_0^2 \geq c_1^2 \quad \text{if} \quad b - 3a^2d^2 \geq 0,$$

which holds for large enough n. For example, for estimating the mean η in the i.i.d. case where $a = b = 1$ and $d^2 = 1/n$, the above condition reduces to $n \geq 3$. Also note that this is a sufficient — not a necessary condition. This together with the previous inequality assures that the desired inequality holds for the first-order terms virtually for all n. The second-order terms, being negligible in comparison, can be ignored.

The equality $\text{MSE}(\widehat{\theta}_{\text{R}}) = \text{MSE}(\widehat{\theta}_{\text{GT}(c_2)})$ follows since their asymptotic expansions are identical up to the order $O(1/\nu^2)$. The following simulations show that, in fact, $\widehat{\theta}_{\text{GT}(c_2)}$ has a smaller MSE than $\widehat{\theta}_{\text{R}}$ in almost all cases. □

5 Simulation Comparisons Between Estimators

5.1 Simulation Results

In this section we report the results of simulations for the MSEs of the four estimators of θ. We performed simulations for three parametric functions: median $(a = 1, b = 0)$, mean $(a = 1, b = 1)$ and second moment $(a = 2, b = 4)$. Four combinations of (μ, σ) were investigated for $\mu = 0, 1$ and $\sigma = 1, 2$.

In each replication we generated $\widehat{\mu} \sim N(\mu, \sigma^2/n)$ and $S^2 \sim \sigma^2\chi^2_{n-1}/(n-1)$ independent of each other for given (μ, σ) and $n = 10, 20, 50$. A total of $N = 10^9$ replications were performed in each case. Next we computed all four estimators of θ in each case. Finally, we computed the mean and variance of all N replications of each estimator from which we computed the MSEs. For the second moment we report $\sqrt{\text{MSE}}$ to be on the same scale as the MSEs for the mean and median. The results are given in Table 1 for the median, in Table 2 for the mean and in Table 3 for the second moment. For a given parametric function (median, mean or the second moment) and for each combination of μ, σ and n, the smallest MSE is shown in bold.

We see that in the case of the median, either $\widehat{\theta}_{\text{R}}$ or $\widehat{\theta}_{\text{GT}(c_2)}$ has the smallest MSE while in the case of the second moment, either $\widehat{\theta}_{\text{GT}(c_2)}$ or $\widehat{\theta}_{\text{MM}}$ has the smallest MSE. On the other hand, in the case of the mean, $\widehat{\theta}_{\text{MM}}$ has the smallest MSE in all cases that were studied. Overall, $\widehat{\theta}_{\text{MM}}$ has the smallest MSE in 20 out of 36 cases, $\widehat{\theta}_{\text{GT}(c_2)}$ has the smallest MSE in 14 out of 36 cases and $\widehat{\theta}_{\text{R}}$ has the smallest MSE in 10 out of 36 cases (the total number of wins exceeds the number of cases because of the ties). In a head-to-head comparison, $\widehat{\theta}_{\text{R}}$ beats $\widehat{\theta}_{\text{GT}(c_2)}$ in 2 cases, $\widehat{\theta}_{\text{GT}(c_2)}$ beats $\widehat{\theta}_{\text{R}}$ in 26 cases and in the remaining 8 cases they are tied. Even in the two cases where $\widehat{\theta}_{\text{R}}$ beats $\widehat{\theta}_{\text{GT}(c_2)}$, the difference in their MSEs is only 0.0001, which is not statistically significant. Thus, $\widehat{\theta}_{\text{GT}(c_2)}$ is preferred over $\widehat{\theta}_{\text{R}}$. Between $\widehat{\theta}_{\text{GT}(c_2)}$ and $\widehat{\theta}_{\text{MM}}$, the preferred estimator depends on the parametric function to be estimated. For estimating the median, $\widehat{\theta}_{\text{GT}(c_2)}$ is preferred, while for estimating the mean, $\widehat{\theta}_{\text{MM}}$ is preferred. For estimating the second moment if σ is small, $\widehat{\theta}_{\text{MM}}$ is preferred, while if σ is large, $\widehat{\theta}_{\text{GT}(c_2)}$ is preferred. However, these MSEs are much larger in magnitude compared to the MSEs of the estimators of the median and mean; thus none of the estimators is very useful for estimating the second moment. Finally we note that $\widehat{\theta}_0$ has the highest MSE in all cases, often by orders of magnitude and so should not be used.

Table 1: Simulated MSE for Estimating the Median ($a = 1, b = 0$)

$\mu = 0$						
		$\sigma = 1$			$\sigma = 2$	
$\widehat{\theta}$	$n = 10$	$n = 20$	$n = 50$	$n = 10$	$n = 20$	$n = 50$
$\widehat{\theta}_0$	0.1058	0.0514	0.0202	0.5084	0.2229	0.0834
$\widehat{\theta}_{\text{R}}$	**0.0991**	**0.0493**	**0.0198**	0.3842	**0.1887**	**0.0774**
$\widehat{\theta}_{\text{GT}(c_2)}$	**0.0991**	**0.0493**	**0.0198**	**0.3839**	**0.1887**	**0.0774**
$\widehat{\theta}_{\text{MM}}$	0.1189	0.0545	0.0207	0.7827	0.2815	0.919
$\mu = 1$						
$\widehat{\theta}_0$	0.7820	0.3794	0.1493	3.7564	1.6471	0.6160
$\widehat{\theta}_{R}$	**0.7319**	**0.3642**	**0.1466**	2.8387	**1.3944**	**0.5720**
$\widehat{\theta}_{\text{GT}(c_2)}$	0.7320	**0.3642**	**0.1466**	**2.8368**	1.3945	**0.5720**
$\widehat{\theta}_{\text{MM}}$	0.8783	0.4030	0.1530	5.7835	2.0800	0.6790

The smallest MSE for each combination of μ, σ and n is shown in bold.

Table 2: Simulated MSE for Estimating the Mean ($a = 1, b = 1$)

$\mu = 0$						
		$\sigma = 1$			$\sigma = 2$	
$\widehat{\theta}$	$n = 10$	$n = 20$	$n = 50$	$n = 10$	$n = 20$	$n = 50$
$\widehat{\theta}_0$	0.4830	0.2215	0.0843	2504.57	109.61	20.043
$\widehat{\theta}_{\text{R}}$	0.3078	0.1740	0.0762	27.085	19.919	10.631
$\widehat{\theta}_{\text{GT}(c_2)}$	0.3067	0.1736	0.0761	25.003	18.545	10.428
$\widehat{\theta}_{\text{MM}}$	**0.3008**	**0.1701**	**0.0752**	**24.613**	**17.053**	**0.2659**
$\mu = 1$						
$\widehat{\theta}_0$	3.5686	1.6364	0.6227	19142	809.99	148.10
$\widehat{\theta}_{\text{R}}$	2.2745	1.2859	0.5629	200.15	147.18	78.556
$\widehat{\theta}_{\text{GT}(c_2)}$	2.2659	1.2830	0.5626	184.75	137.02	77.006
$\widehat{\theta}_{\text{MM}}$	**2.2224**	**1.2568**	**0.5555**	**181.87**	**126.00**	**68.467**

The smallest MSE for each combination of μ, σ and n is shown in bold.

5.2 Accuracy of Asymptotic Expansions of MSE

It is of interest to assess the accuracy of the asymptotic expansions of the MSEs of the estimators by comparing them to their respective simulation estimates. We do not report the results for $\widehat{\theta}_{\text{MM}}$ since the asymptotic expansion for its MSE is not available when d^2 is fixed (which is the case here with $d^2 = 1/n$). The results for the median, mean and the second moment are given in Tables 4, 5 and 6, respectively.

We calculated the percentage errors between the asymptotic expansions and the respective simulation estimates. The main conclusions are as follows. In general, the expansions of the MSEs of the estimators are accurate (errors $< 10\%$) for the median and mean, but not so accurate for the second moment. The

Table 3: Simulated $\sqrt{\text{MSE}}$ for Estimating the Second Moment ($a = 2, b = 4$)

$\widehat{\theta}$	$\sigma = 1$, $n = 10$	$\sigma = 1$, $n = 20$	$\sigma = 1$, $n = 50$	$\sigma = 2$, $n = 10$	$\sigma = 2$, $n = 20$	$\sigma = 2$, $n = 50$
$\mu = 0$						
$\widehat{\theta}_0$	50.046	10.470	4.4769	9.229×10^{16}	1.018×10^{10}	255986
$\widehat{\theta}_{\text{R}}$	5.2044	4.4631	3.2606	3934	6163	5175
$\widehat{\theta}_{\text{GT}(c_2)}$	5.0002	4.3063	3.2293	**2940**	**2810**	3051
$\widehat{\theta}_{\text{MM}}$	**4.9611**	**4.1295**	**3.0440**	2967	2944	**2772**
$\mu = 1$						
$\widehat{\theta}_0$	369.79	77.372	33.080	6.483×10^{17}	7.505×10^{10}	1.8228×10^{6}
$\widehat{\theta}_{\text{R}}$	38.455	32.980	24.093	29660	45424	38159
$\widehat{\theta}_{\text{GT}(c_2)}$	36.947	31.821	23.862	**21726**	**20763**	22543
$\widehat{\theta}_{\text{MM}}$	**36.658**	**30.513**	**22.492**	21921	21754	**20486**

The smallest $\sqrt{\text{MSE}}$ for each combination of μ, σ and n is shown in bold.

Table 4: Asymptotic MSE for Estimating the Median ($a = 1$, $b = 0$)

$\widehat{\theta}$	$\sigma = 1$, $n = 10$	$\sigma = 1$, $n = 20$	$\sigma = 1$, $n = 50$	$\sigma = 2$, $n = 10$	$\sigma = 2$, $n = 20$	$\sigma = 2$, $n = 50$
$\mu = 0$						
$\widehat{\theta}_0$	0.1058	0.0513	0.0202	0.5084	0.2229	0.0834
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	0.0980	0.0493	0.0198	0.3878	0.1887	0.0774
$\mu = 1$						
$\widehat{\theta}_0$	0.7820	0.3794	0.1493	3.7563	1.6470	0.6160
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	0.7315	0.3642	0.1466	2.8657	1.3946	0.5720

Table 5: Asymptotic MSE for Estimating the Mean ($a = 1$, $b = 1$)

$\widehat{\theta}$	$\sigma = 1$, $n = 10$	$\sigma = 1$, $n = 20$	$\sigma = 1$, $n = 50$	$\sigma = 2$, $n = 10$	$\sigma = 2$, $n = 20$	$\sigma = 2$, $n = 50$
$\mu = 0$						
$\widehat{\theta}_0$	0.4726	0.2203	0.0842	234.28	70.059	18.629
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	0.3013	0.1726	0.0761	23.951	19.266	10.580
$\mu = 1$						
$\widehat{\theta}_0$	3.4922	1.6277	0.6221	1731.1	517.67	137.65
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	2.2260	1.2756	0.5620	176.98	142.35	78.179

Table 6: Asymptotic $\sqrt{\text{MSE}}$ for Estimating the Second Moment ($a = 2$, $b = 4$)

	$\mu = 0$					
	$\sigma = 1$			$\sigma = 2$		
$\widehat{\theta}$	$n = 10$	$n = 20$	$n = 50$	$n = 10$	$n = 20$	$n = 50$
$\widehat{\theta}_0$	15.306	8.3701	4.3161	119951	42486	14282
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	4.8940	4.3893	3.2527	5410	6462	4976
	$\mu = 1$					
$\widehat{\theta}_0$	113.10	61.847	31.892	886322	313930	105528
$\widehat{\theta}_{\text{R}}$, $\widehat{\theta}_{\text{GT}(c_2)}$	36.162	32.433	24.035	39972	47748	36771

expansions are more accurate when σ and b are small (e.g., for estimating the median for which $b = 0$ versus for estimating the second moment for which $b = 4$) and n is large. Here the n-values used, 10, 20 and 50, are not large, which partly explains the inaccuracies in asymptotic expansions.

6 Numerical Example

We give a numerical example to illustrate the different values that the various estimators take for actual data. For this purpose instead of using real data we did a systematic comparison using simulated normal data (which when exponentiated yield lognormal data) with $\mu = 1$ and $\sigma = 1, 2, 3$. We used two different sample sizes, $n = 25$ and $n = 50$. Thus we focused on the effects of σ and n on the differences between the various estimators. The results are summarized in Table 7.

Table 7: Estimates of η Using Different Estimators for Numerical Example

(μ, σ)	(1, 1)	(1, 2)	(1, 3)	(1, 1)	(1, 2)	(1, 3)
η	4.4817	20.09	244.7	4.4817	20.09	244.7
	$n = 25$			$n = 50$		
$\overline{Y}$	1.022	1.443	3.067	1.158	0.982	2.336
$\overline{Z}$	4.299	39.412	410.73	4.441	13.558	157.31
$\exp(\overline{Y})$	2.7787	4.2334	21.477	3.1836	2.6698	10.340
$\widehat{\theta}_0$	4.5028	27.57	388.2	4.5588	21.71	336.2
$\widehat{\theta}_{\text{F}}$	4.4643	24.56	300.4	4.5475	20.08	274.2
$\widehat{\theta}_{\text{ES}}$	4.2941	21.38	245.0	4.4823	18.55	241.5
$\widehat{\theta}_{\text{R}}$	4.1547	18.99	205.8	4.4230	17.23	214.8
$\widehat{\theta}_{\text{GT}(c_2)}$	4.1518	18.69	196.8	4.4225	17.11	209.2
$\widehat{\theta}_{\text{MB}}$	4.4634	24.13	285.3	4.5476	19.94	267.2
$\widehat{\theta}_{\text{MM}}$	4.1334	16.31	144.7	4.4116	15.16	153.1

The values of μ, σ and the associated $\eta = \exp(\mu + \sigma^2/2)$ are listed at the top of each column. The estimates of η can be compared with the true values of η. The following observations may be made based on this example.

1. The naive estimator $\exp(\overline{Y})$ underestimates the mean η. Another naive estimator $\overline{Z}$ is highly variable, especially as σ increases. Although unbiased, it overestimates η for $n = 25$ and underestimates η for $n = 50$.

2. The simple adjustment estimator $\widehat{\theta}_0$ overestimates η in all cases.

3. All other estimators are fairly stable and are close to the true η for $\sigma = 1$. As σ increases, the differences between the estimators become more pronounced. The two estimators $\widehat{\theta}_{\mathrm{GT}(c_2)}$ and $\widehat{\theta}_{\mathrm{MM}}$, which generally had the smallest MSE in analytical and simulation comparisons underestimate η in all cases.

4. The effect of increasing n is generally to reduce the biases.

7 Conclusions

We have compared a number of estimators for the parametric function (1) and showed using asymptotic MSE and small sample simulation comparisons that the two best estimators are Shen and Zhu's (2008) minimum MSE estimator (as generalized to our setting) and our proposed estimator with Rukhin's (1986) constant c_2. For estimating the mean, Shen and Zhu's estimator is preferred, while our estimator is preferred for estimating the median. For estimating the second moment, all of the estimators perform rather poorly, but Shen and Zhu's estimator is preferred when σ is small, while our estimator is preferred when σ is large. The simple adjustment estimator should not be used as its MSE is usually much larger in small samples.

We have focused on the point estimation of θ. There has been also much work done on confidence interval estimation of θ which is reviewed in Chapter 3 of Crow and Shimizu (1988). A key reference is a paper by Land (1972). We will report our results on this problem in a separate paper.

Appendix

To derive the asymptotic expansions of the expected values and MSEs of the estimators we make use of the following basic expansions.

Result 1. The bias correction factors $g_1(t)$ and $g_2(t)$ from (7) and (13) and their squares are given up to the order $1/\nu^2$ by

$$g_1(t) = e^t \left[1 - \frac{t^2}{2}\frac{1}{\nu} + \left(\frac{t^2}{4} + \frac{t^3}{2} + \frac{t^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right], \tag{A.1}$$

$$g_2(t) = e^t \left[1 - \frac{t^2}{2}\frac{1}{\nu} + \left(\frac{t^3}{6} + \frac{t^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right], \tag{A.2}$$

$$g_1^2(t) = e^{2t} \left[1 - \frac{t^2}{\nu} + \left(t^2 + \frac{4t^3}{3} + \frac{t^4}{2}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right], \tag{A.3}$$

$$g_2^2(t) = e^{2t} \left[1 - \frac{t^2}{\nu} + \left(\frac{t^3}{3} + \frac{t^4}{2}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right]. \tag{A.4}$$

Proof of Result 1. To emphasize the dependence of $g_1(t)$ and $g_2(t)$ on ν we will denote them by $g_1(t,\nu)$ and $g_2(t,\nu)$, respectively. By starting from the power series representation for $g_1(t,\nu)$ given by (7), we get

$$\begin{aligned} g_1(t,\nu) &= 1 + t + \frac{\nu}{\nu+1}\frac{t^2}{2!} + \frac{\nu^2}{(\nu+1)(\nu+2)}\frac{t^3}{3!} + \cdots \\ &= 1 + t + \frac{1}{1+x}\frac{t^2}{2!} + \frac{1}{(1+x)(1+2x)}\frac{t^3}{3!} + \cdots \\ &= 1 + t + \sum_{j=2}^{\infty}\left[\prod_{i=1}^{j-1}(1+ix)\right]^{-1}\frac{t^j}{j!}, \end{aligned}$$

where we have put $x = 1/\nu$. We regard this last expression as a continuous extension $\widetilde{g}_1(t,x)$ of $g_1(t,\nu)$ which we treat as a continuous differentiable function of x for $x \in (0,1)$. We can then expand $\widetilde{g}_1(t,x)$ around $x = 0$ in Taylor series. First we have

$$\widetilde{g}_1(t,x)|_{x=0} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t.$$

Next to evaluate the derivatives of $\widetilde{g}_1(t,x)$ with respect to x, denote the coefficient of $t^j/j!$ in the infinite series for $\widetilde{g}_1(t,x)$ by

$$f_j(x) = \left[\prod_{i=1}^{j-1}(1+ix)\right]^{-1} \quad \text{so that} \quad \ln f_j(x) = -\sum_{i=1}^{j-1}\ln(1+ix).$$

We have

$$\frac{d\ln f_j(x)}{dx} = -\sum_{i=1}^{j-1}\frac{i}{1+ix}. \tag{A.5}$$

Therefore

$$\frac{df_j(x)}{dx} = f_j(x)\frac{d\ln f_j(x)}{dx} = -\left[\prod_{i=1}^{j-1}(1+ix)\right]^{-1}\sum_{i=1}^{j-1}\frac{i}{1+ix}$$

and

$$\left.\frac{df_j(x)}{dx}\right|_{x=0} = -\sum_{i=1}^{j-1} i = \frac{j(j-1)}{2}. \tag{A.6}$$

Hence

$$\left.\frac{d\widetilde{g}_1(t,x)}{dx}\right|_{x=0} = -\sum_{j=2}^{\infty}\frac{j(j-1)}{2}\frac{t^j}{j!} = -\sum_{j=2}^{\infty}\frac{t^2}{2}\frac{t^{j-2}}{(j-2)!} = -\frac{t^2}{2}e^t.$$

Next to evaluate the second derivative of $f_j(x)$ we use the formula

$$\frac{d^2 f_j(x)}{dx^2} = f_j(x)\frac{d^2\ln f_j(x)}{dx^2} + \frac{1}{f_j(x)}\left[\frac{df_j(x)}{dx}\right]^2. \tag{A.7}$$

From (A.5) we get

$$\frac{d^2\ln f_j(x)}{dx^2} = \sum_{i=1}^{j-1}\frac{i^2}{(1+ix)^2}.$$

Hence

$$\left.\frac{d^2\ln f_j(x)}{dx^2}\right|_{x=0} = \sum_{i=1}^{j-1} i^2 = \frac{j(j-1)(2j-1)}{6}.$$

Substituting this expression along with $f_j(0) = 1$ and $df_j(x)/dx|_{x=0} = j(j-1)/2$ from (A.6) in (A.7) we get

$$\left.\frac{d^2 f_j(x)}{dx^2}\right|_{x=0} = \frac{j(j-1)(2j-1)}{6} + \frac{j^2(j-1)^2}{4} = \frac{j(j-1)(j+1)(3j-2)}{12}.$$

Therefore

$$\begin{aligned}\left.\frac{d^2\widetilde{g}_1(t,x)}{dx^2}\right|_{x=0} &= \sum_{j=2}^{\infty}\frac{j(j-1)(j+1)(3j-2)}{12}\frac{t^j}{j!}\\ &= \sum_{j=2}^{\infty}\frac{(j+1)(3j-2)}{(j-2)!}\frac{t^j}{12}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=2}^{\infty} \frac{3j(j-1)}{j!} \frac{t^{j+2}}{12} + \sum_{j=1}^{\infty} \frac{16j}{j!} \frac{t^{j+2}}{12} + \sum_{j=0}^{\infty} \frac{12}{j!} \frac{t^{j+2}}{12} \\
&= \frac{1}{4} \sum_{j=2}^{\infty} \frac{t^{j+2}}{(j-2)!} + \frac{4}{3} \sum_{j=1}^{\infty} \frac{t^{j+2}}{(j-1)!} + \sum_{j=0}^{\infty} \frac{t^{j+2}}{j!} \\
&= \frac{t^4}{4} \sum_{j=0}^{\infty} \frac{t^j}{j!} + \frac{4t^3}{3} \sum_{j=0}^{\infty} \frac{t^j}{j!} + t^2 \sum_{j=0}^{\infty} \frac{t^j}{j!} = e^t \left(\frac{t^4}{4} + \frac{4t^3}{3} + t^2 \right).
\end{aligned}
$$

So the final expansion for $\widetilde{g}_1(t,x)$ is

$$
\widetilde{g}_1(t,x) = e^t \left[1 - \frac{t^2}{2} x + \left(\frac{t^4}{4} + \frac{4t^3}{3} + t^2 \right) \frac{x^2}{2} + o(x^2) \right]
$$

or equivalently,

$$
g_1(t,\nu) = e^t \left[1 - \frac{t^2}{2} \frac{1}{\nu} + \left(\frac{t^4}{8} + \frac{2t^3}{3} + \frac{t^2}{2} \right) \frac{1}{\nu^2} + o\left(\frac{1}{\nu^2} \right) \right].
$$

Next we find the expansion for $g_2(t,\nu)$. From (13) we see that

$$
\begin{aligned}
\exp\{-\nu[\exp(-t/\nu) - 1]\} &= \exp\left\{ -\nu \left[1 - \frac{t}{\nu} + \frac{t^2}{2\nu^2} - \frac{t^3}{6\nu^3} - 1 + o\left(\frac{1}{\nu^3} \right) \right] \right\} \\
&= \exp\left\{ t - \frac{t^2}{2} \frac{1}{\nu} + \frac{t^3}{6} \frac{1}{\nu^2} + o\left(\frac{1}{\nu^2} \right) \right\} \\
&= e^t \exp\left\{ - \frac{t^2}{2} \frac{1}{\nu} + \frac{t^3}{6} \frac{1}{\nu^2} + o\left(\frac{1}{\nu^2} \right) \right\} \\
&= e^t \left[1 - \frac{t^2}{2} \frac{1}{\nu} + \left(\frac{t^3}{6} + \frac{t^4}{8} \right) \frac{1}{\nu^2} + o\left(\frac{1}{\nu^2} \right) \right].
\end{aligned}
$$

The expressions (A.3) and (A.4) are readily obtained by squaring the expressions for $g_1(t,\nu)$ and $g_2(t,\nu)$ given in (A.1) and (A.2), respectively, and keeping only the terms up to the order $1/\nu^2$. □

Next we state another result that will be useful in the following.

Result 2.

$$
\left(1 - \frac{c\sigma^2}{2\nu} \right)^{-\nu} = e^{c\sigma^2/2} \left[1 + \frac{c^2\sigma^4}{8} \frac{1}{\nu} + \left(\frac{c^3\sigma^6}{24} + \frac{c^4\sigma^8}{128} \right) \frac{1}{\nu^2} + o\left(\frac{1}{\nu^2} \right) \right]. \tag{A.8}
$$

Proof of Result 2. Let $\gamma = c\sigma^2/2$. Then

$$
\begin{aligned}
\ln \left(1 - \frac{\gamma}{\nu} \right)^{-\nu/\gamma} &= -\frac{\nu}{\gamma} \ln \left(1 - \frac{\gamma}{\nu} \right) \\
&= -\frac{\nu}{\gamma} \left[-\frac{\gamma}{\nu} - \frac{\gamma^2}{2\nu^2} - \frac{\gamma^3}{3\nu^3} + o\left(\frac{1}{\nu^3} \right) \right] \\
&= 1 + \frac{\gamma}{2\nu} + \frac{\gamma^2}{3\nu^2} + o\left(\frac{1}{\nu^2} \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(1 - \frac{\gamma}{\nu} \right)^{-\nu/\gamma} &= \exp\left(1 + \frac{\gamma}{2\nu} + \frac{\gamma^2}{3\nu^2} + o\left(\frac{1}{\nu^2} \right) \right) \\
&= e^1 \exp\left(1 + \frac{\gamma}{2\nu} + \frac{\gamma^2}{3\nu^2} + o\left(\frac{1}{\nu^2} \right) \right) \\
&= e \left[1 - \frac{\gamma}{2\nu} + \frac{\gamma^2}{3\nu^2} + \frac{1}{2} \left(-\frac{\gamma}{2\nu} \right)^2 + o\left(\frac{1}{\nu^2} \right) \right]
\end{aligned}
$$

$$= e\left[1+\frac{\gamma}{2\nu}+\frac{11\gamma^2}{24\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].$$

It follows that

$$\begin{aligned}\left(1-\frac{\gamma}{\nu}\right)^{-\nu} &= e^{\gamma}\left[1+\frac{\gamma}{2\nu}+\frac{11\gamma^2}{24\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]^{\gamma}\\ &= e^{\gamma}\left[1+\frac{\gamma^2}{2\nu}+\frac{11\gamma^3}{24\nu^2}+\frac{\gamma(\gamma-1)}{2}\left(\frac{\gamma}{2\nu}\right)^2+o\left(\frac{1}{\nu^2}\right)\right]\\ &= e^{\gamma}\left[1+\frac{\gamma^2}{2}\frac{1}{\nu}+\left(\frac{\gamma^3}{3}+\frac{\gamma^4}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].\end{aligned}$$

Substituting back $\gamma = c\sigma^2/2$ leads to the desired result. □

Next we state a result about the expected values of some functions of S^2. We already have noted Finney's result (8).

Result 3. For a given constant c we have

$$E\left[g_2\left(cS^2/2\right)\right] = e^{c\sigma^2/2}\left[1-\left(\frac{c^2\sigma^4}{8}+\frac{c^3\sigma^6}{16}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right], \tag{A.9}$$

$$E\left[g_1^2\left(cS^2/2\right)\right] = e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{4}\frac{1}{\nu}+\frac{c^4\sigma^8}{32}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right], \tag{A.10}$$

$$E\left[g_2^2\left(cS^2/2\right)\right] = e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c^2\sigma^4}{4}-\frac{c^3\sigma^6}{8}+\frac{c^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]. \tag{A.11}$$

Proof of Result 3.

First consider (A.9). From (A.2) we get

$$E\left[g_2(cS^2/2)\right] = E\left[e^{cS^2/2}\left\{1-\frac{c^2S^4}{8}\frac{1}{\nu}+\left(\frac{c^3S^6}{48}+\frac{c^4S^8}{128}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right\}\right].$$

The expected values of $e^{cS^2/2}$, $S^4e^{cS^2/2}$, $S^6e^{cS^2/2}$ and $S^8e^{cS^2/2}$ are obtained from (6) by substituting $p = 0, 2, 3$ and 4. Thus we get

$$\begin{aligned}E\left[g_2(cS^2/2)\right] &= \left(1-\frac{c\sigma^2}{2\nu}\right)^{-\nu}\left[1-\frac{c^2}{8\nu}\frac{\Gamma(\nu+2)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^2\left(1-\frac{c\sigma^2}{2\nu}\right)^{-2}\right.\\ &\left.+\frac{c^3}{48\nu^2}\frac{\Gamma(\nu+3)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^3\left(1-\frac{c\sigma^2}{2\nu}\right)^{-3}+\frac{c^4}{128\nu^2}\frac{\Gamma(\nu+4)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^4\left(1-\frac{c\sigma^2}{2\nu}\right)^{-4}+o\left(\frac{1}{\nu^2}\right)\right].\end{aligned} \tag{A.12}$$

Now for $p = 2, 3, 4$, we have

$$\begin{aligned}\frac{\Gamma(\nu+p)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^p\left(1-\frac{c_0\sigma^2}{2\nu}\right)^{-p} &= \sigma^{2p}\frac{(\nu+p-1)\cdots\nu\Gamma(\nu)}{\nu^p\Gamma(\nu)}\left(1-\frac{c_0\sigma^2}{2\nu}\right)^{-p}\\ = \sigma^{2p}\left(1+\frac{p-1}{\nu}\right)\cdots\left(1+\frac{1}{\nu}\right)&\left[1+p\frac{c_0\sigma^2}{2\nu}-\frac{p(p+1)}{2}\left(\frac{c_0\sigma^2}{2\nu}\right)^2+o\left(\frac{1}{\nu^2}\right)\right].\end{aligned} \tag{A.13}$$

Keeping the terms only up to the order $1/\nu^2$ we see that the second term inside the square brackets in (A.12) equals

$$-\frac{c^2\sigma^4}{8\nu}\left(1+\frac{1}{\nu}\right)\left[1+\frac{c\sigma^2}{\nu}+o\left(\frac{1}{\nu}\right)\right] = -\frac{c^2\sigma^4}{8\nu}\left[1+\frac{c\sigma^2}{\nu}+\frac{1}{\nu}+o\left(\frac{1}{\nu}\right)\right]$$

$$= -\frac{c^2\sigma^4}{8}\frac{1}{\nu} - \left(\frac{c^3\sigma^6}{8} + \frac{c^2\sigma^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right).$$

Next the third term inside the square brackets in (A.12) equals

$$\frac{c^3\sigma^6}{48\nu^2}\left(1+\frac{1}{\nu}\right)\left(1+\frac{2}{\nu}\right)\left[1+\frac{3c\sigma^2}{2\nu}+o\left(\frac{1}{\nu}\right)\right] = \frac{c^3\sigma^6}{48}\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right).$$

Finally, the fourth term inside the square brackets in (A.12) equals

$$\frac{c^4\sigma^8}{128}\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right).$$

Using Result 2, adding these terms and simplifying we get the final expression in (A.9).

Next consider (A.10). From (A.3) and Result 2 we have

$$\begin{aligned}
E\left[g_1^2(cS^2/2)\right] &= E\left[e^{cS^2}\left\{1-\frac{c^2S^4}{4}\frac{1}{\nu}+\left(\frac{c^2S^4}{4}+\frac{c^3S^6}{6}+\frac{c^4S^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right\}\right]\\
&= \left(1-\frac{c\sigma^2}{\nu}\right)^{-\nu}\left[1-\frac{c^2}{4\nu}\frac{\Gamma(\nu+2)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^2\left(1-\frac{c\sigma^2}{\nu}\right)^{-2}+\frac{c^2}{4\nu^2}\frac{\Gamma(\nu+2)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^2\left(1-\frac{c\sigma^2}{\nu}\right)^{-2}\right.\\
&\quad\left.+\frac{c^3}{6\nu^2}\frac{\Gamma(\nu+3)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^3\left(1-\frac{c\sigma^2}{\nu}\right)^{-3}+\frac{c^4}{32\nu^2}\frac{\Gamma(\nu+4)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^4\left(1-\frac{c\sigma^2}{\nu}\right)^{-4}+o\left(\frac{1}{\nu^2}\right)\right]\\
&= e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{2}\frac{1}{\nu}+\left(\frac{c^3\sigma^6}{3}+\frac{c^4\sigma^8}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\left[1-\frac{c^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c^3\sigma^6}{3}+\frac{c^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\\
&= e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{4}\frac{1}{\nu}+\frac{c^4\sigma^8}{32}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}$$

Finally consider (A.11). From (A.4) and Result 2 we have

$$\begin{aligned}
E\left[g_2^2\left(cS^2/2\right)\right] &= E\left[e^{c_0S^2}\left\{1-\frac{c_0^2S^4}{4}\frac{1}{\nu}+\left(\frac{c_0^3S^6}{24}+\frac{c_0^4S^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right\}\right]\\
&= \left(1-\frac{c\sigma^2}{\nu}\right)^{-\nu}\left[1-\frac{c^2}{4}\frac{\Gamma(\nu+2)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^2\left(1-\frac{c\sigma^2}{\nu}\right)^{-2}\frac{1}{\nu}\right.\\
&\quad\left.+\left\{\frac{c^3}{24}\frac{\Gamma(\nu+3)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^3\left(1-\frac{c\sigma^2}{\nu}\right)^{-3}+\frac{c^4}{32}\frac{\Gamma(\nu+4)}{\Gamma(\nu)}\left(\frac{\sigma^2}{\nu}\right)^4\left(1-\frac{c\sigma^2}{\nu}\right)^{-4}\right\}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\\
&= \left(1-\frac{c\sigma^2}{\nu}\right)^{-\nu}\left[1-\frac{c^2\sigma^4}{4}\frac{1}{\nu}-\left(\frac{c^2\sigma^4}{4}+\frac{c^3\sigma^6}{2}\right)\frac{1}{\nu^2}+\frac{c^3\sigma^6}{24}\frac{1}{\nu^2}+\frac{c^4\sigma^8}{32}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\\
&= e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{2}\frac{1}{\nu}+\left(\frac{c^3\sigma^6}{3}+\frac{c^4\sigma^8}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\\
&\quad\times\left[1-\frac{c^2\sigma^4}{4}\frac{1}{\nu}-\left(\frac{c^2\sigma^4}{4}+\frac{11c^3\sigma^6}{24}-\frac{c^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\\
&= e^{c\sigma^2}\left[1+\frac{c^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c^2\sigma^4}{4}-\frac{c^3\sigma^6}{8}+\frac{c^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}$$

This completes the proof of Result 3. □

DERIVATIONS OF MSE EXPRESSIONS UNDER THE ASSUMPTION d^2 IS FIXED

DERIVATION OF (25) FOR MSE$(\widehat{\theta}_{\mathrm{F}})$.

Since $\widehat{\theta}_{\rm F}$ is an unbiased estimator of θ, from (A.10) in Result 3

$$\begin{aligned}
\mathrm{MSE}(\widehat{\theta}_{\rm F}) &= E(\widehat{\theta}_{\rm F}^2) - \theta^2 \quad \text{(since } \widehat{\theta}_{\rm F} \text{ is unbiased)} \\
&= E\left[\exp(2a\widehat{\mu})\right] \cdot E\left[g_2^2(c_0S^2/2)\right] - \theta^2 \\
&= e^{2a\mu+\delta} \cdot e^{c_0\sigma^2}\left[1 + \frac{c_0^2\sigma^4}{4}\frac{1}{\nu} + \frac{c_0^4\sigma^8}{32}\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] - \theta^2 \\
&= \theta^2 e^{\delta}\left[1 - e^{-\delta} + \frac{c_0^2\sigma^4}{4}\frac{1}{\nu} + \frac{c_0^4\sigma^8}{32}\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}$$

Derivation of (26) for $\mathrm{MSE}(\widehat{\theta}_{\rm GT})$.
First we derive the expected value of $\widehat{\theta}_{\rm GT}$ from (A.9) in Result 3

$$\begin{aligned}
E(\widehat{\theta}_{\rm GT}) &= E\left[\exp(a\widehat{\mu})\right] \cdot E\left[g_2(c_0S^2/2)\right] \\
&= e^{a\mu+\delta/2} \cdot e^{c_0\sigma^2/2}\left[1 - \left(\frac{c_0^2\sigma^4}{8} + \frac{c_0^3\sigma^6}{16}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\
&= \theta\left[1 - \left(\frac{c_0^3\sigma^6}{16} + \frac{c_0^2\sigma^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}$$

Next we evaluate $E(\widehat{\theta}_{\rm GT}^2)$ by using (A.11) in Result 3

$$\begin{aligned}
E(\widehat{\theta}_{\rm GT}^2) &= E[\exp(2a\widehat{\mu})] \cdot E[g_2^2(c_0S^2/2)] \\
&= e^{2a\mu+\delta} \cdot e^{c_0\sigma^2}\left(1 + \frac{c_0^2\sigma^4}{4}\frac{1}{\nu} + \left(-\frac{c_0^2\sigma^4}{4} - \frac{c_0^3\sigma^6}{8} + \frac{c_0^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right) \\
&= \theta^2 e^{\delta}\left[1 + \frac{c_0^2\sigma^4}{4}\frac{1}{\nu} - \left(\frac{c_0^2\sigma^4}{4} + \frac{c_0^3\sigma^6}{8} - \frac{c_0^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}$$

Substituting these expressions in

$$\mathrm{MSE}(\widehat{\theta}_{\rm GT}) = E(\widehat{\theta}_{\rm GT}^2) - 2\theta E(\widehat{\theta}_{\rm GT}) + \theta^2,$$

ignoring the terms of order higher than $1/\nu^2$ and simplifying we get the final result (26).

Derivation of (27) for $\mathrm{MSE}(\widehat{\theta}_0)$.
First we get an expression for $E(\widehat{\theta}_0)$. Using Result 2 it follows that

$$\begin{aligned}
E(\widehat{\theta}_0) &= E\left[\exp(a\widehat{\mu})\right] E\left[\exp(c_0S^2/2)\right] \\
&= e^{a\mu+a^2d^2\sigma^2/2}\left(1 - \frac{c_0\sigma^2}{2\nu}\right)^{-\nu} \quad \text{(from (6))} \\
&= e^{a\mu+\delta/2}e^{c_0\sigma^2/2}\left[1 + \frac{c_0^2\sigma^4}{8}\frac{1}{\nu} + \left(\frac{c_0^3\sigma^6}{24} + \frac{c_0^4\sigma^8}{128}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \quad \text{(from (A.8))}.
\end{aligned}$$

Putting $c_0 + a^2d^2 = b$ and $\exp(a\mu + b\sigma^2/2) = \theta$ we get

$$E(\widehat{\theta}_0) = \theta\left[1 + \frac{c_0^2\sigma^4}{8}\frac{1}{\nu} + \left(\frac{c_0^4\sigma^8}{128} + \frac{c_0^3\sigma^6}{24}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].$$

Next consider

$$
\begin{aligned}
E(\widehat{\theta}_0^2) &= E\left(e^{2a\widehat{\mu}}\right) E\left[\exp(c_0 S^2)\right] \\
&= e^{2a\mu+2a^2d^2\sigma^2}\left(1-\frac{c_0\sigma^2}{\nu}\right)^{-\nu} \\
&= \theta^2 e^{\delta}\left[1+\frac{c_0^2\sigma^4}{2}\frac{1}{\nu}+\left(\frac{c_0^3\sigma^6}{3}+\frac{c_0^4\sigma^8}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}
$$

Substituting these two expressions in

$$
\mathrm{MSE}(\widehat{\theta}_0) = E(\widehat{\theta}_0^2) - 2\theta E(\widehat{\theta}_0) + \theta^2,
$$

ignoring the terms of order higher than $1/\nu^2$ and simplifying we get the final result (27).

Derivation of (28) for $\mathrm{MSE}(\widehat{\theta}_{\mathrm{R}})$.
Using (8) we can write

$$
\begin{aligned}
E(\widehat{\theta}_{\mathrm{R}}) &= E[\exp(a\widehat{\mu})]E[g_1(c_2S^2/2)] \\
&= \exp(a\mu + a^2d^2\sigma^2/2)\exp(c_2\sigma^2/2) \\
&= \exp(a\mu+\delta/2)\exp(c_1\sigma^2/2)\exp(-c_1\sigma^2/2(\nu+1)) \\
&= \exp(a\mu+b\sigma^2/2)\exp(-b\sigma^2/2+\delta/2+(b-3a^2d^2)\sigma^2/2)\exp(-c_1\sigma^2/2(\nu+1)) \\
&= \theta\exp(-\delta)\exp(-c_1\sigma^2/2(\nu+1)).
\end{aligned}
$$

Expanding the second exponential in terms of $1/\nu$ we get

$$
\begin{aligned}
\exp\left\{\frac{c_1\sigma^2}{2(\nu+1)}\right\} &= 1-\frac{c_1}{2}\frac{\sigma^2}{\nu+1}+\frac{c_1^2}{8}\frac{\sigma^4}{(\nu+1)^2}+o\left(\frac{1}{(\nu+1)^2}\right) \\
&= 1-\frac{c_1}{2}\frac{\sigma^2}{\nu}+\frac{c_1}{2}\frac{\sigma^2}{\nu^2}+\frac{c_1^2}{8}\frac{\sigma^4}{\nu^2}+o\left(\frac{1}{\nu^2}\right) \\
&= 1-\frac{c_1}{2}\frac{\sigma^2}{\nu}+\left(\frac{c_1\sigma^2}{2}+\frac{c_1^2\sigma^4}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right).
\end{aligned}
$$

Hence

$$
E(\widehat{\theta}_{\mathrm{R}}) = \theta e^{-\delta}\left[1-\frac{c_1}{2}\frac{\sigma^2}{\nu}+\left(\frac{c_1\sigma^2}{2}+\frac{c_1^2\sigma^4}{8}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].
$$

Next derive an expression for $E(\widehat{\theta}_{\mathrm{R}}^2)$. From (A.10), we have

$$
\begin{aligned}
E(\widehat{\theta}_{\mathrm{R}}^2) &= E[\exp(2a\widehat{\mu})]E[g_1^2(c_2S^2/2)] \\
&= e^{2a\mu+2\delta}e^{c_2\sigma^2}\left[1+\frac{c_2^2\sigma^4}{4}\frac{1}{\nu}+\frac{c_2^4\sigma^8}{32}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right] \\
&= e^{2a\mu+2\delta}e^{c_1\sigma^2}e^{-\frac{c_1\sigma^2}{\nu+1}}\left[1+\frac{c_1^2\sigma^4}{4}\left(1-\frac{1}{\nu+1}\right)^2\frac{1}{\nu}+\frac{c_1^4\sigma^8}{32}\left(1-\frac{1}{\nu+1}\right)^4\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right] \\
&= \theta^2e^{-\delta}\left[1-\frac{c_1\sigma^2}{\nu+1}+\frac{c_1^2\sigma^4}{2(\nu+1)^2}+o\left(\frac{1}{\nu^2}\right)\right]\left[1+\frac{c_1^2\sigma^4}{4}\frac{1}{\nu}-\frac{c_1^2\sigma^4}{4}\frac{2}{\nu+1}\frac{1}{\nu}+\frac{c_1^4\sigma^8}{32}\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right] \\
&= \theta^2e^{-\delta}\left[1-\frac{c_1\sigma^2}{\nu}+\left(c_1\sigma^2+\frac{c_1^2\sigma^4}{2}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right]\left[1+\frac{c_1^2\sigma^4}{4}\frac{1}{\nu}+\left(-\frac{c_1^2\sigma^4}{2}+\frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right] \\
&= \theta^2e^{-\delta}\left[1-\left(c_1\sigma^2-\frac{c_1^2\sigma^4}{4}\right)\frac{1}{\nu}+\left(c_1\sigma^2-\frac{c_1^3\sigma^6}{4}+\frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2}+o\left(\frac{1}{\nu^2}\right)\right].
\end{aligned}
$$

Then

$$\begin{aligned}\mathrm{MSE}(\widehat{\theta}_{\mathrm{R}}) &= E(\widehat{\theta}^2_{\mathrm{R}}) - 2\theta E(\widehat{\theta}_{\mathrm{R}}) + \theta^2 \\ &= \theta^2 e^{-\delta}\left[1 - \left(c_1\sigma^2 - \frac{c_1^2\sigma^4}{4}\right)\frac{1}{\nu} + \left(c_1\sigma^2 - \frac{c_1^3\sigma^6}{4} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &\quad - 2\theta^2 e^{-\delta}\left[1 - \frac{c_1}{2}\frac{\sigma^2}{\nu} + \left(\frac{c_1\sigma^2}{2} + \frac{c_1^2\sigma^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] + \theta^2 \\ &= \theta^2 e^{-\delta}\left[e^{\delta} - 1 + \frac{c_1^2\sigma^4}{4}\frac{1}{\nu} + \left(-\frac{c_1^2\sigma^4}{4} - \frac{c_1^3\sigma^6}{4} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].\end{aligned}$$

Derivation of (29) for $\mathrm{MSE}(\widehat{\theta}_{\mathrm{GT}(c_2)})$.
Using (A.9) we can write

$$\begin{aligned}&E(\widehat{\theta}_{\mathrm{GT}(c_2)}) = E[\exp(a\widehat{\mu})]E[g_2(c_2S^2/2)] \\ &= e^{a\mu + a^2d^2\sigma^2/2}e^{c_2\sigma^2/2}\left[1 - \left(\frac{c_2^2\sigma^4}{8} + \frac{c_2^3\sigma^6}{16}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &= e^{a\mu + \frac{a^2d^2\sigma^2}{2}}e^{\frac{c_1\sigma^2}{2}}e^{-\frac{c_1\sigma^2}{2(\nu+1)}}\left[1 - \left(\frac{c_1^2\sigma^4}{8}\left(1 - \frac{1}{\nu+1}\right)^2 + \frac{c_1^3\sigma^6}{16}\left(1 - \frac{1}{\nu+1}\right)^3\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &= \theta e^{-\delta}\left[1 - \frac{c_1}{2}\frac{\sigma^2}{\nu} + \left(\frac{c_1\sigma^2}{2} + \frac{c_1^2\sigma^4}{8}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right]\left[1 - \left(\frac{c_1^2\sigma^4}{8} + \frac{c_1^3\sigma^6}{16}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &= \theta e^{-\delta}\left[1 - \frac{c_1\sigma^2}{2}\frac{1}{\nu} + \left(\frac{c_1\sigma^2}{2} - \frac{c_1^3\sigma^6}{16}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].\end{aligned}$$

Next derive an expression for $E(\widehat{\theta}^2_{\mathrm{GT}(c_2)})$. From (A.11), we have

$$\begin{aligned}&E(\widehat{\theta}^2_{\mathrm{GT}(c_2)}) = E[\exp(2a\widehat{\mu})]E[g_2^2(c_2S^2/2)] \\ &= e^{2a\mu+2\delta}e^{c_2\sigma^2}\left[\left(1 + \frac{c_2^2\sigma^4}{4}\frac{1}{\nu} + \left(-\frac{c_2^2\sigma^4}{4} - \frac{c_2^3\sigma^6}{8} + \frac{c_2^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right)\right] \\ &= \theta^2 e^{-\delta}\left[1 - \frac{c_1\sigma^2}{\nu+1} + \frac{c_1^2\sigma^4}{2(\nu+1)^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &\times \left[1 + \frac{c_1^2\sigma^4}{4}\frac{1}{\nu} - \frac{c_1^2\sigma^4}{4}\frac{2}{\nu+1}\frac{1}{\nu} + \left(-\frac{c_1^2\sigma^4}{4} - \frac{c_1^3\sigma^6}{8} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &= \theta^2 e^{-\delta}\left[1 - \frac{c_1\sigma^2}{\nu} + \left(c_1\sigma^2 + \frac{c_1^2\sigma^4}{2}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right]\left[1 + \frac{c_1^2\sigma^4}{4}\frac{1}{\nu} + \left(-\frac{3c_1^2\sigma^4}{4} - \frac{c_1^3\sigma^6}{8} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &= \theta^2 e^{-\delta}\left[1 - \left(c_1\sigma^2 - \frac{c_1^2\sigma^4}{4}\right)\frac{1}{\nu} + \left(c_1\sigma^2 - \frac{c_1^2\sigma^4}{4} - \frac{3c_1^3\sigma^6}{8} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].\end{aligned}$$

Then

$$\begin{aligned}\mathrm{MSE}(\widehat{\theta}_{\mathrm{GT}(c_2)}) &= E(\widehat{\theta}^2_{\mathrm{GT}(c_2)}) - 2\theta E(\widehat{\theta}_{\mathrm{GT}(c_2)}) + \theta^2 \\ &= \theta^2 e^{-\delta}\left[1 - \left(c_1\sigma^2 - \frac{c_1^2\sigma^4}{4}\right)\frac{1}{\nu} + \left(c_1\sigma^2 - \frac{c_1^2\sigma^4}{4} - \frac{3c_1^3\sigma^6}{8} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] \\ &\quad - 2\theta^2 e^{-\delta}\left[1 - \frac{c_1\sigma^2}{2}\frac{1}{\nu} + \left(\frac{c_1\sigma^2}{2} - \frac{c_1^3\sigma^6}{16}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right] + \theta^2 \\ &= \theta^2 e^{-\delta}\left[e^{\delta} - 1 + \frac{c_1^2\sigma^4}{4}\frac{1}{\nu} + \left(-\frac{c_1^2\sigma^4}{4} - \frac{c_1^3\sigma^6}{4} + \frac{c_1^4\sigma^8}{32}\right)\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right)\right].\end{aligned}$$

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